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ON SAKAGUCHI FUNCTIONS

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Let $S_s(\alpha)$ $(0 \le \alpha < 1/2)$ be the class of functions $f(z) = z + \cdots$ which are analytic in the unit disk and satisfy there $\operatorname{Re}\{zf'(z)/(f(z)-f(-z))\} > \alpha$. In the present paper, we find the sharp lower bound on $\operatorname{Re}\{(f(z)-f(-z))/z\}$ and investigate two subclasses $S_0(\alpha)$ and $T_0(\alpha)$ of $S_s(\alpha)$. We derive sharp distortion inequalities and some properties of the partial sums for functions in the classes $S_0(\alpha)$ and $T_0(\alpha)$.

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1. Introduction. Let A be the class of functions $f(z) = z + \cdots$ which are analytic in the unit disk $E = \{z : |z| < 1\}$. We denote by S the subclass of A consisting of functions which are univalent in E. A function $f(z) \in A$ is said to be in the class $S_s(\alpha)$ if it satisfies

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)-f(-z)}\right\} > \alpha \quad (z \in E) \tag{1.1}$$

for some α $(0 \le \alpha < 1/2)$. Further, a function $f(z) \in A$ is said to be in the class $T_s(\alpha)$ if and only if $zf'(z) \in S_s(\alpha)$ for some α $(0 \le \alpha < 1/2)$. We denote $S_s(0) = S_s$. Sakaguchi [3] introduced the class S_s and proved that $S_s \subset C(\subset S)$, where C is the class of close-to-convex functions. The class S_s has been studied by several authors (e.g., Stankiewicz [4] and Wu [6]).

For the class $S_s(\alpha)$, Owa et al. [2] have proved the following theorem.

Theorem 1.1. *If $f(z) \in S_s(\alpha)$ with $1/4 \le \alpha < 1/2$, then*

$$\operatorname{Re}\left\{\frac{f(z)-f(-z)}{z}\right\} > \frac{2}{3-4\alpha} \quad (z \in E). \tag{1.2}$$

The following two lemmas were shown by Cho et al. in [1].

Lemma 1.2. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in A$ satisfies*

$$\sum_{n=2}^{\infty} \{2(n-1)|a_{2n-2}| + (2n-1-2\alpha)|a_{2n-1}|\} \le 1-2\alpha \tag{1.3}$$

for some α $(0 \le \alpha < 1/2)$, then $f(z) \in S_s(\alpha)$.

Lemma 1.3. *If $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in A$ satisfies*

$$\sum_{n=2}^{\infty}\{4(n-1)^2|a_{2n-2}|+(2n-1)(2n-1-2\alpha)|a_{2n-1}|\}\le 1-2\alpha \tag{1.4}$$

for some α $(0\le\alpha<1/2)$, then $f(z)\in T_s(\alpha)$.

In view of Lemmas 1.2 and 1.3, Cho et al. [1] introduced the subclass $S_0(\alpha)$ of $S_s(\alpha)$ consisting of functions which satisfy inequality (1.3) in Lemma 1.2, and the subclass $T_0(\alpha)$ of $T_s(\alpha)$ consisting of functions which satisfy inequality (1.4) in Lemma 1.3. It is easy to see that $T_0(\alpha)\subset S_0(\alpha)$ for $0\le\alpha<1/2$.

The following theorems (Theorems 1.4, 1.5, and 1.6) are the main results of [1].

Theorem 1.4. *If $f(z)\in S_0(\alpha)$ with $0\le\alpha<1/2$, then*

$$\operatorname{Re}\frac{f(z)}{z}>\frac{1}{2(1-\alpha)}\quad(z\in E). \tag{1.5}$$

Theorem 1.5. *If $f(z)\in S_0(\alpha)$ with $0\le\alpha<1/2$, then*

$$\begin{aligned}|z|-\frac{1-2\alpha}{2}|z|^2-\frac{1-2\alpha}{3-2\alpha}|z|^3&\le|f(z)|\\&\le|z|+\frac{1-2\alpha}{2}|z|^2+\frac{1-2\alpha}{3-2\alpha}|z|^3,\\1-(1-2\alpha)|z|-\frac{3(1-2\alpha)}{3-2\alpha}|z|^2&\le|f'(z)|\\&\le 1+(1-2\alpha)|z|+\frac{3(1-2\alpha)}{3-2\alpha}|z|^2,\end{aligned} \tag{1.6}$$

for $z\in E$.

