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L_p-INVERSE THEOREM FOR MODIFIED BETA OPERATORS

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We obtain a converse theorem for the linear combinations of modified beta operators whose weight function is the Baskakov operators. To prove our inverse theorem, we use the technique of linear approximating method, namely, Steklov mean.

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1. Introduction. For $f \in L_p[0,\infty)$, $p \geq 1$, modified beta operators with the weight function of Baskakov operators are defined as

$$B_n(f,x) = \frac{(n-1)}{n} \sum_{v=0}^{\infty} b_{n,v}(x) \int_0^{\infty} p_{n,v}(t) f(t)\,dt, \quad x \in [0,\infty), \tag{1.1}$$

where

$$\begin{aligned} b_{n,v}(x) &= \frac{1}{B(v+1,n)} x^v (1+x)^{-n-v-1}, \\ p_{n,v}(x) &= \binom{n+v-1}{v} x^v (1+x)^{-n-v}, \end{aligned} \tag{1.2}$$

and $B(v+1,n)$ being the beta function (see, e.g., [3]).

It is easily verified that the operators B_n are linear positive operators. Also, $B_n(1,x) = 1$. It turns out that the order of approximation for the operators (1.1) is at best $O(n^{-1})$ howsoever smooth the function may be. With the aim of improving the order of approximation, we have to slack the positive condition of these operators for which we may take appropriate linear combinations of the operators (1.1). Now we consider the linear combinations $B_n(f,k,x)$ of the operators $B_{d_j n}(f,x)$ as

$$B_n(f,k,x) = \sum_{j=0}^{k} C(j,k) B_{d_j n}(f,x), \tag{1.3}$$

where

$$C(j,k) = \prod_{\substack{i=0 \\ i \neq j}}^{k} \frac{d_j}{d_j - d_i}, \quad k \neq 0, \; C(0,0) = 1 \tag{1.4}$$

and $d_0, d_1, d_2, \ldots, d_k$ are $(k+1)$ arbitrary but fixed distinct positive integers.

Throughout this note, let $0 < a_1 < a_2 < a_3 < b_3 < b_2 < b_1 < \infty$, $0 < a < b < \infty$, and $I_i = [a_i, b_i]$, $i = 1,2,3$.

For $f \in L_p[0,\infty)$, $1 \le p < \infty$, the Steklov mean $f_{\eta,m}$ of mth order corresponding to f is defined as

$$f_{\eta,m}(t) = \eta^{-m}\int_{-\eta/2}^{\eta/2}\int_{-\eta/2}^{\eta/2}\cdots\int_{-\eta/2}^{\eta/2}\left[f(t)+(-1)^{m-1}\Delta^m_{\sum_{i=1}^m t_i}f(t)dt_1,dt_2,\dots,dt_m\right], \tag{1.5}$$

where $t \in I$ and $\Delta_h^m f(t)$ is the mth order forward difference of the function f with step length h. It follows from [5, 7] that

(i) $f_{\eta,m}$ has derivatives up to order m, $f_{\eta,m}^{(m-1)} \in \mathrm{AC}(I_1)$, and $f_{\eta,m}^{(m-1)}$ exists a.e. and belongs to $L_p(I_1)$;

(ii) $\|f_{\eta,m}^{(r)}\|_{L_p(I_2)} \le K_1\eta^{-r}\omega_r(f,\eta,p,I_1)$, $r = 1,2,\dots,m$;

(iii) $\|f - f_{\eta,m}\|_{L_p(I_2)} \le K_2\omega_m(f,\eta,p,I_1)$;

(iv) $\|f_{\eta,m}\|_{L_p(I_2)} \le K_3\|f\|_{L_p(I_1)}$;

(v) $\|f_{\eta,m}^{(m)}\|_{L_p(I_2)} \le K_4\eta^{-m}\|f\|_{L_p(I_1)}$.

In this note, we obtain an inverse theorem in L_p-approximation for the linear combinations of the operators (1.1).

2. Auxiliary results.

In this section, we give certain results which are necessary to prove the inverse result.

