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DERIVATIONS ON BANACH ALGEBRAS

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Let D be a derivation on a Banach algebra; by using the operator D^2, we give necessary and sufficient conditions for the separating ideal of D to be nilpotent. We also introduce an ideal $M(D)$ and apply it to find out more equivalent conditions for the continuity of D and for nilpotency of its separating ideal.

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1. Introduction. Let A be a Banach algebra. By a derivation on A, we mean a linear mapping $D : A \to A$, which satisfies $D(ab) = aD(b) + D(a)b$ for all a and b in A. The separating space of D is the set

$$S(D) = \{a \in A : \exists \{a_n\} \subset A;\ a_n \longrightarrow 0,\ D(a_n) \longrightarrow a\}. \tag{1.1}$$

The set $S(D)$ is a closed ideal of A which, by the closed-graph theorem, is zero if and only if D is continuous.

Definition 1.1. A closed ideal J of A is said to be a separating ideal if, for each sequence $\{a_n\}$ in A, there is a natural N such that

$$\overline{(Ja_n \cdots a_1)} = \overline{(Ja_N \cdots a_1)} \quad (n \geq N). \tag{1.2}$$

The separating space of a derivation on A is a separating ideal [2, Chapter 5]; it also satisfies the same property for the left products.

The following assertions are of the most famous conjectures about derivations on Banach algebras:

(C1) every derivation on a Banach algebra has a nilpotent separating ideal;
(C2) every derivation on a semiprime Banach algebra is continuous;
(C3) every derivation on a prime Banach algebra is continuous;
(C4) every derivation on a Banach algebra leaves each primitive ideal invariant.

Clearly, if (C1) is true, then the same for (C2) and (C3). Mathieu and Runde in [5] proved that (C1), (C2), and (C3) are equivalent. The conjecture (C4) is known as the noncommutative Singer-Wermer conjecture, and it has been proved in [1] that if each of the conjectures (C1), (C2), or (C3) hold, then (C4) is also true. The conjectures (C1), (C2), and (C3) are still open even if A is assumed to be commutative, but (C4) is true in the commutative case, see [7]. These conjectures are also related to some other famous open problems; the reader is referred to [1, 3, 4, 5, 9] for more details.

In the next section, we deal with (C1), and although, for a derivation D on a Banach algebra, the operators D^n, $n = 2,3,\ldots$, are more complicated, by considering D^2, we easily give some equivalent conditions for $S(D)$ to be nilpotent. As a consequence, we reprove some of the results in [8]. At the end of the next section, we introduce an ideal related to a derivation and apply it to obtain some equivalent conditions for continuity of D and for nilpotency of $S(D)$.

We recall that $S(D)$ is nilpotent if and only if $S(D)\cap R$ is nilpotent, see [1, Lemma 4.2].

2. The results. From now on, A is a Banach algebra, and R and L denote the Jacobson radical and the nil radical of A, respectively, (see [6, Chapter 4] for definitions). Note that D is a derivation on A, and $S(D)$ is the separating ideal of D. If B_i's, $i = 1,2,\ldots,n$, are subsets of A, then $B_1B_2\cdots B_n$ denotes the linear span of the set $\{b_1b_2\cdots b_n : b_i \in B_i,\ \text{for } i = 1,2,\ldots,n\}$, and if all of B_i's coincide with each other, we denote this set by B^n.

Theorem 2.1. *Let J be a closed left ideal of A. Then, $S(D)\cap J$ is nilpotent if and only if $D^2 |_{\overline{\bigcap_{n=1}^{\infty}(S(D)\cap J)^n}}$ is continuous.*

Proof. Suppose that D^2 is continuous on $\overline{\bigcap_{n=1}^{\infty}(S(D)\cap J)^n}$. Consider a in $S(D)\cap J$, then for each $n\in\mathbb{N}$, $a^n \in (S(D)\cap J)^n$, and since $S(D)$ is a separating ideal, there exists $N\in\mathbb{N}$ such that

$$\overline{S(D)a^n} = \overline{S(D)a^N} \quad (n \geq N). \tag{2.1}$$

Hence, by the Mittag-Leffler theorem [2, Theorem A.1.25] and the fact that $S(D)a^n \subseteq (S(D)\cap J)^n$, we have

$$\overline{S(D)a^N} = \bigcap_{n=1}^{\infty}\overline{S(D)a^n} = \overline{\bigcap_{n=1}^{\infty}S(D)a^n} \subseteq \overline{\bigcap_{n=1}^{\infty}(S(D)\cap J)^n}. \tag{2.2}$$

Now, let $\{x_n\}\subseteq A$, $x_n \to 0$, and $D(x_n)\to a^{N+1}$. Take $y_n = x_na^{N+1}$, then $y_n \in S(D)a^N \subseteq \overline{\bigcap_{n=1}^{\infty}(S(D)\cap J)^n}$, $y_n \to 0$, and $D(y_n)\to a^{2(N+1)}$, and by the hypothesis, $D^2(y_n)\to 0$ and $D^2({y_n}^2)\to 0$. On the other hand,

$$D^2({y_n}^2) = y_nD^2(y_n) + 2(Dy_n)^2 + D^2(y_n)y_n \longrightarrow 2a^{4(N+1)}. \tag{2.3}$$

Therefore, $a^{4N+4} = 0$, that is, $S(D)\cap J$ is a nil and hence a nilpotent ideal by closedness [6, Theorem 4.4.11]. The converse is trivial. □

Remark 2.2. (i) Note that in Theorem 2.1, we can replace J by a right ideal, see [2, Theorem 5.2.24].