Theorem 1.6. *If $f(z)\in T_0(\alpha)$ with $0\le\alpha<1/2$, then*

$$\begin{aligned}|z|-\frac{1-2\alpha}{4}|z|^2-\frac{1-2\alpha}{3(3-2\alpha)}|z|^3&\le|f(z)|\\&\le|z|+\frac{1-2\alpha}{4}|z|^2+\frac{1-2\alpha}{3(3-2\alpha)}|z|^3,\\1-\frac{1-2\alpha}{2}|z|-\frac{1-2\alpha}{3-2\alpha}|z|^2&\le|f'(z)|\\&\le 1+\frac{1-2\alpha}{2}|z|+\frac{1-2\alpha}{3-2\alpha}|z|^2,\end{aligned} \tag{1.7}$$

for $z\in E$.

In the present paper, we improve the above results and find the sharp bounds.

2. Main results

Theorem 2.1. *If* $f(z) \in S_s(\alpha)$ $(0 \le \alpha < 1/2)$*, then*

$$\operatorname{Re}\left\{\frac{f(z)-f(-z)}{z}\right\} > 4^{\alpha} \quad (z \in E). \tag{2.1}$$

The result is sharp.

Proof. For $f(z) \in S_s(\alpha)$, it follows from (1.1) that the function

$$g(z) = \frac{f(z)-f(-z)}{2} \tag{2.2}$$

is an odd starlike function of order $2\alpha \in [0,1)$, that is, $g(-z) = -g(z)$ and

$$\operatorname{Re}\frac{zg'(z)}{g(z)} > 2\alpha \quad (z \in E). \tag{2.3}$$

Since every odd function in S is the square-root transform of some function in S, there exists an $h(z) \in S$ such that $g(z) = \sqrt{h(z^2)}$. Further, $h(z)$ is also starlike of order 2α, or equivalently

$$\frac{zh'(z)}{h(z)} - 1 \prec 2(1-2\alpha)\frac{z}{1-z}, \tag{2.4}$$

where the symbol $\prec$ stands for subordination.

Applying a result due to Suffridge [5, Theorem 3] to (2.4), we have

$$\int_0^z \left(\frac{h'(t)}{h(t)} - \frac{1}{t}\right) dt \prec 2(1-2\alpha)\int_0^z \frac{dt}{1-t}, \tag{2.5}$$

which yields that

$$\frac{h(z)}{z} \prec \frac{1}{(1-z)^{2(1-2\alpha)}}. \tag{2.6}$$

Thus, there exists an analytic function $w(z)$ in E such that $|w(z)| \le |z|$ and

$$\frac{h(z)}{z} = \left(\frac{1}{1-w(z)}\right)^{2(1-2\alpha)} \quad (z \in E), \tag{2.7}$$

and hence

$$\frac{g(z)}{z} = \left(\frac{1}{1-w(z^2)}\right)^{1-2\alpha} \quad (z \in E). \tag{2.8}$$

With the help of the elementary inequality $\mathrm{Re}(t^{\beta}) \geq (\mathrm{Re}\, t)^{\beta}$ $(0 < \beta \leq 1$ and $\mathrm{Re}\, t > 0)$, it follows from (2.8) that

$$\begin{aligned} \mathrm{Re}\,\frac{g(z)}{z} &\geq \left\{\mathrm{Re}\left(\frac{1}{1-w(z^2)}\right)\right\}^{1-2\alpha} \geq \left(\frac{1}{1+|w(z^2)|}\right)^{1-2\alpha} \\ &\geq \left(\frac{1}{1+|z|^2}\right)^{1-2\alpha} > \frac{1}{2^{1-2\alpha}} \quad (z \in E) \end{aligned} \tag{2.9}$$

for $0 \leq \alpha < 1/2$. This proves (2.1).

