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THE SPECTRUM OF A CLASS OF ALMOST PERIODIC OPERATORS

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For almost Mathieu operators, it is shown that the occurrence of Cantor spectrum and the existence, for every point in the spectrum and suitable phase parameters, of at least one localized eigenfunction which decays exponentially are inconsistent properties.

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1. Introduction. In a series of papers [14, 15, 16, 17, 18, 19] the author has developed an approach to study the spectrum of the simplest kind of nontrivial almost periodic operators, which is heavily based on C^*-algebraic methods. This approach originated in the belief that the involvement of irrational rotation C^*-algebras in the investigation of almost Mathieu operators would yield an interdependence between the occurrence of localized eigenfunctions and the topological nature of the spectrum of these operators. In the sequel, we are going to establish such a connection. For almost Mathieu operators which are defined by

$$(H(\alpha,\beta,\theta)\xi)_n = \xi_{n+1}+\xi_{n-1}+2\beta\cos(2\pi\alpha n+\theta)\xi_n, \quad \xi\in\ell^2(\mathbb{Z}), \tag{1.1}$$

where α, β, and θ are real parameters, the following version of localization has been established by Fröhlich et al. (see [7, page 6 and Section 3]).

Consider an irrational number α which satisfies the following Diophantine condition: there exists a constant $c>0$ such that $|n\alpha-m|\geq c/n^2$ for all $m,n\in\mathbb{Z}$ and $n\neq 0$. Then there exists a constant $\beta_0>0$ such that for any $\beta\geq\beta_0$, the following condition holds:

(L′) there exists a subset $N_1\subset\mathbb{R}$ which has Lebesgue measure zero, and a constant $0<r<1$, such that the following condition holds true: if for $\xi=\{\xi_n\}_{n\in\mathbb{Z}}$ there are numbers $a>0$, $\chi\in\mathbb{R}$, and $\theta\in\mathbb{R}\backslash N_1$ such that $|\xi_n|\leq an^2$ and

$$\xi_{n+1}+\xi_{n-1}+2\beta\cos(2\pi\alpha n+\theta)\xi_n=\chi\xi_n, \quad \text{for } n\in\mathbb{Z}, \tag{1.2}$$

then ξ decays exponentially of order r as $|n|\to\infty$, that is, there exists a constant $b>0$, such that $|\xi_n|\leq br^{|n|}$ for $n\in\mathbb{Z}$.

The property (L′) entails that there exists a subset $N_2 \subset \mathbb{R}$ containing N_1, which also has Lebesgue measure zero, such that for every $\theta \in \mathbb{R}\backslash N_2$, the operator $H(\alpha,\beta,\theta)$ has pure point spectrum with eigenfunctions decaying exponentially of order r.

The work of Fröhlich et al. is paralleled to some extent by the work of Sinaĭ [21] (see also [4, Section 5.5]). It is well known that for irrational α, the spectrum of $H(\alpha,\beta,\theta)$ does not depend on θ. We will denote this spectrum by $\mathrm{Sp}(\alpha,\beta)$. In this paper, we are concerned with the following condition for the localization of eigenfunctions:

(L) for every $\chi \in \mathrm{Sp}(\alpha,\beta)$, there exists a θ such that the difference equation

$$\xi_{n+1} + \xi_{n-1} + 2\beta\cos(2\pi\alpha n + \theta)\xi_n = \chi\xi_n \tag{1.3}$$

has a nontrivial solution which decays exponentially as $|n| \to \infty$.

While condition (L′) has been established for the parameters stipulated above, no set of parameters has been found yet for which condition (L) holds true. The objective of this paper is to prove the following theorem.

Theorem 1.1. *The validity of condition (L) and the occurrence of Cantor spectrum are inconsistent for almost Mathieu operators.*

In [1, 3, 9, 10, 11], the occurrence of Cantor spectrum has been established under various conditions where property (L′) does not hold. In a number of papers (cf. [12, 20]), it has been conjectured that $\mathrm{Sp}(\alpha,\beta)$ should always be a Cantor set.

