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SUBORDINATION CRITERIA FOR STARLIKENESS AND CONVEXITY

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For functions p analytic in the open unit disc $U = \{z : |z| < 1\}$ with the normalization $p(0) = 1$, we consider the families $\mathcal{P}[A,-1]$, $-1 < A \le 1$, consisting of p such that $p(z)$ is subordinate to $(1+Az)/(1-z)$ in U and $\mathcal{P}(1,b)$, $b > 0$, consisting of p, which have the disc formulation $|p-1| < b$ in U. We then introduce subordination criteria for the choice of $p(z) = zf'(z)/f(z)$, where f is analytic in U and normalized by $f(0) = f'(0) - 1 = 0$. We also obtain starlikeness and convexity conditions for such functions f and consequently extend and, in some cases, improve the corresponding previously known results.

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1. Introduction. Let $\mathcal{A}$ denote the class of functions that are analytic in the open unit disc $U = \{z : |z| < 1\}$. In the sequel, we assume that p in $\mathcal{A}$ is normalized by $p(0) = 1$ and f in $\mathcal{A}$ is normalized by $f(0) = f'(0) - 1 = 0$.

For $0 < b \le a$, the function $p \in \mathcal{A}$ is said to be in $\mathcal{P}(a,b)$ if and only if

$$\left|p(z) - a\right| < b, \quad z \in U. \tag{1.1}$$

Without loss of generality, we omit the trivial case $p(z) = 1$ and assume that $|1-a| < b$.

For $-1 \le B < A \le 1$, the function $p \in \mathcal{A}$ is said to be in $\mathcal{P}[A,B]$ if and only if

$$p(z) \prec \frac{1+Az}{1+Bz}, \quad z \in U. \tag{1.2}$$

Here the symbol "$\prec$" stands for *subordination*. For the functions f and g in $\mathcal{A}$, we say that f is subordinate to g in U, denoted by $f \prec g$, if there exists a Schwarz function w in $\mathcal{A}$ with $|w(z)| < 1$ and $w(0) = 0$ such that $f(z) = g(w(z))$ in U.

For $0 < b \le a$, there is a correspondence between $\mathcal{P}(a,b)$ and $\mathcal{P}[A,B]$; namely,

$$\mathcal{P}(a,b) \equiv \mathcal{P}\left[\frac{b^2 - a^2 + a}{b}, \frac{1-a}{b}\right]. \tag{1.3}$$

Two subclasses that have been studied extensively (e.g., see [2, 10]) are $\mathcal{P}(1,b)$ and $\mathcal{P}[A,-1]$. The class $\mathcal{P}(1,b)$, which is defined using the disc formulation, has an alternative characterization in terms of subordination, where

$$p \in \mathcal{P}(1,b) \Longleftrightarrow p(z) \prec 1+bz. \tag{1.4}$$

In this paper, we study the subordination criteria for functions $p(z) = zf'(z)/f(z)$ in $\mathcal{A}$, where $f \in \mathcal{A}$. We also obtain starlikeness and convexity conditions for such functions $f \in \mathcal{A}$ and consequently extend and, in some cases, improve the corresponding previously known results. The significance of the above choice for p is evident if we recall that $f \in \mathcal{A}$ is said to be starlike of order α, $0 \le \alpha \le 1$ if $(zf'(z)/f(z)) \in \mathcal{P}(1,1-\alpha)$, and $f \in \mathcal{A}$ is said to be convex of order α, $0 \le \alpha \le 1$ if $(1+zf''/f') \in \mathcal{P}(1,1-\alpha)$. Finally, we note that all functions, starlike or convex, of order α_2 are, respectively, starlike or convex of order α_1 if $0 \le \alpha_1 \le \alpha_2 \le 1$.

2. Main results. First, we introduce a subordination criterion for $p(z) = zf'(z)/f(z)$ in $\mathcal{P}[A,-1]$. To prove our first theorem, we need the following celebrated result, which is due to Miller and Mocanu [3].

Lemma 2.1. *Let q be univalent in the unit disc U and let ϕ and ψ be analytic in a domain $\mathcal{C}$ containing $q(U)$ with $\psi(\omega) \neq 0$ for $\omega \in q(U)$. Set $Q(z) = zq'(z)\psi(q(z))$ and $h(z) = \phi(q(z)) + Q(z)$. Also, suppose that Q is starlike univalent in U and $\Re(zh'(z)/Q(z)) = \Re[\phi'(q(z))/\psi(q(z)) + zQ'(z)/Q(z)] > 0$ in U. If p is analytic in U, $p(0) = q(0)$, $q(U) \in \mathcal{C}$, and $\phi(p(z)) + zp'(z)\psi(p(z)) \prec h(z)$, then $p \prec q$, and q is the best dominant of the subordination.*