To establish the sharpness, we consider the function

$$f_0(z) = \frac{z}{(1+z^2)^{1-2\alpha}}. \tag{2.10}$$

It is easy to verify that $f_0(z) \in S_s(\alpha)$ and

$$\mathrm{Re}\left\{\frac{f_0(z) - f_0(-z)}{z}\right\} = 2\,\mathrm{Re}\left\{\frac{1}{(1+z^2)^{1-2\alpha}}\right\} \longrightarrow 4^{\alpha} \tag{2.11}$$

as $z = \mathrm{Re}\, z \to 1$. The proof of the theorem is now complete. □

Remark 2.2. This theorem improves and extends Theorem 1.1.

As an immediate consequence of Theorem 2.1, we have the following corollary.

Corollary 2.3. *If $f(z) \in S_s(\alpha)$ with $0 \leq \alpha < 1/2$ and if $f(z)$ is odd, then*

$$\mathrm{Re}\,\frac{f(z)}{z} > \frac{1}{2^{1-2\alpha}} \quad (z \in E). \tag{2.12}$$

The result is sharp with the extremal function $f_0(z)$ given by (2.10). In particular, if $f(z) \in S_s$ and $f(z)$ is odd, then

$$\mathrm{Re}\,\frac{f(z)}{z} > \frac{1}{2} \quad (z \in E) \tag{2.13}$$

and the result is sharp.

Theorem 2.4. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S_0(\alpha)$ with $0 \leq \alpha < 1/2$, then for $z \in E$,*

(a) *the following inequalities hold true:*

$$|z| - \frac{1-2\alpha}{2}|z|^2 \leq |f(z)| \leq |z| + \frac{1-2\alpha}{2}|z|^2. \tag{2.14}$$

Equalities are attained by the function

$$f(z) = z - \frac{1-2\alpha}{2} z^2 \in S_0(\alpha), \tag{2.15}$$

(b) *the following inequalities hold true:*

$$1-2|a_2||z|-\frac{3(1-2\alpha-2|a_2|)}{3-2\alpha}|z|^2 \le |f'(z)| \le 1+2|a_2||z|+\frac{3(1-2\alpha-2|a_2|)}{3-2\alpha}|z|^2. \tag{2.16}$$

Equalities are attained, for example, by

$$f(z)=z-az^2-\frac{1-2\alpha-2a}{3-2\alpha}z^3 \in S_0(\alpha) \tag{2.17}$$

or

$$f(z)=z+az^2+\frac{1-2\alpha-2a}{3-2\alpha}z^3 \in S_0(\alpha), \tag{2.18}$$

where $0 \le a \le (1-2\alpha)/2$.

Proof. (a) Writing inequality (1.3) in Lemma 1.2 as

$$\sum_{n=2}^{\infty}[n-(1+(-1)^{n+1})\alpha]|a_n| \le 1-2\alpha, \tag{2.19}$$

we arrive at $\sum_{n=2}^{\infty}|a_n| \le (1-2\alpha)/2$. Therefore, it follows from $f(z)\in S_0(\alpha)$ that for $z\in E$,

$$\begin{aligned} |f(z)| &\ge |z|-|z|^2\sum_{n=2}^{\infty}|a_n| \ge |z|-\frac{1-2\alpha}{2}|z|^2 \ge 0,\\ |f(z)| &\le |z|+|z|^2\sum_{n=2}^{\infty}|a_n| \le |z|+\frac{1-2\alpha}{2}|z|^2. \end{aligned} \tag{2.20}$$

(b) Since $n-(1+(-1)^{n+1})\alpha \ge n(1-2\alpha/3)$ for $n\ge 3$, it follows from (2.19) that

$$2|a_2|+\left(1-\frac{2\alpha}{3}\right)\sum_{n=3}^{\infty}n|a_n| \le 1-2\alpha \tag{2.21}$$

for $f(z)\in S_0(\alpha)$. Hence, we deduce that for $z\in E$,

$$\begin{aligned} |f'(z)| &\ge 1-2|a_2||z|-|z|^2\sum_{n=3}^{\infty}n|a_n|\\ &\ge 1-2|a_2||z|-\frac{3(1-2\alpha-2|a_2|)}{3-2\alpha}|z|^2 > 0,\\ |f'(z)| &\le 1+2|a_2||z|+\frac{3(1-2\alpha-2|a_2|)}{3-2\alpha}|z|^2. \end{aligned} \tag{2.22}$$

□

Remark 2.5. Note that $|a_2| \leq (1-2\alpha)/2$. Theorem 2.4 is an improvement of Theorem 1.5.