We are going to list several properties that condition (L) implies. These properties will be crucial in the proof of the theorem. The first important observation is that if (L) holds and if μ denotes the integrated density of states for $H(\alpha,\beta,\theta)$, then the logarithmic potential associated with μ takes the constant value $\log|\beta|$ on $\mathrm{Sp}(\alpha,\beta)$. This means that $\mathrm{Sp}(\alpha,\beta)$ is a regular compactum, μ is its equilibrium distribution, and $|\beta|$ is the logarithmic capacity of $\mathrm{Sp}(\alpha,\beta)$. (The basic material from classical potential theory which will be used in this paper has been assembled in Appendix A.) This shows among other things that the integrated density of states as well as the (averaged) Lyapunov index (as defined in [5]) are uniquely determined by $\mathrm{Sp}(\alpha,\beta)$. However, considerably more can be shown. The following assertion gives a characterization of the level curves of the conductor potential associated with $\mathrm{Sp}(\alpha,\beta)$ in terms of the spectra of perturbed operators, which are bounded but not selfadjoint.

Assertion 1.2. If (L) holds, then a complex number z is contained in the spectrum of the operator

$$(H_\delta(\alpha,\beta)\xi)_n = \xi_{n+1} + \xi_{n-1} + \beta(\delta e^{2\pi\alpha ni} + \delta^{-1}e^{-2\pi\alpha ni})\xi_n, \quad \xi \in \ell^2(\mathbb{Z}), \tag{1.4}$$

if and only if $\int \log|z-s|\,d\mu(s) = \log|\beta| + |\log|\delta||$.

In order to prove this assertion, we will consider the C^*-algebra generated by the family of operators $\{H_\delta(\alpha,\beta)/\delta \in \mathbb{R}\backslash\{0\}\}$, which is an irrational rotation C^*-algebra with rotation number α. This will put us in a position to study the resolvent of these operators in terms of certain series expansions which arise naturally with the irrational rotation C^*-algebra. These series expansions can be looked upon as noncommutative versions of Fourier series in two variables. The exponential behavior of these series expansions at infinity is then expressed in terms of subharmonic functions. Finally, potential theoretic arguments can be invoked to accomplish the proof of Assertion 1.2.

Our second assertion, whose proof relies heavily on the first one, establishes the claimed connection between condition (L) and the topological nature of the spectrum of almost Mathieu operators.

Assertion 1.3. If (L) holds, then any open and closed subset of $\mathrm{Sp}(\alpha,\beta)$ is not a Cantor set.

In order to render this paper accessible to a wider audience, we will include the exposition material which has been published by the author in [14, 15, 16, 17, 18, 19]. The organization of this paper is as follows. In Section 2, we briefly discuss the irrational rotation C^*-algebra in the context of our approach. In Section 3, we present a notion of multiplicity for elements in $\mathrm{Sp}(\alpha,\beta)$ which was developed in [15]. In Section 4, we study the resolvent of the operators $H_\delta(\alpha,\beta)$ (according to [17]). In Section 5, we give the proofs of Assertions 1.2 and 1.3. In Appendix A, we present some material from classical potential theory. In Appendix B, we state and prove a result about conductor potentials of regular compact subsets of the real line, which is vital for the proof of Assertion 1.3.

2. The irrational rotation C^*-algebra.

Throughout the paper, α denotes an irrational number. An irrational rotation C^*-algebra $\mathcal{A} = \mathcal{A}_\alpha$ is a C^*-algebra which is generated by two unitaries u and v satisfying the relation $uv = e^{2\pi\alpha i}vu$. Such an algebra is uniquely determined, up to isomorphisms, by the number α. We let $h(\alpha,\beta) = u+u^*+\beta(v+v^*)$. The operator $H(\alpha,\beta,\theta)$ is the image of $h(\alpha,\beta)$ under a specific representation of $\mathcal{A}$ on the Hilbert space $\ell^2(\mathbb{Z})$. If π_θ is the representation of $\mathcal{A}$ which is determined on the generators u and v by

$$\begin{aligned} (\pi_\theta(u)\xi)_n &= \xi_{n+1}, \\ (\pi_\theta(v)\xi)_n &= e^{-(2\pi\alpha n+\theta)i}\xi_n, \end{aligned} \tag{2.1}$$

then $\pi_\theta(h(\alpha,\beta)) = H(\alpha,\beta,\theta)$. The symmetries of the operator $h(\alpha,\beta)$ can be expressed in terms of certain symmetries on $\mathcal{A}$. These are (uniquely determined) involutive conjugate linear automorphisms σ_u and σ_v of $\mathcal{A}$ and anti-automorphisms $\tilde{\sigma}_u$ and $\tilde{\sigma}(v)$ of $\mathcal{A}$ such that

$$\begin{aligned} \sigma_u(u) = \tilde{\sigma}_u(u) = u^*, &\qquad \sigma_u(v) = \tilde{\sigma}_u(v) = v, \\ \sigma_v(u) = \tilde{\sigma}_v(u) = u, &\qquad \sigma_v(v) = \tilde{\sigma}_v(v) = v^*. \end{aligned} \tag{2.2}$$