(ii) The argument of Theorem 2.1 shows that if J is not assumed to be closed and if D^2 is continuous on $\overline{\bigcap_{n=1}^{\infty}(S(D)\cap J)^n}$, then $S(D)\cap J$ will be a nil ideal.

Corollary 2.3. *The set $S(D)$ is nilpotent if and only if $D^2\,|_{\overline{\bigcap_{n=1}^{\infty}(S(D)\cap R)^n}}$ is continuous.*

Proof. If $S(D)$ is nilpotent, then the result is obvious. Conversely, by Theorem 2.1, $S(D)\cap R$ is nilpotent, and by [1, Lemma 4.2], $S(D)$ is nilpotent. □

Corollary 2.4. *If $\dim(\bigcap_{n=1}^{\infty}(S(D)\cap R)^n)<\infty$, then $S(D)$ is nilpotent.*

The assertions of the following theorem were proved by Villena in [8], see also [9, Theorem 4.4]. Using Theorem 2.1, we can reprove them in a different way.

Theorem 2.5. *The derivation D is continuous if one of the following assertions hold:*

(a) *A is semiprime and $\dim(R\cap(\bigcap_{n=1}^{\infty}A^n))<\infty$;*

(b) *A is prime and $\dim(\bigcap_{n=1}^{\infty}(aA\cap R)^n)<\infty$ for some $a\in A$ with $a^2\neq 0$;*

(c) *A is an integral domain and $\dim(\bigcap_{n=1}^{\infty}(aA\cap R)^n)<\infty$ for some nonzero $a\in A$.*

Proof. (a) By Corollary 2.4, $S(D)$ is nilpotent, and since A is semiprime, D is continuous.

(b) Without loss of generality, we may assume that A has an identity. By assumption, $\bigcap_{n=1}^{\infty}(aA\cap R\cap S(D))^n$ is finite dimensional; thus, D^2 is continuous on this space, and by Remark 2.2(ii), $aA\cap R\cap S(D)$ is a nil right ideal; therefore, $a(S(D)\cap R)$ is a nil right ideal, and by [6, Theorem 4.4.11], $a(S(D)\cap R)\subseteq L=\{0\}$. Thus, $AaA(S(D)\cap R)=\{0\}$, where AaA is the ideal generated by a. Since $a^2\neq 0$ and A is prime, it follows that $S(D)\cap R=\{0\}$ and hence $S(D)\subseteq L=\{0\}$.

(c) The same argument as in (b) shows that $a(S(D)\cap R)=\{0\}$, and since A is an integral domain, $S(D)\cap R=\{0\}$ and D is continuous. □

In the sequel, we give other equivalent conditions for $S(D)$ to be nilpotent, but first we introduce the set

$$M(D)=\{x\in S(D)\cap R : D(x)\in R\}. \tag{2.4}$$

Obviously, $M(D)$ is an ideal of A and $(S(D)\cap R)^2\subseteq M(D)$. The following theorems show that this ideal can help us to study the continuity of a derivation or nilpotency of its separating ideal.

Theorem 2.6. *The derivation D is continuous if and only if $M(D)=\{0\}$.*

Proof. Clearly, if D is continuous, then $M(D) = \{0\}$. Conversely, let $M(D) = \{0\}$; then, $(S(D)\cap R)^2 = \{0\}$. Therefore, $(S(D)\cap R)$ and hence $S(D)$ is a nilpotent ideal. Therefore, $S(D) \subseteq L$; we also have $D(L) \subseteq L$ by [1, Lemma 4.1]; thus, $D(S(D)) \subseteq R$, that is, $S(D) \subseteq M(D) = \{0\}$ and D is continuous. □

Theorem 2.7. *The following assertions are equivalent:*

(a) *$S(D)$ is nilpotent;*
(b) *$M(D)$ is a nil ideal;*
(c) *$\bigcap_{n=1}^{\infty} M(D)^n = \{0\}$.*

Proof. Clearly, (a) implies (b). Suppose that (b) holds, then $(S(D)\cap R)^2$ is a nil ideal; therefore, $S(D)$ is a nilpotent ideal and (a) holds. Now, if $S(D)$ is nilpotent, then $\bigcap_{n=1}^{\infty}(S(D)^n) = \{0\}$ and this implies (c). Finally, if $\bigcap_{n=1}^{\infty} M(D)^n = \{0\}$, then by Theorem 2.1 and Remark 2.2 $M(D) = M(D)\cap S(D)$ is a nil ideal and (c) implies (b). □

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