Theorem 2.6. *If $f(z) \in T_0(\alpha)$ with $0 \leq \alpha < 1/2$, then for $z \in E$, the following inequalities hold true:*

$$|z| - \frac{1-2\alpha}{4}|z|^2 \leq \left|f(z)\right| \leq |z| + \frac{1-2\alpha}{4}|z|^2, \tag{2.23}$$

$$1 - \frac{1-2\alpha}{2}|z| \leq \left|f'(z)\right| \leq 1 + \frac{1-2\alpha}{2}|z|. \tag{2.24}$$

The results are sharp with the extremal function $f(z) = z - ((1-2\alpha)/4)z^2 \in T_0(\alpha)$.

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in T_0(\alpha)$. Writing inequality (1.4) in Lemma 1.3 as

$$\sum_{n=2}^{\infty} n\left[n - (1+(-1)^{n+1})\alpha\right]\left|a_n\right| \leq 1 - 2\alpha, \tag{2.25}$$

we get $\sum_{n=2}^{\infty} |a_n| \leq (1-2\alpha)/4$. From this, we easily have (2.23).

Noting that $f(z) \in T_0(\alpha)$ if and only if $zf'(z) \in S_0(\alpha)$, (2.24) follows directly from Theorem 2.4(a). □

Remark 2.7. Theorem 2.6 improves Theorem 1.6.

Theorem 2.8. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S_0(\alpha)$ with $0 \leq \alpha < 1/2$, and let*

$$s_1(z) = z, \qquad s_m(z) = z + \sum_{n=2}^{m} a_n z^n \quad (m \geq 2). \tag{2.26}$$

Then for $z \in E$, the following inequalities hold true:

$$\operatorname{Re}\frac{f(z)}{s_m(z)} > \frac{m + (1+(-1)^{m+1})\alpha}{m+1-(1+(-1)^m)\alpha}, \tag{2.27}$$

$$\operatorname{Re}\frac{s_m(z)}{f(z)} > \frac{m+1-(1+(-1)^m)\alpha}{m+2-(3+(-1)^m)\alpha}. \tag{2.28}$$

The results are sharp for each $m \geq 1$.

Proof. Let

$$\lambda_n = \frac{n - (1+(-1)^{n+1})\alpha}{1-2\alpha} \quad (n \geq 2). \tag{2.29}$$

Then $\lambda_{n+1} > \lambda_n > 1$ $(n \geq 2)$ and it follows from (2.19) that

$$\sum_{n=2}^{m} |a_n| + \lambda_{m+1} \sum_{n=m+1}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} \lambda_n |a_n| \leq 1 \quad (m \geq 2). \tag{2.30}$$

If we let

$$p_1(z) = \lambda_{m+1}\left\{\frac{f(z)}{s_m(z)} - \left(1 - \frac{1}{\lambda_{m+1}}\right)\right\}, \tag{2.31}$$

then

$$p_1(z) = 1 + \frac{\lambda_{m+1}\sum_{n=m+1}^{\infty} a_n z^{n-1}}{1+\sum_{n=2}^{m} a_n z^{n-1}} \quad (m \ge 2) \tag{2.32}$$

and from (2.30), we have

$$\begin{aligned} \left|\frac{p_1(z)-1}{p_1(z)+1}\right| &= \left|\frac{\lambda_{m+1}\sum_{n=m+1}^{\infty} a_n z^{n-1}}{2+2\sum_{n=2}^{m} a_n z^{n-1} + \lambda_{m+1}\sum_{n=m+1}^{\infty} a_n z^{n-1}}\right| \\ &\le \frac{\lambda_{m+1}\sum_{n=m+1}^{\infty} |a_n|}{2-2\sum_{n=2}^{m} |a_n| - \lambda_{m+1}\sum_{n=m+1}^{\infty} |a_n|} \le 1 \quad (z \in E). \end{aligned} \tag{2.33}$$

Thus, we conclude that $\operatorname{Re} p_1(z) > 0$ $(z \in E)$, that is,

$$\operatorname{Re}\frac{f(z)}{s_m(z)} > 1 - \frac{1}{\lambda_{m+1}} \quad (z \in E). \tag{2.34}$$

This proves (2.27) for $m \ge 2$.