Furthermore, there is a (uniquely determined) automorphism ρ with period four such that

$$\rho(u) = v^*, \qquad \rho(v) = u. \tag{2.3}$$

The operator $h(\alpha,\beta)$ is always a fixed point for σ_u, σ_v, $\tilde{\sigma}_u$, and $\tilde{\sigma}_v$, and $h(\alpha,\beta)$ is a fixed point for ρ if and only if $\beta = 1$. Since σ_u and σ_v commute, likewise $\tilde{\sigma}_u$ and $\tilde{\sigma}_v$, $\sigma = \sigma_u \circ \sigma_v = \tilde{\sigma}_u \circ \tilde{\sigma}_v$ is an involutive automorphism of $\mathcal{A}$. Notice that $\rho^2 = \sigma$.

There is a unique tracial state τ on $\mathcal{A}$, that is, τ is a state which has the trace property $\tau(ab) = \tau(ba)$ for all $a,b \in \mathcal{A}$. Furthermore, if μ denotes the integrated density of states for $H(\alpha,\beta,\theta)$, then we have for any continuous function f on $\mathrm{Sp}(\alpha,\beta)$ the identity

$$\tau(f(h(\alpha,\beta))) = \int f(t)\,d\mu(t). \tag{2.4}$$

Let $w_{pq} = e^{-pq\pi\alpha i}u^p v^q$. Notice that $w_{pq}^* = w_{-p,-q}$, $\sigma_u(w_{pq}) = \tilde{\sigma}_u(w_{pq}) = w_{-p,q}$, $\sigma_v(w_{pq}) = \tilde{\sigma}_v(w_{pq}) = w_{p,-q}$, and $\rho(w_{pq}) = w_{q,-p}$. For any element $a \in \mathcal{A}$, let $\hat{a}_{pq} = \tau(w_{-p,-q}a)$. We call this number the Fourier coefficient of a at the position (p,q). The series $\sum_{p,q\in\mathbb{Z}} \hat{a}_{pq} w_{pq}$ converges to the element a in the Hilbert space norm associated with τ. We will call this series the Fourier series of a.

Proposition 2.1. *Suppose that $a \in \mathcal{A}$ has a finite Fourier series. Then the Fourier series $\sum_{p,q\in\mathbb{Z}} c_{pq}(z)w_{pq}$ of the resolvent $(a-z)^{-1}$ has the following property: for every compact subset K of the resolvent set of a, the double sequence $\{\sup|c_{pq}(z)| : z \in K\}_{p,q\in\mathbb{Z}}$ decays exponentially as $|p|$ and $|q|$ approach infinity.*

Proof. Suppose that the Fourier coefficients of a vanish for $|p|,|q| \geq n$. Then for complex numbers x and y with modules close to one, the spectrum of the operator

$$a(x,y) = \sum_{|p|,|q|\leq n} \hat{a}_{pq} x^p y^q w_{pq} \tag{2.5}$$

is contained in $\mathbb{C}\backslash K$, and we have

$$\left(a(x,y) - z\right)^{-1} = \sum_{p,q\in\mathbb{Z}} c_{pq}(z) x^p y^q w_{pq}. \tag{2.6}$$

The series on the right-hand side of this identity is absolutely convergent. Thus,

$$|c_{pq}(z)|\,|x|^p|y|^q = |\tau((a(x,y)-z)^{-1}w_{-p,-q})| \le \|(a(x,y)-z)^{-1}\| \quad (2.7)$$

for $z \in K$. Suitable choices for $|x|$ and $|y|$ conclude the argument. □

3. Point spectrum and a certain multiplicity for points in the spectrum. In the sequel, we assume throughout that $\beta \neq 0$. We call a state φ on the C^*-algebra $\mathcal{A}$ an eigenstate of $h(\alpha,\beta)$ for $\chi \in \mathrm{Sp}(\alpha,\beta)$ if the identity

$$\varphi(h(\alpha,\beta)a) = \chi\varphi(a), \quad \forall a \in \mathcal{A}, \quad (3.1)$$

holds. The general theory of C^*-algebras yields that for every $\chi \in \mathrm{Sp}(\alpha,\beta)$, there exists at least one eigenstate of $h(\alpha,\beta)$ for χ. Since $h(\alpha,\beta)$ is a selfadjoint operator and a state is a selfadjoint functional, any eigenstate φ also satisfies the following identity:

$$\varphi(h(\alpha,\beta)a) = \varphi(ah(\alpha,\beta)) \quad \forall a \in \mathcal{A}. \quad (3.2)$$

