IJMMS 2003:33, 2097–2107
PII. S0161171203210589
http://ijmms.hindawi.com

WEAKLY PERIODIC AND SUBWEAKLY PERIODIC RINGS

AMBER ROSIN and ADIL YAQUB

Received 30 October 2002

Our objective is to study the structure of subweakly periodic rings with a particular emphasis on conditions which imply that such rings are commutative or have a nil commutator ideal. Related results are also established for weakly periodic (as well as periodic) rings.

2000 Mathematics Subject Classification: 16U80, 16D70.

Throughout, R represents a ring and $\mathscr{C}(R)$ denotes the commutator ideal of R. For any x,y in R, $[x,y]=xy-yx$ is the usual commutator. A *word* $w(x,y)$ is a product in which each factor is x or y. The *empty word* is defined to be 1. We now state formally the definition of a *subweakly periodic* ring.

Definition 1. A ring R is called *subweakly periodic* if every x in $R\backslash J$ can be written in the form $x=a+b$, where a is nilpotent and b is potent.

In the preparation for the proofs of the main theorems, we first prove the following lemmas.

Lemma 2. *Suppose that R is any ring with the property that for all x,y in R, there exist words $w(x,y)$, $w'(x,y)$ depending on x and y such that*

$$w(x,y)[xy,yx]=0=[xy,yx]w'(x,y)\quad (x,y\in R). \tag{1}$$

Then the commutator ideal $\mathscr{C}(R)$ is contained in J. In particular, if R is also semisimple, then R is commutative.

Proof. The semisimple ring R/J is isomorphic to a subdirect sum of primitive rings R_i, $i\in\Gamma$, each of which clearly satisfies (1).

Case 1. Suppose that R_i is a division ring. We claim that R_i must be commutative. Suppose not. Let $x,y\in R_i$ be such that $[x,y]\neq 0$. As $x\neq 0$ and $y\neq 0$, we must have $w(x,y)\neq 0$, and so by (1), $[xy,yx]=0$. Since $[x,y]\neq 0$ implies that $[x,y+1]\neq 0$, we may repeat the above argument, with $y+1$ playing the role of y, to conclude that $[x(y+1),(y+1)x]=0$, and hence $[xy+x,yx+x]=0$. Combining this with $[xy,yx]=0$, we obtain $[xy,x]+[x,yx]=0$, and thus $[xy-yx,x]=0$, that is, $[x,y]$ commutes with x. Interchanging x and y in the above argument, we see that $[y,x]$ commutes with y, and hence

$$[x,y]\text{ commutes with both } x \text{ and } y. \tag{2}$$

Observe that $[xy,x]=x[y,x]$, and hence

$$x=[xy,x][y,x]^{-1}. \tag{3}$$

By (2), $[xy,x]$ commutes with both x and xy, and hence $[xy,x]$ commutes with $x^{-1}(xy)=y$. Again, by (2), $[y,x]$ commutes with y, and hence $[y,x]^{-1}$ also commutes with y. The net result is that both factors of the right-hand side of (3) commute with y, and hence $[x,y]=0$, contradiction. This contradiction proves that the division ring R_i is commutative.

Case 2. The primitive ring R_i is not a division ring. Since (1) is inherited by all subrings and all homomorphic images of the ground ring R, it follows, by Jacobson's density theorem [4, page 33], that for some $n>1$ and some division ring D, the complete matrix ring D_n of all $n\times n$ matrices over D satisfies (1). This, however, is false as can be seen by taking $x=E_{11}$, $y=E_{11}+E_{12}$ $(x,y\in D_n)$. Indeed, in this case, any word $w(x,y)$ must be x or y (since $x^2=x$, $y^2=y$, $xy=y$, and $yx=x$), which implies that

$$w(x,y)[xy,yx]=w(x,y)[y,x]=x[y,x]=x-y\neq 0 \tag{4}$$

or

$$w(x,y)[xy,yx]=w(x,y)[y,x]=y[y,x]=x-y\neq 0. \tag{5}$$

This contradiction shows that each R_i must be a division ring, and hence must be commutative (by Case 1). Therefore, R/J is commutative, and hence $\mathscr{C}(R)\subseteq J$. This proves Lemma 2. □

Lemma 3. *Suppose that R is a ring which satisfies the "word" hypothesis (1) of Lemma 2. Suppose, further, that J is commutative. Then the commutator ideal of R is nil, and hence N is an ideal of R.*

