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OSCILLATION FOR ADVANCED DIFFERENTIAL EQUATIONS WITH OSCILLATING COEFFICIENTS

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Some sufficient conditions are established for the oscillation of all solutions of the advanced differential equation $x'(t) - p(t)x(t+\tau) = 0$, $t \geq t_0$, where the coefficient $p(t) \in C([t_0, \infty), R)$ is oscillatory, and τ is a positive constant.

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1. Introduction. Consider the following advanced differential equation:

$$x'(t) - p(t)x(t+\tau) = 0, \quad t \geq t_0, \tag{1.1}$$

where the coefficient $p(t) \in C([t_0, \infty), R)$ is an oscillatory function, that is, $p(t)$ takes both positive and negative values, and τ is a positive constant.

The problem of establishing sufficient conditions for the oscillation of all solutions of (1.1) has been the subject of many investigations. For example, see [1, 2, 3, 4, 5, 6] and the references cited therein. With respect to the oscillation of difference equation with oscillating coefficients, readers can refer to [7, 8, 9, 10].

By a solution of (1.1), we mean a continuously differentiable function defined on $[t_0, \infty)$ such that (1.1) is satisfied for $t \geq t_0$. As is customary, a solution of (1.1) is called oscillatory if it has arbitrarily large zeros. Otherwise it is called nonoscillatory. Equation (1.1) is called oscillatory if all its solutions are oscillatory.

For (1.1), Ladas and Stavroulakis [4] proved that all solutions of (1.1) oscillate if

$$p(t) \geq 0, \qquad \liminf_{t\to\infty} \int_t^{t+\tau} p(s)ds > \frac{1}{e}. \tag{1.2}$$

For the result, see also Kusano [3] and Koplatadze and Chanturiya [1]. Recently, Li and Zhu [6] improved the above result to the following form.

Theorem 1.1. *Suppose that there exist a $t_1 > t_0 + \tau$ and a positive integer k such that*

$$\begin{gathered} p_k(t) \geq \frac{1}{e^k}, \quad q_k(t) \geq \frac{1}{e^k}, \quad t \geq t_1 + k\tau, \\ \int_{t_1+k\tau}^{\infty} p(t)\left[\exp\left(e^{k-1}p_k(t) - \frac{1}{e}\right) - 1\right]dt = \infty. \end{gathered} \tag{1.3}$$

Then every solution of (1.1) oscillates. Here, $p(t) \in C([t_0,\infty),[0,\infty))$ and the sequences $\{p_n(t)\}$ and $\{q_n(t)\}$ of functions are defined as follows:

$$\begin{aligned} p_1(t) &= \int_t^{t+\tau} p(s)ds, \\ p_n(t) &= \int_t^{t+\tau} p(s)p_{n-1}(s)ds, \quad n \geq 2,\ t \geq t_0, \\ q_1(t) &= \int_{t-\tau}^{t} p(s)ds, \quad t \geq t_0 + \tau, \\ q_n(t) &= \int_{t-\tau}^{t} p(s)q_{n-1}(s)ds, \quad n \geq 2,\ t \geq t_0 + n\tau. \end{aligned} \tag{1.4}$$

For the studies of the oscillation of (1.1), all the previous works, such as Ladas and Stavroulakis [4], Kusano [3], Li and Zhu [6], and Koplatadze and Chanturiya [1], are under the assumption that the coefficient $p(t)$ has constant sign, that is, $p(t) \in C([t_0,\infty),R_+)$. These investigations, in general, make use of the observation that if $x(t)$ is an eventually positive solution of (1.1), then $x'(t) = p(t)x(t+\tau) \geq 0$ for all large t so that $x(t)$ is eventually nondecreasing. However, when the coefficient $p(t)$ is oscillatory, that is, $p(t)$ takes both positive and negative values, the monotonicity does not hold any longer. All known results in the literature cannot be applied to the case where the coefficient $p(t)$ is oscillatory. Then, a natural question arises on how to investigate the oscillation of (1.1) when the coefficient $p(t)$ is oscillatory.

This problem, to the best of our knowledge, does not have any results up to now. The aim of this paper is to solve the problem. Our work is the continuity of that in [6], and the result obtained is of significance because a large number of oscillation criteria for higher-order as well as nonlinear differential equations can be reduced to oscillation criteria for equations of form (1.1).