Suppose that φ is a state on $\mathcal{A}$, and for any $p,q \in \mathbb{Z}$, let $x_{pq} = \varphi(w_{pq})$. Then φ satisfies condition (3.1) if and only if

$$\begin{aligned}\cos(\pi\alpha q)&(x_{p-1,q}+x_{p+1,q})\\ &+\beta\cos(\pi\alpha p)(x_{p,q-1}+x_{p,q+1}) = \chi x_{pq} \quad \text{for any } p,q \in \mathbb{Z}.\end{aligned} \quad (3.3)$$

Also, φ satisfies condition (3.2) if and only if

$$\sin(\pi\alpha q)(x_{p-1,q}-x_{p+1,q}) - \beta\sin(\pi\alpha p)(x_{p,q-1}-x_{p,q+1}) = 0 \quad \text{for any } p,q \in \mathbb{Z}. \quad (3.4)$$

So, if φ is an eigenstate of $h(\alpha,\beta)$ for χ, then the double sequence $\{x_{pq}\}$ solves the difference equations (3.3) and (3.4). Notice that the combined system (3.3) and (3.4) is redundant.

We are now going to explain how the solutions of the combined system (3.3) and (3.4) can be generated by certain recursions (see [15, 18]). To this end, we consider a modified system where certain phase angles have been introduced in the coefficients

$$\begin{aligned}\cos(\pi\alpha q+\theta_2)(x_{p-1,q}+x_{p+1,q}) + \beta\cos(\pi\alpha p+\theta_1)(x_{p,q-1}+x_{p,q+1}) &= \chi x_{pq},\\ \sin(\pi\alpha q+\theta_2)(x_{p-1,q}-x_{p+1,q}) - \beta\sin(\pi\alpha p+\theta_1)(x_{p,q-1}-x_{p,q+1}) &= 0,\end{aligned} \quad (3.5)$$

where θ_1 and θ_2 satisfy the condition

$$\frac{\theta_1+\theta_2}{\pi}, \frac{\theta_1-\theta_2}{\pi} \notin \mathbb{Z}+a\mathbb{Z}. \quad (3.6)$$

For any $p,q\in\mathbb{Z}$, we define 4×4 matrices A_{pq} and B_{pq} having the property that a double sequence $\{x_{pq}\}$ solves system (3.5) if and only if

$$\begin{pmatrix} x_{p+1,q+1}\\ x_{p+1,q}\\ x_{p,q+1}\\ x_{pq}\end{pmatrix} = A_{pq}\begin{pmatrix} x_{p,q+1}\\ x_{p,q}\\ x_{p-1,q+1}\\ x_{p-1,q}\end{pmatrix}, \qquad \begin{pmatrix} x_{p+1,q+1}\\ x_{p,q+1}\\ x_{p+1,q}\\ x_{pq}\end{pmatrix} = B_{pq}\begin{pmatrix} x_{p+1,q}\\ x_{pq}\\ x_{p+1,q-1}\\ x_{p,q-1}\end{pmatrix}. \tag{3.7}$$

Let $A_{pq}=A_{pq}(\chi,\beta)=(a_{k\ell})_{1\le k,\ell\le 4}$, where

$$\begin{aligned} a_{11} &= \frac{\chi\sin(\pi\alpha p+\theta_1)}{\sin[\pi\alpha(p+q+1)+\theta_1+\theta_2]},\\ a_{12} &= -\frac{\beta\sin(2\pi\alpha p+2\theta_1)}{\sin[\pi\alpha(p+q+1)+\theta_1+\theta_2]},\\ a_{13} &= -\frac{\sin[\pi\alpha(p-q-1)+\theta_1-\theta_2]}{\sin[\pi\alpha(p+q+1)+\theta_1+\theta_2]},\\ a_{21} &= -\frac{\beta\sin(2\pi\alpha p+2\theta_1)}{\sin[\pi\alpha(p-q)+\theta_1+\theta_2]},\\ a_{22} &= \frac{\chi\sin(\pi\alpha p+\theta_1)}{\sin[\pi\alpha(p-q)+\theta_1-\theta_2]},\\ a_{24} &= -\frac{\sin[\pi\alpha(p+q)+\theta_1+\theta_2]}{\sin[\pi\alpha(p-q)+\theta_1-\theta_2]},\\ a_{k\ell} &= 1 \quad \text{for } (k,\ell)\in\{(3,1),(4,2)\}, \end{aligned} \tag{3.8}$$