Proof. By Lemma 2, every commutator $[x,y]$ is in J. Since, by hypothesis, J is commutative,

$$[[x,y],[z,w]]=0 \quad \forall x,y,z,w\in R. \tag{6}$$

Observe that (6) represents a polynomial identity which is satisfied by all the elements of the ground ring R. Moreover, the greatest common divisor of all the coefficients of this polynomial is 1. Furthermore, (6) is not satisfied by any 2×2 matrix ring over $\mathrm{GF}(p)$ for any prime p, as a consideration of the following commutators shows:

$$[x,y]=[E_{11},E_{12}], \qquad [z,w]=[E_{22},E_{21}]. \tag{7}$$

It follows from [1] that the commutator ideal $\mathscr{C}(R)$ is nil, and hence N is an ideal of R. This proves Lemma 3. □

Lemma 4. *Suppose that R is a subweakly periodic ring which satisfies all the hypotheses of Lemma 3. Then for any x in R,*

$$x \in J \quad \text{or} \quad x - x^n \in N \quad \text{for some integer } n > 1. \tag{8}$$

Proof. Let $x \in R$, $x \notin J$. Then, by Definition 1,

$$x = a + b, \quad a \in N,\ b^n = b, \text{ where } n = n(b) > 1. \tag{9}$$

Now, by Lemma 3, N is an ideal of R, and since $a \in N$,

$$x^n = (a+b)^n = a_0 + b^n, \quad a_0 \in N. \tag{10}$$

Thus,

$$x - x^n = a + b - a_0 - b^n = a - a_0 \quad (\text{since } b^n = b), \tag{11}$$

and hence $x - x^n \in N$. This proves Lemma 4. □

Lemma 5. *Suppose that R is a subweakly periodic ring which satisfies all the hypotheses of Lemma 4. Suppose that $\sigma : R \to S$ is a homomorphism of R onto a ring S. Then the set N' of nilpotents of S is contained in $\sigma(J)$, and hence N' is a commutative set.*

Proof. Suppose that $s \in N'$ with $s^k = 0$. Let $d \in R$ such that $\sigma(d) = s$. If $d \in J$, then $s = \sigma(d) \in \sigma(J)$, and the lemma follows. So, suppose that $d \notin J$. Then by Lemma 4,

$$d - d^n \in N \quad \text{for some integer } n > 1. \tag{12}$$

It is readily verified that

$$d - d^{k+1}d^{k(n-2)} = (d - d^n) + d^{n-1}(d - d^n) + \cdots + (d^{n-1})^{k-1}(d - d^n). \tag{13}$$

Combining (12), (13), and the fact that N is an ideal of R, we see that

$$d - d^{k+1}d^{k(n-2)} \in N, \tag{14}$$

and hence

$$s - s^{k+1}s^{k(n-2)} = \sigma(d - d^{k+1}d^{k(n-2)}) \in \sigma(N). \tag{15}$$

This implies (since $s^k = 0$) that $s \in \sigma(N)$. Thus, $N' \subseteq \sigma(N)$. By Lemma 3, N is an ideal of R, and hence $N \subseteq J$, which implies $N' \subseteq \sigma(N) \subseteq \sigma(J)$. Finally, since J is commutative, $\sigma(J)$ is commutative, and thus N' is a commutative set.

□

Lemma 6. *Suppose that R is any ring which satisfies the "word" hypothesis (1) of Lemma 2. Then the set E of all idempotents of R is contained in the center of R.*

Proof. Suppose that $e^2 = e \in R$, $x \in R$, and $f = e + ex - exe$. Then

$$f^2 = f, \qquad ef = f, \qquad fe = e, \qquad [e,f] = ex - exe. \tag{16}$$

By the "word" hypothesis (1) of Lemma 2, there exists a word $w(e,f)$ such that

$$w(e,f)[ef,fe] = 0. \tag{17}$$

By (16), we see that $w(e,f) = e$ or $w(e,f) = f$, and hence (17) is equivalent to

$$e[ef,fe] = 0 \quad \text{or} \quad f[ef,fe] = 0. \tag{18}$$

These two equations, in turn, are equivalent to $e - f = 0$, and hence

$$ex = exe. \tag{19}$$

Now, let $f' = e + xe - exe$. Again, by the second part of the "word" hypothesis (1) of Lemma 2, there exists a word $w'(e,f')$ such that

$$[ef', f'e]w'(e,f') = 0. \tag{20}$$

An argument similar to the one above shows that (20) is equivalent to $e - f' = 0$, and hence

$$xe = exe. \tag{21}$$

Combining (19) and (21), we obtain the lemma. □

Lemma 7. *Suppose that R is a subweakly periodic ring which satisfies all the hypotheses of Lemma 4. Then, for any x in R, x is in J or $x^q = x^q e$ for some idempotent e and some $q \geq 1$.*