2. Main result. As a starting point, we introduce a lemma that is required for the proof of our main results.

Lemma 2.1. *Suppose that $r \geq 0$ and that $\varphi(\cdot)$ is a nonnegative function in R with $\varphi(0) = 0$. Then $\varphi(r)re^x \geq \varphi(r)x + \varphi(r)\ln(er + 1 - \operatorname{sign}(r))$ for any*

$x \in R$, where the function $\operatorname{sign}(\cdot)$ *is the usual sign function, that is,*

$$\operatorname{sign}(r) = \begin{cases} -1 & r < 0, \\ 0 & r = 0, \\ 1 & r > 0. \end{cases} \tag{2.1}$$

The proof of Lemma 2.1 is trivial and is omitted here.

Remark 2.2. In Lemma 2.1, the condition "$\varphi(0) = 0$" is only required to ensure the inequality holding for $r = 0$. When $r > 0$, this condition is unnecessary.

The main result of this paper is the following.

Theorem 2.3. *Let* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *be two sequences in* $[t_0, \infty)$ *satisfying*

$$a_n + 2\tau \le b_n \le a_{n+1} - 2\tau. \tag{2.2}$$

Assume that

$$p(t) \ge 0, \quad \text{for } t \in \cup_{n=1}^{\infty}[a_n, b_n]. \tag{2.3}$$

Define function $P(t)$ *as follows:*

$$P(t) = \begin{cases} p(t), & t \in \cup_{n=1}^{\infty}[a_n, b_n - \tau], \\ 0, & \text{otherwise.} \end{cases} \tag{2.4}$$

If

$$\int_{t_0}^{\infty} P(t) \ln\left[e \int_{t-\tau}^{t} P(s)ds + 1 - \operatorname{sign}\left(\int_{t-\tau}^{t} P(s)ds \right) \right] dt = \infty, \tag{2.5}$$

then every solution of (1.1) is oscillatory.

Proof. Assume, for the sake of contradiction, that there exists an eventually positive solution $x(t)$ of (1.1). Without loss of generality, we suppose that $x(t) > 0$ for $t \ge a_2$. With respect to function $P(t)$, we have the following assertions:

(i) $\int_{t-\tau}^{t} P(s)ds < 1$, $t \ge a_2$;

(ii) $\lim_{t\to\infty} \sup \int_{t-\tau}^{t} P(s)ds > 0$.

We now first prove (i). Indeed, according to the definition of $P(t)$, we derive

$$0 \le P(t) \le p(t), \quad t \in \cup_{n=2}^{\infty}[a_n, b_n]. \tag{2.6}$$

Then, by (1.1), one can see

$$x'(t) - P(t)x(t+\tau) \ge 0, \quad t \in \cup_{n=2}^{\infty}[a_n, b_n], \tag{2.7}$$

and so $x(t)$ is nondecreasing for $t \in \cup_{n=2}^{\infty}[a_n, b_n]$.

If $t \in \cup_{n=2}^{\infty}[a_n+\tau, b_n-\tau]$, then $[t-\tau, t] \subset \cup_{n=2}^{\infty}[a_n, b_n-\tau]$. Integrating (2.7) from $t-\tau$ to t and noticing the nondecreasing nature of $x(t)$ give

$$\begin{aligned} x(t) &\geq x(t-\tau) + \int_{t-\tau}^{t} P(s)x(s+\tau)ds \\ &> \int_{t-\tau}^{t} P(s)x(s+\tau)ds \\ &\geq x(t)\int_{t-\tau}^{t} P(s)ds, \end{aligned} \tag{2.8}$$

which implies

$$\int_{t-\tau}^{t} P(s)ds < 1, \quad t \in \cup_{n=2}^{\infty}[a_n+\tau, b_n-\tau]. \tag{2.9}$$

If $t \notin \cup_{n=2}^{\infty}[a_n+\tau, b_n-\tau]$, then one can consider the following three cases.