and $a_{k\ell}=0$ for the remaining entries of the matrix. Furthermore, let

$$B_{qp}=B_{qp}(\chi,\beta)=A_{pq}\left(\frac{\chi}{\beta},\beta^{-1}\right). \tag{3.9}$$

Condition (3.6) ensures that the denominators in these formulae do not vanish. Apart from being invertible, the matrices A_{pq} and B_{pq} satisfy the following identity:

$$PB_{p,q+1}PA_{pq}=A_{p,q+1}PB_{p-1,q+1}P, \quad P=\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}. \tag{3.10}$$

There are exactly four linearly independent solutions of (3.5), which can be generated in the following manner: given any numbers x_{00}, x_{10}, x_{01}, and x_{11}, one can use the formulae in (3.7), as recursions on the two-dimensional lattice, to compute the values x_{pq}:

$$\begin{array}{ccc} \circ\circ & \xrightarrow{A_{pq}} & \circ\circ\ast \\ \circ\circ & & \circ\circ\ast \end{array} \qquad \begin{array}{ccc} \circ\circ & \xrightarrow{A_{pq}^{-1}} & \ast\circ\circ \\ \circ\circ & & \ast\circ\circ \end{array}$$
$$\begin{array}{ccc} & & \ast\ast \\ \circ\circ & \xrightarrow{B_{pq}} & \circ\circ \\ \circ\circ & & \circ\circ \end{array} \qquad \begin{array}{ccc} & & \circ\circ \\ \circ\circ & \xrightarrow{B_{pq}^{-1}} & \circ\circ \\ \circ\circ & & \ast\ast \end{array}. \tag{3.11}$$

(The four circles to the left represent the four input parameters, while the stars to the right represent the last two output parameters.) Since there are infinitely many ways to reach a position (p,q) by a finite succession of those four basic recursions, departing at the positions $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$, we face the question whether this procedure produces consistent results. Identity (3.10) ensures that the outcome is independent indeed from the specific path we chose to reach the position (p,q).

We now consider $\{(\theta_1^{(n)},\theta_2^{(n)})\}_{n\in\mathbb{N}}$ as a sequence of pairs of nonvanishing phase angles which converges to $(0,0)$. Moreover, we assume that $\theta_1^{(n)}$ and $\theta_2^{(n)}$ satisfy condition (3.6), and $\theta_1^{(n)}/\theta_2^{(n)}$ approaches a number c as $n\to\infty$. Given arbitrary values x_{00}, x_{10}, x_{01}, and x_{11}, the solutions of system (3.5) with phase angles $\theta_1^{(n)}$ and $\theta_2^{(n)}$ and those initial values converge for each point (p,q) in the lattice $\mathbb{Z}^2$ to a solution of the combined system (3.3) and (3.4). Now, consider the case where $x_{00}=x_{01}=x_{11}=0$ but $x_{10}\neq 0$. The limit of the solutions associated with the sequence $\{(\theta_1^{(n)},\theta_2^{(n)})\}_{n\in\mathbb{N}}$ vanishes at $(-1,0)$ depending on whether the constant c equals one or not. This shows that the combined system (3.3) and (3.4) has at least five linearly independent solutions.