Proof. Let $x \in R$. By Lemma 4,

$$x \in J \quad \text{or} \quad x - x^n \in N \quad \text{for some integer } n > 1. \tag{22}$$

Suppose that $x \notin J$. Then, $(x - x^n)^q = 0$ for some positive integer q, and hence

$$x^q = x^{q+1}g(x) \quad \text{for some polynomial } g(\lambda) \in \mathbb{Z}[\lambda]. \tag{23}$$

By reiterating, we see that

$$x^q = x^q(xg(x)) = x^q(xg(x))^2 = \cdots = x^q(xg(x))^q. \tag{24}$$

Let $e = (xg(x))^q$. It is readily verified that $e^2 = e$, and thus by (24), the lemma is proved. □

The following three lemmas are well known and are stated without proofs.

Lemma 8. *If $[x,y]$ commutes with x, then for all positive integers k,*

$$[x^k, y] = kx^{k-1}[x,y]. \tag{25}$$

Lemma 9. *Let R be a subdirectly irreducible ring. Then the only central idempotent elements of R are 0 and 1 (if $1 \in R$).*

Lemma 10. *Let R be a weakly periodic ring. Then the Jacobson radical J of R is* $\mathrm{nil}(J \subseteq N)$.

This lemma was proved in [2].

We are now in a position to prove our main theorems.

Theorem 11. *Let R be a subweakly periodic ring such that the following two conditions hold:*

(i) *for all x, y in R, there exist words $w(x,y)$ and $w'(x,y)$ depending on x and y such that $w(x,y)[xy,yx] = 0 = [xy,yx]w'(x,y)$;*

(ii) *the Jacobson radical J is commutative.*

Then R is commutative. (In particular, this theorem holds if R is weakly periodic (or periodic).)

Proof. By Lemma 7, we have for any x in R,

$$x \in J \quad \text{or} \quad x^q = x^q e, \quad e^2 = e, \; q \geq 1. \tag{26}$$

As is well known, the ground ring R can be written as

$$R \cong \text{a subdirect sum of subdirectly irreducible rings } R_i \quad (i \in \Gamma). \tag{27}$$

Let $\sigma_i : R \to R_i$ be the natural homomorphism of R onto R_i. Let $x_i \in R_i$, and suppose that $x \in R$ such that $\sigma_i(x) = x_i$. By (26), we see that

$$x_i \in \sigma_i(J) \quad \text{or} \quad {x_i}^q = {x_i}^q e_i \quad \text{with } e_i = \sigma_i(e), \; e^2 = e \in R, \; q \geq 1. \tag{28}$$

Moreover, by the proof of Lemma 7, we see that we can take $e = (xg(x))^q$ for some polynomial $g(\lambda) \in \mathbb{Z}[\lambda]$. Also, by Lemma 6, e is a central idempotent of R, and hence

$$e_i = (x_i g(x_i))^q \text{ is a central idempotent of } R_i. \tag{29}$$

We now distinguish two cases.

Case 1. The ring R_i does not have an identity. Suppose that there is an $x_i \notin \sigma_i(J)$. Then by (28), (29), and Lemma 9, ${x_i}^q = 0$ for some $q \geq 1$. Thus, x_i is nilpotent, and hence by Lemma 5, $x_i \in \sigma_i(J)$. This contradiction shows that

$x_i \in \sigma_i(J)$ for all $x_i \in R_i$. By hypothesis, J is commutative, and consequently, $R_i = \sigma_i(J)$ is commutative as well.