Case 1. If $t \in \cup_{n=2}^{\infty}(b_n-\tau, b_n]$, then, by definition (2.4) of $P(t)$,

$$\begin{aligned} \int_{t-\tau}^{t} P(s)ds &\leq \int_{b_n-2\tau}^{t} P(s)ds \\ &= \int_{b_n-2\tau}^{b_n-\tau} P(s)ds + \int_{b_n-\tau}^{t} P(s)ds \\ &= \int_{b_n-2\tau}^{b_n-\tau} P(s)ds < 1. \end{aligned} \tag{2.10}$$

Note that (2.9) is used in getting the last inequality above.

Case 2. If $t \in \cup_{n=2}^{\infty}[a_n, a_n+\tau)$, then

$$\int_{t-\tau}^{t} P(s)ds = \int_{t-\tau}^{a_n} P(s)ds + \int_{a_n}^{t} P(s)ds = \int_{a_n}^{t} P(s)ds \leq \int_{a_n}^{a_n+\tau} P(s)ds < 1. \tag{2.11}$$

Case 3. If $t \in \cup_{n=2}^{\infty}(b_n, a_{n+1}]$, then

$$\int_{t-\tau}^{t} P(s)ds = 0 < 1. \tag{2.12}$$

Thus, we complete the proof of assertion (i).

Next, we will show assertion (ii). If we have $\lim_{t\to\infty}\int_{t-\tau}^{t} P(s)ds = 0$, so there exists a $T \geq t_0$ such that

$$\int_{t-\tau}^{t} P(t)dt < \frac{1}{e} \quad \text{for } t \geq T. \tag{2.13}$$

It follows from (2.5) that

$$\int_{a_2}^{\infty} P(t)\ln\left[e\int_{t-\tau}^{t} P(s)ds + 1 - \operatorname{sign}\left(\int_{t-\tau}^{t} P(s)ds\right)\right]dt = \infty. \tag{2.14}$$

Therefore,

$$\begin{aligned}\infty &= \int_{a_2}^{\infty} P(t)\ln\left[e\int_{t-\tau}^{t} P(s)ds + 1 - \operatorname{sign}\left(\int_{t-\tau}^{t} P(s)ds\right)\right]dt \\ &= \sum_{n=2}^{\infty}\int_{a_n}^{a_{n+1}} P(t)\ln\left[e\int_{t-\tau}^{t} P(s)ds + 1 - \operatorname{sign}\left(\int_{t-\tau}^{t} P(s)ds\right)\right]dt \\ &= \sum_{n=2}^{\infty}\left(\int_{a_n}^{b_n-\tau} + \int_{b_n-\tau}^{a_{n+1}}\right)P(t)\ln\left[e\int_{t-\tau}^{t} P(s)ds + 1 - \operatorname{sign}\left(\int_{t-\tau}^{t} P(s)ds\right)\right]dt \\ &= \sum_{n=2}^{\infty}\int_{a_n}^{b_n-\tau} P(t)\ln\left[e\int_{t-\tau}^{t} P(s)ds + 1 - \operatorname{sign}\left(\int_{t-\tau}^{t} P(s)ds\right)\right]dt.\end{aligned} \tag{2.15}$$

And so there exists a positive integer sequence $\{n_i\}_{i=1}^{\infty}$ such that

$$\int_{a_{n_i}}^{b_{n_i}-\tau} P(t)\ln\left[e\int_{t-\tau}^{t} P(s)ds + 1 - \operatorname{sign}\left(\int_{t-\tau}^{t} P(s)ds\right)\right]dt > 0, \quad i = 1,2,\ldots. \tag{2.16}$$

Since $P(t)$ is nonnegative and continuous in $t \in [a_{n_i}, b_{n_i} - \tau]$, $i \geq 1$, there exists an interval $[k_i, l_i] \subset [a_{n_i}, b_{n_i} - \tau], i \geq 1$, such that

$$P(t)\ln\left[e\int_{t-\tau}^{t} P(s)ds + 1 - \operatorname{sign}\left(\int_{t-\tau}^{t} P(s)ds\right)\right]dt > 0, \quad t \in [k_i, l_i],\ i = 1,2,\ldots. \tag{2.17}$$