Lemma 2.1 [3]. *Let the mth order moment be defined as*

$$T_{n,m}(x) = \frac{(n-1)}{n}\sum_{v=0}^{\infty} b_{n,v}(x)\int_0^\infty p_{n,v}(t)(t-x)^m dt, \tag{2.1}$$

then $T_{n,0}(x) = 1$ and $T_{n,1}(x) = (1+3x)/(n-2)$ and there holds the recurrence relation

$$\begin{aligned}(n-m-2)T_{n,m+1}(x) &= x(1+x)\left[T^{(1)}_{n,m}(x) + 2mT_{n,m-1}(x)\right]\\ &\quad + \left[(1+2x)(m+1)+x\right]T_{n,m}(x), \quad n > m+2.\end{aligned} \tag{2.2}$$

Consequently for each $x \in [0,\infty)$,

$$\mu_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right). \tag{2.3}$$

Lemma 2.2. *Let $h \in L_1[0,\infty)$ have a compact support, then, for $n \in \mathbb{N}$,*

$$\left\|\frac{n-1}{n}\int_0^\infty b_{n,v}(x)p_{n,v}(t)\left(\frac{v}{n}-t\right)^m h(t)dt\right\|_{L_1[0,\infty)} \le K_5 n^{-m/2}\|h\|_{L_1[0,\infty)}, \tag{2.4}$$

where the constant K_5 is independent of n and h.

Proof. Applying Fubini's theorem and Holder's inequality, we obtain

$$
\begin{aligned}
&\frac{n-1}{n}\int_0^\infty\int_0^\infty\sum_{v=0}^\infty b_{n,v}(x)p_{n,v}(t)\left|\frac{v}{n}-x\right|^m|h(t)|\,dt\,dx\\
&\quad=\int_0^\infty\sum_{v=0}^\infty\left\{\int_0^\infty\frac{n-1}{n}b_{n,v}(x)\left|\frac{v}{n}-x\right|^m dx\right\}p_{n,v}(t)|h(t)|\,dt\\
&\quad\le\int_0^\infty\sum_{v=0}^\infty\left\{\int_0^\infty\frac{n-1}{n}b_{n,v}(x)\left(\frac{v}{n}-x\right)^{2m} dx\right\}^{1/2}p_{n,v}(t)|h(t)|\,dt\\
&\quad=\int_0^\infty\sum_{v=0}^\infty\left\{\sum_{j=0}^{2m}\binom{2m}{j}\left(\frac{v}{n}\right)^{2m-j}(-1)^j\int_0^\infty\frac{n-1}{n}b_{n,v}(x)x^j dx\right\}^{1/2}\\
&\qquad\qquad\times p_{n,v}(t)|h(t)|\,dt\\
&\quad=\int_0^\infty\sum_{v=0}^\infty\left\{\sum_{j=0}^{2m}\binom{2m}{j}\left(\frac{v}{n}\right)^{2m-j}(-1)^j\frac{(n+v)!}{n\cdot v!(n-2)!}B(v+j+1,n-j)\right\}^{1/2}\\
&\qquad\qquad\times p_{n,v}(t)|h(t)|\,dt\\
&\quad=\int_0^\infty\sum_{v=0}^\infty\left\{\sum_{j=0}^{2m}\binom{2m}{j}\left(\frac{v}{n}\right)^{2m-j}(-1)^j\right.\\
&\qquad\qquad\left.\times\frac{(v/n+j/n)(v/n+(j-1)/n)\cdots(v/n+1/n)}{(1-2/n)(1-3/n)\cdots(1-j/n)}\right\}^{1/2}\\
&\qquad\qquad\times p_{n,v}(t)|h(t)|\,dt.
\end{aligned}
\tag{2.5}
$$

Now, we use the identities

$$
\begin{aligned}
&\prod_{i=1}^{j}\left(\frac{v}{n}+\frac{i}{n}\right)=\left(\frac{v}{n}\right)^j+\left(\frac{v}{n}\right)^{j-1}\frac{1}{n}P_1(j)+\left(\frac{v}{n}\right)^{j-2}\frac{1}{n^2}P_2(j)+\cdots,\\
&\frac{1}{\prod_{i=2}^{j}(1-\frac{i}{n})}=1+\frac{1}{n}Q_1(j)+\frac{1}{n^2}Q_2(j)+\cdots,
\end{aligned}
\tag{2.6}
$$

where $P_k(j)$ and $Q_k(j)$ are polynomials in j of degree $2k$.