Case 2. The ring R_i has an identity 1. Let $x_i \notin \sigma_i(J)$. By (28) and (29), ${x_i}^q = {x_i}^q e_i$, where $e_i = (x_i g(x_i))^q$ is a central idempotent of R_i. Now, by Lemma 9, $e_i = 0$ or $e_i = 1$. If $e_i = 1$, then $(x_i g(x_i))^q = 1$, which implies that x_i is a unit of R_i. On the other hand, if $e_i = 0$, then ${x_i}^q = 0$, and hence, by Lemma 5, $x_i \in \sigma_i(J)$. The net result is

$$\forall x_i \in R_i,\ x_i \in \sigma_i(J) \quad \text{or} \quad x_i \text{ is a unit in } R_i. \tag{30}$$

Next, we claim that the set N_i of nilpotents of R_i is an ideal of R_i. To prove this, first recall that by Lemma 5, N_i is a commutative set, and hence N_i is closed with respect to subtraction. Now, suppose that $a_i \in N_i$, $x_i \in R_i$. By Lemma 5, $a_i \in \sigma_i(J)$, and therefore, $a_i = \sigma_i(j)$ for some $j \in J$. Suppose that $x \in R$ is such that $x_i = \sigma_i(x)$. Then $a_i x_i = \sigma_i(j)\sigma_i(x) = \sigma(jx) \in \sigma_i(J)$ since $j \in J$. Since J is commutative, $\sigma_i(J)$ is commutative, and hence

$$[a_i x_i, a_i] = 0 \quad (a_i \in N_i,\ x_i \in R_i). \tag{31}$$

An easy induction shows that (31) implies that $(a_i x_i)^q = {a_i}^q {x_i}^q$ for all positive integers q, and hence $a_i x_i$ is nilpotent. Similarly, $x_i a_i \in N_i$, and thus

$$N_i \text{ is a commutative ideal of } R_i. \tag{32}$$

Our next goal is to show that

$$[[a_i, u_i], u_i] = 0 \quad \text{for all units } u_i \text{ in } R_i \text{ and all } a_i \in \sigma_i(J). \tag{33}$$

To prove this, suppose that $a_i = \sigma_i(j)$ for some $j \in J$. Then j is quasiregular, and hence $\sigma_i(j)$ is quasiregular as well, that is, a_i is quasiregular. Therefore, ${u_i}' = 1 + a_i$ is a unit in R_i. By hypothesis (i), there exists a word $w(u_i, {u_i}')$ such that $w(u_i, {u_i}')[u_i {u_i}', {u_i}' u_i] = 0$. Since both u_i and ${u_i}'$ are units in R_i, $w(u_i, {u_i}')$ must also be a unit, and hence $[u_i {u_i}', {u_i}' u_i] = 0$, which is equivalent to

$$[u_i(1 + a_i), (1 + a_i)u_i] = 0. \tag{34}$$

Note that $\sigma_i(J)$ is a commutative ideal of R_i, and hence $[u_i a_i, a_i u_i] = 0$. So, (34) implies that $[u_i, a_i u_i] + [u_i a_i, u_i] = 0$, which is equivalent to $[u_i, [a_i, u_i]] = 0$. This proves (33). Next, we prove that

$$\sigma_i(J) \text{ is contained in the center of } R_i. \tag{35}$$

Suppose not. Then

$$[a_i, b_i] \neq 0 \quad \text{for some } a_i \in \sigma_i(J),\ b_i \in R_i. \tag{36}$$

By (30), $b_i \in \sigma_i(J)$ or b_i is a unit of R_i. As $\sigma_i(J)$ is commutative, (36) implies that $b_i \notin \sigma_i(J)$. Therefore,

$$b_i \text{ is a unit of } R_i. \tag{37}$$

Moreover, by (36), we cannot have both $[a_i, 2b_i] = 0$ and $[a_i, 3b_i] = 0$. We assume, without loss of generality, that $[a_i, 2b_i] \neq 0$. Letting $2b_i$ play the role of b_i in the argument which led to (37), we see that

$$2b_i \text{ is a unit of } R_i. \tag{38}$$

Let $b \in R$ such that $\sigma_i(b) = b_i$. By Lemma 4, $b \in J$ or $b - b^n \in N$ for some integer $n > 1$. Since $b_i \notin \sigma_i(J)$, $b \notin J$, and hence

$$b - b^n \in N \quad \text{for some integer } n > 1. \tag{39}$$

Applying a similar argument to $2b_i$ yields

$$2b - (2b)^m \in N \quad \text{for some integer } m > 1. \tag{40}$$

Now, by (39) and (40),

$$b_i - {b_i}^n \in N_i, \qquad 2b_i - (2b_i)^m \in N_i. \tag{41}$$

For any $x_i \in R_i$, let $\overline{x}_i = x_i + N_i \in R_i/N_i$. Then (41) implies that

$$(\overline{b}_i)^n = \overline{b}_i, \quad (2\overline{b}_i)^m = 2\overline{b}_i, \quad (n > 1,\ m > 1). \tag{42}$$