From this, we know

$$\int_{t-\tau}^{t} P(t)dt > \frac{1}{e}, \quad t \in [k_i, l_i],\ i = 1,2,\ldots, \tag{2.18}$$

which is contrary to (2.13). So, assertion (ii) is true. Accordingly, there exist a constant $d > 0$ and a sequence $\{t_n\}$ of points such that $t_n \to \infty$ as $n \to \infty$, $t_n \in [a_n, b_n - \tau]$, and

$$\int_{t_n-\tau}^{t_n} P(t)dt \geq d, \quad n = 1,2,\ldots. \tag{2.19}$$

We now set

$$\lambda(t) = \frac{x'(t)}{x(t)}, \quad t \geq a_2. \tag{2.20}$$

Then, by (2.7),

$$\lambda(t) \geq 0, \quad t \in \cup_{n=2}^{\infty}[a_n, b_n]. \tag{2.21}$$

Integrating (2.20) from $t \geq a_2$ to $t+\tau$ and noting (1.1), one can obtain

$$\lambda(t) = p(t)\exp\left(\int_t^{t+\tau}\lambda(s)ds\right), \quad t \geq a_2, \tag{2.22}$$

$$\lambda(t)\int_{t-\tau}^{t}P(s)ds = p(t)\int_{t-\tau}^{t}P(s)ds\exp\left(\int_t^{t+\tau}\lambda(s)ds\right), \quad t \geq a_2. \tag{2.23}$$

Notice that $P(t)$ is nonnegative and continuous restricted to $\cup_{n=2}^{\infty}[a_n, b_n-\tau]$. We can show that $\int_{t-\tau}^{t}P(s)ds = 0$ for any $t \in \cup_{n=2}^{\infty}[a_n, b_n-\tau]$ is equivalent to $P(t) = 0$ for any $t \in \cup_{n=2}^{\infty}[a_n, b_n-\tau]$. Now, we simply verify this as follows.

If $\int_{t-\tau}^{t}P(s)ds = 0$ for any $t \in \cup_{n=2}^{\infty}[a_n, b_n-\tau]$, then there must be $P(t) = 0$ for any $t \in \cup_{n=2}^{\infty}[a_n, b_n-\tau]$. We only prove $P(t) = 0$ at the point a_n and the proofs for the other points are similar and are omitted. Otherwise, $P(a_n) > 0$. Since $P(t)$ is right-continuous at the point a_n, there exists a $\delta > 0$ such that $P(t) > 0$ for $t \in [a_n, a_n+\delta)$. Hence, for any given $\epsilon \in (0, \min\{\tau, \delta\})$, $\int_{a_n+\epsilon-\tau}^{a_n+\epsilon}P(s)ds = \int_{a_n}^{a_n+\epsilon}P(s)ds > 0$, which contradicts the known assumption.

Conversely, if $P(t) = 0$ for any $t \in \cup_{n=2}^{\infty}[a_n, b_n-\tau]$, then, according to definition (2.4) of $P(t)$, one can see that $P(t) \equiv 0$ for $t \geq a_2$, which means that $\int_{t-\tau}^{t}P(s)ds = 0$ for any $t \in \cup_{n=2}^{\infty}[a_n, b_n-\tau]$, and so the conclusion above holds.

Therefore, by using Lemma 2.1 with $r = \int_{t-\tau}^{t}P(s)ds$, $\varphi(r) = P(t)$, and $x = \int_t^{t+\tau}\lambda(s)ds$, we derive

$$\begin{aligned}
&\lambda(t)\int_{t-\tau}^{t}P(s)ds \\
&\quad \geq P(t)\int_{t-\tau}^{t}P(s)ds\exp\left(\int_t^{t+\tau}\lambda(s)ds\right) \\
&\quad \geq P(t)\int_t^{t+\tau}\lambda(s)ds + P(t)\ln\left[e\int_{t-\tau}^{t}P(s)ds + 1 - \operatorname{sign}\left(\int_{t-\tau}^{t}P(s)ds\right)\right],
\end{aligned} \tag{2.24}$$

that is, for $t \in \cup_{n=2}^{\infty}[a_n, b_n-\tau]$,

$$\begin{aligned}
&\lambda(t)\int_{t-\tau}^{t}P(s)ds \\
&\quad \geq P(t)\int_t^{t+\tau}\lambda(s)ds + P(t)\ln\left[e\int_{t-\tau}^{t}P(s)ds + 1 - \operatorname{sign}\left(\int_{t-\tau}^{t}P(s)ds\right)\right].
\end{aligned} \tag{2.25}$$