Using the fact $\sum_{j=0}^{2m}\binom{2m}{j}(-1)^j j^k=0$, $k<2n$, we have

$$
\begin{aligned}
&\left\|\frac{n-1}{n}\int_0^\infty\sum_{v=0}^\infty b_{n,v}(x)p_{n,v}(t)\left(\frac{v}{n}-x\right)^m h(t)\,dt\right\|_{L_1[0,\infty)}\\
&\quad\le K_6\int_0^\infty\sum_{v=0}^\infty\left\{\left(\frac{v}{n}\right)^{2m}\frac{1}{n^m}+\frac{1}{n}\left(\frac{v}{n}\right)^{2m-1}\frac{1}{n^{m-1}}+\cdots+\frac{1}{n^{2m}}\right\}^{1/2}p_{n,v}(t)|h(t)|\,dt.
\end{aligned}
\tag{2.7}
$$

Now applying Holder's inequality for summation and by the compactness of h, we obtain the required result. □

Lemma 2.3. *Let $h \in L_p[0,\infty)$, $p > 1$, have a compact support, $i, j \in \mathbb{N} \cup \{0\}$. Then for $m > 0$, there holds*

$$\begin{aligned}&\left\| \frac{n-1}{n} \int_0^\infty \sum_{v=0}^\infty b_{n,v}(x) p_{n,v}(t) \left(\frac{v}{n} - x\right)^i \int_x^t (t-w)^j h(w)\,dw\,dt \right\|_{L_p(I_2)} \\ &\quad \le K_7 \{ n^{-(i+j+1)/2} \|h\|_{L_p(I_1)} + n^{-m} \|h\|_{L_p[0,\infty)} \}.\end{aligned} \tag{2.8}$$

Proof. Applying Jensen's inequality repeatedly,

$$\begin{aligned}&\left| \sum_{v=0}^\infty b_{n,v}(x) \left(\frac{v}{n} - x\right)^i \int_0^\infty \frac{(n-1)}{n} p_{n,v}(t) \int_x^t (t-w)^j h(w)\,dw\,dt \right|^p \\ &\quad \le \sum_{v=0}^\infty b_{n,v}(x) \left| \frac{v}{n} - x \right|^{ip} \int_0^\infty \frac{(n-1)}{n} p_{n,v}(t) |t-x|^s \\ &\qquad\qquad \times \left| \int_t^u |h(w)|^p dw \right| dt, \quad s = jp + p - 1 \\ &\quad = \sum_{v=0}^\infty b_{n,v}(x) \left| \frac{v}{n} - x \right|^{ip} \\ &\qquad \times \int_0^\infty \frac{(n-1)}{n} \varphi(t) p_{n,v}(t) |t-x|^s \left| \int_t^u |h(w)|^p dw \right| dt \\ &\qquad + \sum_{v=0}^\infty b_{n,v}(x) \left| \frac{v}{n} - x \right|^{ip} \\ &\qquad \times \int_0^\infty \frac{(n-1)}{n} (1-\varphi(t)) p_{n,v}(t) |t-x|^s \left| \int_t^u |h(w)|^p dw \right| dt.\end{aligned} \tag{2.9}$$

We break the interval $[x,t]$ in the first term as

$$\bigcup_{l=0}^m \left([x, x+(l+1)n^{-1/2}] \cup [x-(l+1)n^{-1/2}, x] \right), \tag{2.10}$$

where $mn^{-1/2} \le \max\{b_1 - a_2, b_2 - a_1\} < (m+1)n^{-1/2}$, which is similar to [4, Theorem 2] and [2, Theorem 3.2]. A typical element of the first term is now L_p-bounded by

$$\begin{aligned}&\frac{n^2}{l^4} \int_{a_2}^{b_2} \left[\sum_{v=0}^\infty b_{n,v}(x) \left| \frac{v}{n} - x \right|^{ip} \int_{x+(l)n^{-1/2}}^{x+(l+1)n^{-1/2}} \frac{(n-1)}{n} p_{n,v}(t) |t-x|^{s+4} dt \right] \\ &\qquad \times \left(\int_{x+(l)n^{-1/2}}^{x+(l+1)n^{-1/2}} \varphi(w) |h(w)|^p dw \right) dx.\end{aligned} \tag{2.11}$$

We apply Holder's inequality for infinite sum, Lemma 2.1, and Fubini's theorem to obtain the required estimate. The presence of the factor $(1-\varphi(t))$ in the second term of (2.9) implies that $|t-x|/\delta > 1$, which gives arbitrary order $O(n^{-m})$. This completes the proof of the lemma. □

Lemma 2.4. *There exist polynomials $q_{i,j,r}(x)$ independent of n and v such that*

$$\{x(1+x)\}^r \frac{d^r}{dx^r}(b_{n,v}(x)) = \sum_{\substack{2i+j\le r\\ i,j\ge 0}} (n+1)^i [v-(n+1)x]^j q_{i,j,r}(x) b_{n,v}(x). \tag{2.12}$$

The proof of the above lemma is similar to [3, Lemma 2.2].