Theorem 2.2. *Let f in $\mathcal{A}$ be so that $f(z)/z \neq 0$ in U. Also, let $\alpha > 0$, $|\beta| \le 1$, and $-1 < A \le 1$ be so that*

$$\frac{\beta(1-\alpha)}{\alpha} + \frac{1}{2}(1+\beta)(1-A) + \frac{(1-\beta)(1-A)}{2(1+A)} \ge 0. \tag{2.1}$$

If

$$\left(\frac{zf'(z)}{f(z)}\right)^{\beta}\left(1+\alpha\frac{zf''(z)}{f'(z)}\right) \prec h(z), \tag{2.2}$$

where

$$h(z) = \left(\frac{1+Az}{1-z}\right)^{\beta-1}\left[(1-\alpha)\frac{1+Az}{1-z} + \frac{\alpha(1+Az)^2 + \alpha(1+A)z}{(1-z)^2}\right], \tag{2.3}$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1-z}. \tag{2.4}$$

Proof. Setting $zf'(z)/f(z) = p(z)$, condition (2.2) can be written as

$$(p(z))^{\beta}[(1-\alpha)+\alpha p(z)]+\alpha z p'(z)^{\beta-1} \prec h(z). \tag{2.5}$$

For $q(z) = (1+Az)/(1-z)$, it is clear that q is univalent in U and $q(U)$ is the region $\Re z > (1-A)/2$. Also, for $\psi(z) = \alpha z^{\beta-1}$ and $\phi(z) = z^{\beta}(1-\alpha+\alpha z)$, we observe that ψ and ϕ satisfy the conditions required by Lemma 2.1. Therefore,

$$\begin{aligned} Q(z) &= zq'(z)\psi(q(z)) = \frac{\alpha(1+A)z(1+Az)^{\beta-1}}{(1-z)^{\beta+1}}, \\ h(z) &= \phi(q(z))+Q(z) = \left(\frac{1+Az}{1-z}\right)^{\beta}\left[1-\alpha+\alpha\frac{1+Az}{1-z}\right] \\ &\quad + \frac{\alpha(1+A)z(1+Az)^{\beta-1}}{(1-z)^{\beta+1}}. \end{aligned} \tag{2.6}$$

Now, the above assumptions yield

$$\begin{aligned} \Re\frac{zQ'(z)}{Q(z)} &= \Re\left[1+(\beta-1)\frac{Az}{1+Az}+(1+\beta)\frac{z}{1-z}\right] \\ &> -1+(1-\beta)\frac{1}{1+|A|}+(1+\beta)\frac{1}{2} \\ &= \frac{(1-|A|)(1-\beta)}{2(1+|A|)} > 0, \\ \Re\frac{zh'(z)}{Q(z)} &= \frac{\beta(1-\alpha)}{\alpha}+(1+\beta)\Re\left(\frac{1+Az}{1-z}\right)+\Re\frac{zQ'(z)}{Q(z)} \\ &> \frac{\beta(1-\alpha)}{\alpha}+\frac{1}{2}(1+\beta)(1-|A|)+\frac{(1-\beta)(1-|A|)}{2(1+|A|)} \geq 0. \end{aligned} \tag{2.7}$$

This completes the proof since all the conditions required by Lemma 2.1 are satisfied. □

We remark that for $\beta = A = 0$ and $\alpha = 1$, the above theorem reinstates the fact that every convex function is starlike of order $1/2$. Also, for $\beta = A = 1$, we obtain [8, Theorem 1], and for $\alpha = \beta = 1$ and $A = 0$ we obtain [8, Theorem 3]. Furthermore, letting $\alpha = -\beta = 1$ in the above theorem, yields the following corollary.

Corollary 2.3. *Let $f \in \mathcal{A}$ and $f(z)/z \neq 0$ in U. If $-1 < |A| \leq 1$ and*

$$\frac{1+zf''(z)/f'(z)}{zf'(z)/f(z)} \prec 1+\frac{(1+A)z}{(1+Az)^2}, \tag{2.8}$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1-z}. \tag{2.9}$$

Remark 2.4. The function $h(z) = 1+(1+A)z/(1+Az)^2$ has interesting mapping properties. Note that h takes real values for real values of z with $h(0) = 1$ and $h(U)$ is symmetric with respect to the real axis. Now, for $D = \{h(e^{i\theta}) : 0 \le \theta < 2\pi\}$ and $d = (1,0)$, observe that

$$\operatorname{mindist}(D,d) = \frac{1}{1+A}. \tag{2.10}$$

Consequently, h maps the unit circle onto the region, which properly contains the region $|\omega - 1| < (1+A)/(1-A)^2$. This is an extension to [9, Theorem 1] which does not extend as for the sharpness. (Also see Obradović and Tuneski [7].)