Notice that $P(t) = 0$ for $t \in \cup_{n=2}^{\infty}(b_n-\tau, a_{n+1})$. Accordingly, we have, for $t \in \cup_{n=2}^{\infty}(b_n-\tau, b_n]$, $\int_{t-\tau}^{t}P(s)ds = \int_{t-\tau}^{b_n-\tau}P(s)ds > 0$. This, together with (2.23) and Lemma 2.1, implies the validity of (2.25). For $t \in \cup_{n=2}^{\infty}(b_n, a_{n+1})$, $\int_{t-\tau}^{t}P(s)ds = 0$, which also indicates that (2.25) holds. So, (2.25) is true for $t \geq a_2$.

Integrating (2.25) from a_2 to t_n for $n > 2$ produces

$$\begin{aligned}&\int_{a_2}^{t_n}\lambda(t)\int_{t-\tau}^{t}P(s)\,ds\,dt-\int_{a_2}^{t_n}P(t)\int_{t}^{t+\tau}\lambda(s)\,ds\,dt\\&\quad\geq\int_{a_2}^{t_n}P(t)\ln\left[e\int_{t-\tau}^{t}P(s)\,ds+1-\operatorname{sign}\left(\int_{t-\tau}^{t}P(s)\,ds\right)\right]dt.\end{aligned}\tag{2.26}$$

Put

$$\begin{aligned}D_1&=\{(s,t)\mid a_2+\tau\leq s\leq t_n,\ s-\tau\leq t\leq s\},\\D_2&=\{(s,t)\mid t_n\leq s\leq t_n+\tau,\ s-\tau\leq t\leq t_n\},\\D_3&=\{(s,t)\mid a_2\leq t\leq a_2+\tau,\ t\leq s\leq a_2+\tau\}.\end{aligned}\tag{2.27}$$

It is clear that $P(t)\lambda(s)\geq 0$ for $(t,s)\in D_2\cup D_3$. Thus,

$$\begin{aligned}\int_{a_2}^{t_n}P(t)\int_{t}^{t+\tau}\lambda(s)\,ds\,dt&=\sum_{i=1}^{3}\int\int_{D_i}P(t)\lambda(s)\,ds\,dt\\&\geq\int\int_{D_1}P(t)\lambda(s)\,ds\,dt=\int_{a_2+\tau}^{t_n}\int_{s-\tau}^{s}P(t)\lambda(s)\,ds\,dt\\&=\int_{a_2+\tau}^{t_n}\lambda(s)\int_{s-\tau}^{s}P(t)\,dt\,ds=\int_{a_2+\tau}^{t_n}\lambda(t)\int_{t-\tau}^{t}P(s)\,ds\,dt.\end{aligned}\tag{2.28}$$

This, together with (2.26), leads to

$$\begin{aligned}&\int_{a_2}^{t_n}P(t)\ln\left[e\int_{t-\tau}^{t}P(s)\,ds+1-\operatorname{sign}\left(\int_{t-\tau}^{t}P(s)\,ds\right)\right]dt\\&\quad\leq\int_{a_2}^{t_n}\lambda(t)\int_{t-\tau}^{t}P(s)\,ds\,dt-\int_{a_2+\tau}^{t_n}\lambda(t)\int_{t-\tau}^{t}P(s)\,ds\,dt\\&\quad=\int_{a_2}^{a_2+\tau}\lambda(t)\int_{t-\tau}^{t}P(s)\,ds\,dt<\infty,\end{aligned}\tag{2.29}$$

whereas, according to (2.5), we have

$$\lim_{n\to\infty}\int_{a_2}^{t_n}P(t)\ln\left[e\int_{t-\tau}^{t}P(s)\,ds+1-\operatorname{sign}\left(\int_{t-\tau}^{t}P(s)\,ds\right)\right]dt=\infty,\tag{2.30}$$

which contradicts (2.29) and completes the proof. □

3. Example. As an application of Theorem 2.3, we consider the oscillation of the following equation:

$$x'(t) - p(t)x(t+1) = 0, \quad t \geq 0, \tag{3.1}$$

where $\tau = 1$ and the function $p(t)$ is 6-periodic one with

$$p(t) = \begin{cases} -t, & 0 \leq t \leq 1, \\ t-2, & 1 < t \leq 4, \\ 6-t, & 4 < t \leq 6. \end{cases} \tag{3.2}$$