Lemma 2.5. *Let $h \in L_p[0,\infty)$, $p \ge 1$ and $\operatorname{supp} h \subset I_2$, then*

$$||B_n^{(2k+2)}(h,\cdot)||_{L_p(I_2)} \le K_8 n^{k+1} \|h\|_{L_p(I_2)}. \tag{2.13}$$

Moreover, if $h^{(2k+1)} \in \mathrm{AC}(I_2)$ and $h^{(2k+2)} \in L_p(I_2)$, then

$$||B_n^{(2k+2)}(h,\cdot)||_{L_p(I_2)} \le K_9 ||h^{(2k+2)}||_{L_p(I_2)}, \tag{2.14}$$

the constants K_8 and K_9 are independent of n and h.

Proof. Since functions $q_{i,j,2k+2}(x)$ and $\{x(1+x)\}^{-(2k+2)}$ are bounded on I_2, it follows from Lemmas 2.2 and 2.4 that, for $h \in L_1[0,\infty)$,

$$||B_n^{(2k+2)}(h,\cdot)||_{L_p(I_2)} \le K_{10} n^{k+1} \|h\|_{L_p(I_2)}. \tag{2.15}$$

If $h \in L_\infty[0,\infty)$, then by Lemmas 2.1 and 2.4, we get

$$||B_n^{(2k+2)}(h,\cdot)||_{L_\infty(I_2)} \le K_{11} n^{k+1} \|h\|_{L_\infty(I_2)}. \tag{2.16}$$

Now applying Riesz-Thorin interpolation theorem [6], we get (2.13). To prove (2.14), we have

$$h(t) = \sum_{r=0}^{2k+1} \frac{(t-x)^r}{r!} h^{(r)}(x) + \frac{1}{(2k+1)!} \int_x^t (t-w)^{2k+1} h^{(2k+2)}(w)\,dw. \tag{2.17}$$

Using Lemmas 2.1 and 2.3, it is easily verified that

$$B_n^{(2k+2)}(h,x) = \frac{(n-1)}{n(2k+1)\{x(1+x)\}^{2k+2}} \sum_{v=0}^{\infty} b_{n,v}(x)$$
$$\cdot \Bigg\{ \sum_{\substack{2i+j\le 2k+2\\ i,j\ge 0}} (n+1)^i \{v-(n+1)x\}^j q_{i,j,2k+2}(x)$$
$$\times \int_0^\infty p_{n,v}(t) \int_x^t (t-w)^{2k+1} h^{(2k+2)}(w)\,dw\,dt \Bigg\}. \tag{2.18}$$

Now applying Lemma 2.3, we obtain the estimate (2.14). □

3. Inverse theorem.

In this section, we prove the following inverse theorem.

Theorem 3.1. *Let* $0 < \alpha < 2k+2$, $f \in L_p[0,\infty)$, $p \ge 1$, *and*

$$\|B_n(f,k,x) - f(x)\|_{L_p(I_1)} = O(n^{-\alpha/2}), \quad n \longrightarrow \infty, \tag{3.1}$$

then

$$\omega_{2k+2}(f,\tau,p,I_2) = O(\tau^\alpha), \quad \tau \longrightarrow 0. \tag{3.2}$$

Proof. We choose a function $g \in C_0^{2k+2}$ such that $\operatorname{supp} h \subset (x_2,y_2)$, $h(t) = 1$ on $[x_3,y_3]$, and $a_1 < x_1 < x_2 < x_3 < a_2 < b_2 < y_3 < y_2 < y_1 < b_1$. Writing $fh = \bar{f}$ for all values of $\gamma \le \tau$, we have

$$\Delta_\gamma^{2k+2} \bar{f}(x) = \Delta_\gamma^{2k+2}(\bar{f}(t) - B_n(\bar{f},k,x)) + \Delta_\gamma^{2k+2} B_n(\bar{f},k,x), \tag{3.3}$$

where Δ_γ^{2k+2} denotes the $(2k+2)$th order forward difference. Applying Jensen's inequality repeatedly and Fubini's theorem for the second term, we have