Our next theorem is on the subordination criterion for $zf'(z)/f(z) \in \mathcal{P}(1,b)$.

Theorem 2.5. *Let $f \in \mathcal{A}$ and $f(z)/z \neq 0$ in U. Also, let $\alpha > 0$, $|\beta| \le 1$, and $0 < b \le 1$ be so that $2\beta + \alpha(1-\beta) + (1-b)(1+b+b\beta) \ge 0$. If*

$$\left(\frac{zf'(z)}{f(z)}\right)^{\beta}\left(1+\alpha\frac{zf''(z)}{f'(z)}\right) \prec \frac{1+(1+2\alpha)bz+\alpha b^2z^2}{(1+bz)^{1-\beta}} = h(z), \tag{2.11}$$

then

$$\frac{zf'(z)}{f(z)} \prec 1+bz. \tag{2.12}$$

Proof. Setting $p(z) = zf'(z)/f(z)$, condition (2.11) may be written as

$$(p(z))^{\beta}[(1-\alpha)+\alpha p(z)] + \alpha z p'(z)(p(z))^{\beta-1} \prec h(z). \tag{2.13}$$

Here, we need once again to make use of Lemma 2.1. Set $q(z) = 1+bz$, $\psi(z) = \alpha z^{\beta-1}$, and $\phi(z) = z^{\beta}(1-\alpha+\alpha z)$. We observe that q is univalent and $q(U)$ is a region, so that its boundary is the circle with radius b and center at $(1,0)$. Using an argument similar to that used to prove Theorem 2.2, we write $Q(z) = \alpha bz(1+bz)^{\beta-1}$ and $h(z) = \phi(q(z)) + Q(z)$. Therefore,

$$\begin{aligned}
\Re\frac{zQ'(z)}{Q(z)} &= \beta + (1-\beta)\Re\frac{1}{1+bz} > \beta + \frac{1-\beta}{1+b} = \frac{1+\beta b}{1+b} \ge 0,\\
\Re\frac{zh'(z)}{Q(z)} &= \Re\left[\frac{\beta(1-\alpha)}{\alpha} + (1+\beta)(1+bz)\right] + \Re\frac{zQ'(z)}{Q(z)}\\
&> \frac{\beta(1-\alpha)}{\alpha} + (1+\alpha)(1-b) + \frac{1+\beta b}{1+b} \ge 0.
\end{aligned} \tag{2.14}$$

Thus, the proof is complete since all the conditions required by Lemma 2.1 are satisfied. □

By letting $\beta = 1$ in Theorem 2.5, we obtain the following corollary, which is an improvement in a result obtained in [6]. For an alternative proof of the following corollary, see Mocanu and Oros [4]. Another generalization of this result is contained in Mocanu and Oros [5].

Corollary 2.6. *Let $f \in \mathcal{A}$ and $f(z)/z \neq 0$ in U. Also, let $\alpha > 0$ and $0 < b \leq 1$. If*

$$\frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)} \prec 1 + (1+2\alpha)bz + \alpha b^2 z^2, \tag{2.15}$$

then

$$\frac{zf'(z)}{f(z)} \prec 1 + bz. \tag{2.16}$$

Recalling Remark 2.4 after Corollary 2.3 for $h(z) = 1 + (1+2\alpha)bz + \alpha b^2 z^2$, observe that h takes real values for real values of z with $h(0) = 1$ and $h(U)$ is symmetric with respect to the real axis. Now, for $D = \{h(e^{i\theta}) : 0 \leq \theta < 2\pi\}$ and $d = (1,0)$, it can be shown that

$$\begin{aligned} \operatorname{mindist}(D,d) &= (1+2\alpha)b - \alpha b^2, \\ \operatorname{Maxdist}(D,d) &= (1+2\alpha)b + \alpha b^2. \end{aligned} \tag{2.17}$$

Therefore, h maps the unit disc U onto a region, which properly contains the region $\{z : |z-1| < (1+\alpha)b\}$. This improves [6, Theorem 1] obtained by Obradović et al.

For $0 \leq \rho < 1$, define $\Omega = \{w : |w-1| \leq 1 - 2\rho + \Re w\}$ and let $\mathcal{S}(\rho)$ consist of functions $f \in \mathcal{A}$ satisfying the condition $zf'/f \in \Omega$. Note that the class $\mathcal{S}(\rho)$ consists of starlike functions. Also, we let $\mathcal{K}(\rho)$ consist of convex functions $f \in \mathcal{A}$ for which $zf' \in \mathcal{S}(\rho)$.