Observe that by (42),

$$\begin{gathered} (2\overline{b}_i)^{(m-1)(n-1)+1} = (2\overline{b}_i)^m = 2\overline{b}_i, \\ (2\overline{b}_i)^{(m-1)(n-1)+1} = 2^{(m-1)(n-1)+1}(\overline{b}_i)^n = 2^{(m-1)(n-1)+1}\overline{b}_i. \end{gathered} \tag{43}$$

Hence,

$$(2^{(m-1)(n-1)+1} - 2)\overline{b}_i = \overline{0}. \tag{44}$$

Also, by (37), $\overline{b}_i$ is a unit of R_i/N_i, and hence the above equation implies that

$$(2^{(m-1)(n-1)+1} - 2)\overline{1} = \overline{0}. \tag{45}$$

Therefore, $(2^{(m-1)(n-1)+1}-2)\cdot 1 \in R_i$ is nilpotent, and hence R_i is not of characteristic zero. Since R_i is subdirectly irreducible, we have

$$\text{characteristic of } R_i \text{ is } p^k, \quad p \text{ prime}, \ k \geq 1. \tag{46}$$

Now, using (37), (42), and (46), we see that the subring $\langle \overline{b}_i \rangle$ generated by the unit $\overline{b}_i$ is a finite commutative ring with identity which has no nonzero nilpotents, and hence

$$\langle \overline{b}_i \rangle \cong \bigoplus_{j=1}^{t} \mathrm{GF}\left(p^{k_j}\right), \quad t \text{ finite, each } k_j \geq 1. \tag{47}$$

Let $\alpha = k_1 k_2 \cdots k_t$, then by (47), $(\overline{b}_i)^{p^{k\alpha}} = \overline{b}_i$. Thus

$$b_i{}^{p^{k\alpha}} - b_i \in N_i \subseteq \sigma_i(J). \tag{48}$$

Now, since $a_i \in \sigma_i(J)$ and $\sigma_i(J)$ is commutative, (48) yields $[b_i{}^{p^{k\alpha}} - b_i, a_i] = 0$. Thus

$$\left[b_i{}^{p^{k\alpha}}, a_i\right] = [b_i, a_i]. \tag{49}$$

Combining (37), (33), Lemma 8, and (46), we see that (49) implies that $[b_i, a_i] = 0$, which contradicts (36). This contradiction proves (35). To complete the proof of the theorem, let $x_i \in R_i$ and let $x \in R$ be such that $\sigma_i(x) = x_i$. By Lemma 3, N is an ideal of R, and therefore $N \subseteq J$. Now, by Lemma 4, $x \in J$, in which case, $x - x^n \in J$ for any integer n or $x - x^n \in N \subseteq J$ for some integer $n > 1$, and hence, by (35), $x_i - x_i{}^n \in \sigma_i(J) \subseteq$ Center of R_i. Thus, by a well-known Herstein [3, Theorem 21], R_i is commutative. Therefore, R is commutative, and this proves the theorem. □

Theorem 12. *Let R be a subweakly periodic ring such that the following two conditions hold:*

(i) *for all x and y in R, there exist relatively prime positive integers $m = m(x,y)$ and $n = n(x,y)$ such that*

$$(xy)^m - (yx)^m \in C, \qquad (xy)^n - (yx)^n \in C, \tag{50}$$

where C is the center of R;

(ii) *the Jacobson radical J is commutative.*

Then R is commutative. (In particular, this theorem holds if R is weakly periodic (or periodic).)

Proof. Since m and n are relatively prime integers, there exist positive integers k and l such that $km - ln = 1$. Let $\gamma = ln$. Then $km = \gamma + 1$. Moreover, by (50),

$$[xy,(yx)^m]=0, \qquad [xy,(yx)^n]=0, \tag{51}$$

which implies

$$[xy,(yx)^{km}]=0, \qquad [xy,(yx)^{ln}]=0. \tag{52}$$

Thus

$$[xy,(yx)^{\gamma+1}]=0, \qquad [xy,(yx)^{\gamma}]=0. \tag{53}$$

Hence

$$(yx)^{\gamma+1}xy=xy(yx)^{\gamma+1}=xy(yx)^{\gamma}(yx)=(yx)^{\gamma}(xy)(yx), \tag{54}$$

and hence

$$(yx)^{\gamma+1}xy-(yx)^{\gamma}(xy)(yx)=0. \tag{55}$$

Thus

$$(yx)^{\gamma}\{yx\cdot xy-xy\cdot yx\}=0, \tag{56}$$

and hence

$$(yx)^{\gamma}[xy,yx]=0. \tag{57}$$

A similar argument shows that $[xy,yx](yx)^{\gamma}=0$, and so

$$(yx)^{\gamma}[xy,yx]=0=[xy,yx](yx)^{\gamma}. \tag{58}$$

Theorem 12 now follows from (58) and Theorem 11 as well. □

Theorem 13. *Let R be a weakly periodic ring and suppose that the following conditions hold:*