If we take

$$f(z) = z - \frac{1-2\alpha}{m+1-(1+(-1)^m)\alpha} z^{m+1} \in S_0(\alpha), \tag{2.35}$$

then $s_m(z) = z$ and

$$\frac{f(z)}{s_m(z)} \longrightarrow \frac{m+(1+(-1)^{m+1})\alpha}{m+1-(1+(-1)^m)\alpha} \tag{2.36}$$

as $z \to 1$. Hence, the bound in (2.27) is best possible for each $m \ge 2$.

Similarly, if we put

$$p_2(z) = (1+\lambda_{m+1})\left(\frac{s_m(z)}{f(z)} - \frac{\lambda_{m+1}}{1+\lambda_{m+1}}\right), \tag{2.37}$$

then by (2.30), we deduce that

$$\begin{aligned} \left|\frac{p_2(z)-1}{p_2(z)+1}\right| &= \left|\frac{-(1+\lambda_{m+1})\sum_{n=m+1}^{\infty} a_n z^{n-1}}{2+2\sum_{n=2}^{m} a_n z^{n-1} - (\lambda_{m+1}-1)\sum_{n=m+1}^{\infty} a_n z^{n-1}}\right| \\ &\le \frac{(1+\lambda_{m+1})\sum_{n=m+1}^{\infty} |a_n|}{2-2\sum_{n=2}^{m} |a_n| - (\lambda_{m+1}-1)\sum_{n=m+1}^{\infty} |a_n|} \\ &\le 1 \end{aligned} \tag{2.38}$$

for $z \in E$. From this, we easily have (2.28) for $m \ge 2$. Further, the bound in (2.28) is best possible for the function $f(z)$ given by (2.35).

For $m = 1$, replacing (2.30) by

$$\lambda_2 \sum_{n=2}^{\infty} |a_n| \le \sum_{n=2}^{\infty} \lambda_n |a_n| \le 1 \tag{2.39}$$

and proceeding as the above, the proof of the theorem is completed. □

Letting $m = 1$ in Theorem 2.8, we have the following corollary.

Corollary 2.9. *Let $f(z) \in S_0(\alpha)$ with $0 \le \alpha < 1/2$, then for $z \in E$, the following inequalities hold true:*

$$\operatorname{Re} \frac{f(z)}{z} > \frac{1+2\alpha}{2}, \tag{2.40}$$

$$\operatorname{Re} \frac{z}{f(z)} > \frac{2}{3-2\alpha}. \tag{2.41}$$

The results are sharp.

Remark 2.10. Inequality (2.40) in Corollary 2.9 is an improvement of Theorem 1.4 for $0 < \alpha < 1/2$ and agrees with Theorem 1.4 for $\alpha = 0$.

Using inequality (2.25) instead of (2.19), the following theorem can be proved on the lines of the proof of Theorem 2.8. We omitted the details of its proof.

Theorem 2.11. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in T_0(\alpha)$ with $0 \le \alpha < 1/2$, and let*

$$s_1(z) = z, \qquad s_m(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (m \ge 2),$$
$$\mu_{m+1} = \frac{(m+1)[m+1-(1+(-1)^m)\alpha]}{1-2\alpha} \quad (m \ge 1). \tag{2.42}$$

Then for $z \in E$, the following inequalities hold true:

$$\operatorname{Re} \frac{f(z)}{s_m(z)} > \frac{\mu_{m+1}-1}{\mu_{m+1}}, \tag{2.43}$$

$$\operatorname{Re} \frac{s_m(z)}{f(z)} > \frac{\mu_{m+1}}{1+\mu_{m+1}}. \tag{2.44}$$

The results are sharp for each $m \ge 1$, with the extremal function

$$f(z) = z - \frac{z^{m+1}}{\mu_{m+1}} \in T_0(\alpha). \tag{2.45}$$

Theorem 2.11, with $m = 1$, leads to the following corollary.

Corollary 2.12. *Let $f(z) \in T_0(\alpha)$ with $0 \le \alpha < 1/2$. Then for $z \in E$,*

$$\operatorname{Re} \frac{f(z)}{z} > \frac{3+2\alpha}{4}, \qquad \operatorname{Re} \frac{z}{f(z)} > \frac{4}{5-2\alpha}. \tag{2.46}$$

The results are sharp.

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