For $0 \leq \rho < \beta \leq 1$, let $\mathcal{M}_\beta(\rho)$ be the largest number for which the disc $\mathcal{D}(\beta, \mathcal{M}_\beta(\rho)) = \{\omega : |\omega - \beta| < \mathcal{M}_\beta(\rho)\}$ lies inside the region Ω. A direct calculation yields

$$\mathcal{M}_\beta(\rho) = \begin{cases} \beta - \rho & \text{if } \rho < \beta < 2 - \rho, \\ 2\sqrt{(1-\rho)(\beta-1)} & \text{if } \beta \geq 2 - \rho, \end{cases} \tag{2.18}$$

Therefore, the disc contains the point 1 for

$$\frac{1+\rho}{2} < \beta < (2-\rho) + \sqrt{\frac{\rho^2 - \rho + 5}{2}} \tag{2.19}$$

and we have justified the following lemma.

Lemma 2.7. *Let $f \in \mathcal{A}$ and $(1+\rho)/2 < \beta < (2-\rho) + \sqrt{\rho^2 - \rho + 5/2}$. If*

$$\left| \frac{zf'(z)}{f(z)} - \beta \right| < \mathcal{M}_\beta(\rho), \tag{2.20}$$

then $f \in \mathcal{S}(\rho)$.

The above lemma in conjunction with Corollary 2.6 yields the following theorem.

Theorem 2.8. *Let $f \in \mathcal{A}$ and $f(z)/z \neq 0$ in U. Also, let $\alpha > 0$ and $0 < b \leq 1$. If*

$$\frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)} \prec 1 + (1+2\alpha)bz + \alpha b^2 z^2, \tag{2.21}$$

then $f \in \mathcal{S}(1-b)$.

With some restrictions on ρ and b, we show that we can do even better than the above theorem in terms of classification of the function f. First, we need the following result due to Jack [1].

Lemma 2.9. *Let ω be a nonconstant analytic function in U with $\omega(0) = 0$. If $|\omega|$ attains its maximum value on the circle $|z| = r$ at some point z_0, then $z_0\omega'(z_0) = k\omega(z_0)$, where $k \geq 1$.*

Theorem 2.10. *For $\alpha > 0$, let $\rho = (\alpha - b(2+3\alpha+\alpha b))/\alpha(1-b)$ and $0 < b \leq (-(3+2\alpha)+\sqrt{9+12\alpha+8\alpha^2})/2\alpha$. If $f \in \mathcal{A}$, $f(z)/z \neq 0$, and*

$$\frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)} \prec 1 + (1+2\alpha)bz + \alpha b^2 z^2, \tag{2.22}$$

then $f \in \mathcal{K}(\rho)$.

Proof. Setting $p(z) = zf'(z)/f(z)$ and $\omega(z) = \alpha z f''(z)/f'(z)$, condition (2.22) may be written as

$$p(z)(1+\omega(z)) \prec 1 + (1+2\alpha)bz + \alpha b^2 z^2 \tag{2.23}$$

or

$$|p(z)(1+\omega(z)) - 1| < (1+2\alpha)b + \alpha b^2, \quad z \in U. \tag{2.24}$$

Therefore, $|p(z) - 1| < b$ and so, by Corollary 2.6, we only need to show that

$$|\omega(z)| < \frac{2(1+\alpha)b + \alpha b^2}{1-b} = T. \tag{2.25}$$

Define $g(z) = \omega(z)/T$. Since $g(0) = 0$ and g is analytic in U, it suffices to show that $|g| < 1$ in U. On the contrary, suppose that there exists $z_0 \in U$, so that $|g(z_0)| = 1$. Then , by Lemma 2.9, there exists $k \geq 1$, so that $z_0 g'(z_0) = kg(z_0)$. Consequently,

$$\begin{aligned} |p(z_0)(1+\omega(z_0)) - 1| &= |p(z_0)(1+Tg(z_0)) - 1| \\ &= |(p(z_0)-1)(1+Tg(z_0)) + Tg(z_0)| \\ &\geq T|g(z_0)| - b(1+T|g(z_0)|) \\ &= (1+2\alpha)T + \alpha T^2. \end{aligned} \tag{2.26}$$

This is a contradiction to the required condition (2.24), and so the proof is complete. □

As a corollary to the above theorem we obtain the following corollary.

Corollary 2.11. *Let $f \in \mathcal{A}$ be so that $f(z)/z \neq 0$ and*

$$\frac{zf'(z)}{f(z)} + \frac{z^2 f''(z)}{f(z)} \prec 1 + 0.5777z + 0.037z^2, \quad z \in U. \tag{2.27}$$

Then, $|zf''(z)/f'(z)| < 0.99987$, and so f is convex.

We note that our Corollary 2.11 is an improvement to [6, Corollary 2(b)].

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