$$\begin{aligned}
\|\Delta_\gamma^{2k+2}\bar{f}\|_{L_p[x_2,y_2]} &\le \|\Delta_\gamma^{2k+2}(\bar{f} - B_n(\bar{f},k,\cdot))\|_{L_p[x_2,y_2]} \\
&\quad + \gamma^{2k+2}\|B_n^{(2k+2)}(\bar{f},k,\cdot)\|_{L_p[x_2,y_2+(2k+2)\gamma]} \\
&\le \|\Delta_\gamma^{2k+2}(\bar{f} - B_n(\bar{f},k,\cdot))\|_{L_p[x_2,y_2]} \\
&\quad + \gamma^{2k+2}\|B_n^{(2k+2)}(\bar{f} - \bar{f}_{\eta,2k+2},k,\cdot)\|_{L_p[x_2,y_2+(2k+2)\gamma]} \\
&\quad + \|B_n^{(2k+2)}(\bar{f}_{\eta,2k+2},k,\cdot)\|_{L_p[x_2,y_2+(2k+2)\gamma]}.
\end{aligned} \tag{3.4}$$

Applying Lemma 2.5 and using the properties of Steklov mean, we get

$$\begin{aligned}
\|\Delta_\gamma^{2k+2}\bar{f}\|_{L_p[x_2,y_2]} &\le \|\Delta_\gamma^{2k+2}(\bar{f} - B_n(\bar{f},k,\cdot))\|_{L_p[x_2,y_2]} \\
&\quad + K_{12}\gamma^{2k+2}(n^{k+1} + \eta^{-(2k+2)})\,\omega_{2k+2}(\bar{f},\eta,p,[x_2,y_2]).
\end{aligned} \tag{3.5}$$

Following [1], we can complete the proof of the theorem if we show that

$$||\Delta_{\gamma}^{2k+2}(\bar{f}-B_n(\bar{f},k,\cdot))||_{L_p[x_2,y_2]} = O(n^{-\alpha/2}), \quad n \longrightarrow \infty, \tag{3.6}$$

therefore,

$$\omega_{2k+2}(\bar{f},\tau,p,[x_2,y_2]) = O(\tau^{\alpha}), \quad \tau \longrightarrow 0. \tag{3.7}$$

For $t \in [x_3,y_3]$, $\bar{f}(t) = f(t)$, thus

$$\omega_{2k+2}(f,\tau,p,I_2) = O(\tau^{\alpha}), \quad \tau \longrightarrow 0 \text{ as required.} \tag{3.8}$$

We will prove (3.6) by the principle of mathematical induction on α. First, we consider the case $\alpha \le 1$, thus

$$\begin{aligned} ||B_n(fh,k,\cdot)-fh||_{L_p[x_2,y_2]} \le{}& ||B_n(h(x)(f(t)-f(x)),k,\cdot)||_{L_p[x_2,y_2]} \\ &+||B_n(f(t)(h(t)-h(x)),k,\cdot)||_{L_p[x_2,y_2]}. \end{aligned} \tag{3.9}$$

Now $|h(t)-h(x)| = |t-x||h'(\xi)|$ for some ξ lying between t and x. Using Lemma 2.1 and the compactness of f to estimate the second term, and the assumption of the theorem for the first term, we get

$$||B_n(fh,k,\cdot)-fh||_{L_p[x_2,y_2]} = O(n^{-1/2})+O(n^{-\alpha/2}). \tag{3.10}$$

This completes the proof of (3.6) for the case $\alpha \le 1$.

Now, assume that (3.6) holds for all values of α satisfying $m-1 < \alpha < m$ and prove that the same holds true for $m < \alpha < m+1$. Thus, we have

$$\omega_{2k+2}(f,\tau,p,[c,d]) = O(\tau^{m-1+\beta}), \quad \tau \longrightarrow 0,\ 0<\beta<1 \tag{3.11}$$

for any $[c,d] \subset (a_1,b_1)$. Let $\phi(t)$ denote the characteristic function of $[x_1,y_1]$. The assumed smoothness of f implies that

$$\begin{aligned} &||B_n(fh,k,\cdot)-fh||_{L_p[x_2,y_2]} \\ &\quad\le \sum_{i=0}^{r-2}\frac{1}{i!}||f^{(i)}(x)B_n((t-x)^i(h(t)-h(x),k,\cdot))||_{L_p[x_2,y_2]} \\ &\qquad+\frac{1}{(r-2)!}\Bigg\|B_n(\phi(t)\Big(h(t)-h(x) \\ &\qquad\qquad\times\Big(\int_x^t(t-w)^{r-2}(f^{(r-1)}(w)-f^{(r-1)}(x))dw\Big),k,\cdot\Big)\Bigg\|_{L_p[x_2,y_2]} \\ &\qquad+||B_n(F(t,x)(1-\phi(t))(h(t)-h(x)),k,\cdot)||_{L_p[x_2,y_2]} \\ &\quad= J_1+J_2+J_3, \end{aligned} \tag{3.12}$$

where $F(t,x) = f(x)-\sum_{i=0}^{r-2}((t-x)^i/i!)f^{(i)}(x)$, $t \in [0,\infty)$, and $x \in [x_2,y_2]$.