(i) *for all x,y in R, there exist words $w(x,y)$ and $w'(x,y)$ such that $(x,y)[xy,yx]=0=[xy,yx]w'(x,y)$;*

(ii) *every commutator $[a,b]$, with a and b nilpotent, is potent (i.e., $[a,b]^k=[a,b]$ for some integer $k>1$).*

Then R is commutative. (In particular, this theorem holds if R is periodic.)

Proof. In view of Theorem 11, it suffices to show that J is commutative. By Lemma 2, the commutator idea $\mathscr{C}(R)\subseteq J$. Also, by Lemma 10, $J\subseteq N$. Hence, $\mathscr{C}(R)\subseteq N$, and thus N is an ideal of R. This implies that $N\subseteq J$, and hence $J=N$. Suppose that a and b are any elements of N. Then, $[a,b]$ is also in N since N is an ideal. Also, $[a,b]$ is potent by hypothesis (ii), and hence $[a,b]=0$

(since the only element which is both potent and nilpotent is zero). Therefore, $N = J$ is commutative. The theorem now follows from Theorem 11. □

Theorem 14. *Let R be a weakly periodic ring such that the following conditions hold:*

(i) *for all x, y in R, there exist relatively prime positive integers m, n such that $(xy)^n - (yx)^n \in C$ and $(xy)^m - (yx)^m \in C$;*

(ii) *for all a, b in N, $[a, b]$ is potent.*

Then R is commutative. (In particular, this theorem holds if R is periodic.)

Proof. A careful examination of the proof of Theorem 12 shows that hypothesis (i) above implies that the "word" hypothesis stated in condition (i) of Theorem 13 holds here as well (see (58)). Thus, all of the hypotheses of Theorem 13 are satisfied, and hence R is commutative. □

We conclude by considering the special case where the "words" involved in the above theorems happen to be the empty words. As an illustration, we consider the status of Theorem 11 when $w(x, y)$ and $w'(x, y)$ are the empty words. The result is the following corollary.

Corollary 15. *Let R be a subweakly periodic ring (in particular, R may be chosen to be weakly periodic (or periodic)) satisfying the following conditions:*

(i) $[xy, yx] = 0$ *for all x, y in R;*

(ii) *J is commutative.*

Then R is commutative.

A similar corollary may be obtained by taking the "words" in Theorem 13 to be the empty words.

Finally, we have the following corollary of Corollary 15.

Corollary 16. *Let R be a subweakly periodic ring (in particular, R may be chosen to be weakly periodic or (periodic)) satisfying the following conditions:*

(i) $[x, y]$ *commutes with x for all x in R;*

(ii) *J is commutative.*

Then R is commutative.

Proof. Let $x, y \in R$. By hypothesis (i) (interchanging x and y), it follows that $[y, x]$ commutes with y, and hence $[x, y]$ commutes with y. Thus, $[x, y]$ commutes with both x and y, and hence with xy. Thus, yx commutes with xy, and Corollary 16 now follows from Corollary 15. □

References

[1] H. E. Bell, *On some commutativity theorems of Herstein*, Arch. Math. (Basel) **24** (1973), 34–38.

[2] J. Grosen, H. Tominaga, and A. Yaqub, *On weakly periodic rings, periodic rings and commutativity theorems*, Math. J. Okayama Univ. **32** (1990), 77–81.

[3] I. N. Herstein, *A generalization of a theorem of Jacobson. III*, Amer. J. Math. **75** (1953), 105–111.

[4] N. Jacobson, *Structure of Rings*, American Mathematical Society Colloquium Publications, vol. 37, American Mathematical Society, Rhode Island, 1964.

Amber Rosin: Department of Mathematics, University of California, Santa Barbara, CA 93106, USA

E-mail address: rosin@math.ucsb.edu

Adil Yaqub: Department of Mathematics, University of California, Santa Barbara, CA 93106, USA

E-mail address: yaqub@math.ucsb.edu