Obviously, $\lim_{t\to\infty} \inf \int_t^{t+\tau} p(t)ds = -1/2 < 0$. Therefore, the results in [1, 2, 3, 4, 5, 6] cannot be applied to (3.1). But, if we denote $a_n = 2 + 6(n-1)$, $b_n = 6n$, $n \geq 1$, then, clearly, $a_n, b_n \in [0, \infty)$,

$$a_n + 2\tau \leq b_n \leq a_{n+1} - 2\tau, \quad n = 1, 2, \ldots, \tag{3.3}$$

and $p(t) \geq 0$ for $t \in \cup_{n=1}^{\infty}[a_n, b_n]$. Furthermore, if we set

$$P(t) = \begin{cases} p(t), & t \in \cup_{n=1}^{\infty}[a_n, b_n - \tau], \\ 0, & \text{otherwise}, \end{cases} \tag{3.4}$$

then we have

$$\begin{aligned}
&\int_{a_n}^{b_n-\tau} P(t) \ln\left[e\int_{t-\tau}^{t} P(s)ds + 1 - \operatorname{sign}\left(\int_{t-\tau}^{t} P(s)ds\right)\right]dt \\
&\quad = \int_2^5 P(t) \ln\left[e\int_{t-\tau}^{t} P(s)ds + 1 - \operatorname{sign}\left(\int_{t-\tau}^{t} P(s)ds\right)\right]dt \\
&\quad = \int_2^4 P(t) \ln\left[e\int_{t-\tau}^{t} P(s)ds + 1 - \operatorname{sign}\left(\int_{t-\tau}^{t} P(s)ds\right)\right]dt \\
&\qquad + \int_4^5 P(t) \ln\left[e\int_{t-\tau}^{t} P(s)ds + 1 - \operatorname{sign}\left(\int_{t-\tau}^{t} P(s)ds\right)\right]dt \\
&\quad = \int_2^4 (t-2) \ln\left[e\left(\int_{t-1}^{2} P(s)ds + \int_2^t P(s)ds\right) + 1 - \operatorname{sign}\left(\int_{t-\tau}^{t} P(s)ds\right)\right]dt \\
&\qquad + \int_4^5 (6-t) \ln\left[e\left(\int_{t-1}^{2} P(s)ds + \int_2^4 P(s)ds + \int_4^t P(s)ds\right)\right. \\
&\qquad\qquad \left. + 1 - \operatorname{sign}\left(\int_{t-\tau}^{t} P(s)ds\right)\right]dt
\end{aligned}$$

$$
\begin{aligned}
&= \int_2^4 (t-2)\ln\left(e\int_2^t (s-2)ds\right)dt + \int_4^5 (6-t)\ln\left(e\left(2+\int_4^t (6-s)ds\right)\right)dt \\
&= \left[\int_2^t (s-2)ds \ln\left(e\int_2^t (s-2)\right)ds - \int_2^t (s-2)ds\right]\Bigg|_2^4 \\
&\quad + \left[\left(2+\int_4^t (6-s)ds\right)\ln\left(e\left(2+\int_4^t (6-s)ds\right)\right) - \left(2+\int_4^t (6-s)ds\right)\right]\Bigg|_4^5 \\
&= 2\ln 2 + \left(\frac{7}{2}\ln\frac{7}{2} - 2\ln 2\right) = \frac{7}{2}\ln\frac{7}{2} > 0,
\end{aligned}
\tag{3.5}
$$

which means that

$$
\int_{a_2}^{\infty} P(t)\ln\left[e\int_{t-\tau}^{t} P(s)ds + 1 - \operatorname{sign}\left(\int_{t-\tau}^{t} P(s)ds\right)\right]dt = \infty. \tag{3.6}
$$

By Theorem 2.3, every solution of (3.1) is oscillatory.

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