The direct theorem in [4] and Lemma 2.1 imply that $J_1, J_3 = O(n^{-(k+1)})$, $n \to \infty$, using Jensen's inequality, mean value theorem on h, and breaking $[x,t]$ as in Lemma 2.3, we have

$$\begin{aligned}
&\int_{x_2}^{y_2} \Big| B_n\Big(\phi(t)\Big(h(t) \\
&\qquad - h(x)\Big(\int_x^t (t-w)^{r-2}\big(f^{(r-1)}(w) - f^{(r-1)}(x)\big)dw\Big), x\Big)\Big)\Big|^p dx \\
&\quad \le K_{13} \int_{x_2}^{y_2}\int_{x_1}^{y_1} W_n(x,t)|t-x|^{rp-1} \\
&\qquad\qquad \times \int_x^t \phi(w)\,\big|f^{(r-1)}(w) - f^{(r-1)}(x)\big|^p dw\,dt\,dx \\
&\quad \le K_{13}\sum_{l=1}^{r}\int_{x_2}^{y_2}\Big\{\int_{x+(l)n^{-1/2}}^{x+(l+1)n^{-1/2}} W_n(x,t)\big(n^2 l^{-4}\big)^p|t-x|^{rp+4p-1} \\
&\qquad\qquad \times \int_x^{x+(l+1)n^{-1/2}} \phi(w)\,\big|f^{(r-1)}(w) \\
&\qquad\qquad\qquad - f^{(r-1)}(x)\big|^p dw\,dt \\
&\qquad + \int_{x-(l+1)n^{-1/2}}^{x-(l)n^{-1/2}} W_n(x-t)\big(n^2 l^{-4}\big)^p|t-x|^{rp+4p-1} \\
&\qquad\qquad \times \int_{x-(l+1)n^{-1/2}}^{x} \phi(w)\,\big|f^{(r-1)}(w) \\
&\qquad\qquad\qquad - f^{(r-1)}(x)\big|^p dw\,dt\Big\} dx \\
&\quad + \int_{x_2}^{y_2}\int_{x_2-n^{-1/2}}^{y_2+n^{-1/2}} W_n(x,t)|t-x|^{rp-1} \\
&\qquad\qquad \times \int_{x-n^{-1/2}}^{x+n^{-1/2}} \phi(w)\,\big|f^{(r-1)}(w) - f^{(r-1)}(t)\big|^p dw\,dt\,dx \\
&\quad \le K_{13}\Big\{\sum_{l=1}^{r}\big(n^2 l^{-4}\big)^p n^{-((r+4)p-1)/2} \\
&\qquad\qquad \times \int_0^{(l+1)n^{-1/2}} \big(\omega(f^{(r-1)}, w, p, [x_1, y_1])\big)^p dw \\
&\qquad\qquad + n^{-(rp-1)/2}\int_0^{n^{-1/2}} \big(\omega(f^{(r-1)}, w, p, [x_1, y_1])\big)^p dw\Big\}.
\end{aligned} \tag{3.13}$$

On using Lemma 2.1, then interchanging integration in x and w, and lastly, using the fact $\omega(f^{(r-1)}, w, p, [x_1, y_1]) = O(\omega^{\beta})$, we find

$$J_2 = O\big(n^{-(r+\beta)/2}\big), \quad n \longrightarrow \infty. \tag{3.14}$$

Combining the estimates of J_1, J_2, and J_3, we obtain (3.6). The proof of (3.6) shows that

$$\omega(f,\tau,p,I_2) = O(\tau^{\alpha}), \quad \alpha < 2k+2,\ \alpha \neq 2,3,\dots,2k+1. \tag{3.15}$$

The above statement implies that it is true for integer values $2,3,\dots,2k+1$ also. To prove this, let $\alpha = \tau$, where r takes values from $2,3,\dots,2k+1$. Since (3.15) is true for $(r,r+1)$, it follows that

$$\omega_{2k+2}(f,\tau,p,I_2) = O(\tau^{r+\theta}) = O(\tau^{r}), \quad 0 < \theta < 1. \tag{3.16}$$

This completes the proof of the theorem. □

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