Suppose that $\{x_{pq}\}$ is any solution of (3.3) and (3.4). Exploiting (3.4) for $(p,q)\in\{(1,0),(-1,0),(0,1),(0,-1)\}$ shows that $x_{11}=x_{1,-1}=x_{-1,1}=x_{-1,-1}$. Moreover, exploiting (3.3) for $p=q=0$ shows that $x_{0,-1}$ is uniquely determined by x_{00}, x_{10}, x_{01}, x_{11}, and $x_{-1,0}$. Observe that the matrices A_{pq} and B_{pq} are well defined even for $\theta_1=\theta_2=0$ whenever $p\neq q$ and $p\neq -q-1$. So, anything that has been said earlier regarding the recursions on the two-dimensional lattice remains intact for $\theta_1=\theta_2=0$ as long as we do not appeal to any formulae involving A_{pq} and B_{pq}, when $p=q$ or $p=-q-1$, or to any formulae involving A_{pq}^{-1} and B_{pq}^{-1}, when $p=-q$ or $p=q+1$. (Observe that for $\theta_1=\theta_2=0$, the matrices $A_{p,-p}$ and $A_{p+1,p}$ are singular.) This entails that any point in the sector $\{(p,q)\in\mathbb{Z}^2/p\geq |q|\}$ can be reached by a finite succession of recursions of the four types described above, departing at the positions $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$, where the first step involves the matrix A_{10}. Any point in the sector $\{(p,q)\in\mathbb{Z}^2/q\geq |p|\}$ can be reached by a finite succession of recursions departing at $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$, where the first step involves the matrix B_{10}. For the remaining two sectors $\{(p,q)\in\mathbb{Z}^2/p\leq -|q|\}$ and $\{(p,q)\in\mathbb{Z}^2/q\leq -|p|\}$, one can use recursions departing at $(0,0)$, $(-1,0)$, $(0,-1)$, and $(-1,-1)$, where the first step involves the matrices $A_{-1,-1}^{-1}$ and $B_{-1,-1}^{-1}$, respectively. We thus conclude that the combined system (3.3) and (3.4) has exactly five linearly independent solutions, which are determined at the positions $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$, and $(-1,0)$. Moreover, the values at any position (p,q) for $|p| > 1$ or $|q| > 1$ can be determined by iterative recursions.

Our next objective is to give a more detailed description of the solutions $\{x_{pq}\}$ of the combined system (3.3) and (3.4) for which $x_{00} = x_{11} = 0$ (according to [15, 16]). The following characterizations can be established with the aid of the recursions described above:

(1) if $x_{00} = x_{11} = x_{10} = x_{-1,0} = 0$, but $x_{01} \neq 0$, then $x_{pq} = 0$ for $|q| \le |p|$, $x_{p,-q} = -x_{pq}$, $x_{-p,q} = x_{pq}$, for $p,q \in \mathbb{Z}$, $x_{p,p+1} = (-\beta)^{-p}x_{01}$ for $p \ge 0$,

(2) if $x_{00} = x_{11} = x_{01} = 0$, but $x_{10} = -x_{-1,0} \neq 0$, then $x_{pq} = 0$ for $|q| \ge |p|$, $x_{-p,q} = -x_{p,q}$, $x_{p,-q} = x_{pq}$, for $p,q \in \mathbb{Z}$, $x_{p+1,p} = (-\beta)^{p}x_{10}$ for $p \ge 0$,

(3) if $x_{00} = x_{11} = 0$, but $x_{10} = x_{-1,0} \neq 0$, then $x_{pp} = 0$ for $p \in Z$, $x_{pq} = x_{-p,q} = x_{p,-q} = x_{-p,-q}$ for $p,q \in \mathbb{Z}$, $x_{p,p+1} = (-\beta)^{-p}x_{01}$, $x_{p+1,p} = (-\beta)^{p}x_{10}$, for $p \ge 0$.

A more specific characterization of the solutions described above can be given if we express them in terms of the parameter χ:

(4) for every $(p,q) \in \mathbb{Z}^2$, $|p| \neq |q|$, there exists a (unique) polynomial $\omega_{pq}(\chi)$ of degree $||p|-|q||-1$ such that if $\{x_{pq}\}$ is a solution of the combined system (3.3) and (3.4) satisfying $x_{00} = x_{11} = 0$, then

$$\begin{aligned} x_{pq} &= \omega_{pq}(\chi)x_{01} \quad \text{for } q > |p|, \\ x_{pq} &= \omega_{pq}(\chi)x_{0,-1} \quad \text{for } q < -|p|, \\ x_{pq} &= \omega_{pq}(\chi)x_{10} \quad \text{for } p > |q|, \\ x_{pq} &= \omega_{pq}(\chi)x_{-1,0} \quad \text{for } p < -|q|. \end{aligned} \tag{3.12}$$

In order to establish this last property, one can use two-dimensional recursions. For instance, to cover the case where $|p| > |q|$, one considers the recursion

$$\begin{aligned} \begin{pmatrix} x_{p,q+1} \\ x_{p-1,q} \end{pmatrix} &= \frac{1}{\beta \sin \pi\alpha(q-p)} \begin{pmatrix} -\sin 2\pi\alpha q & -\beta \sin \pi\alpha(p+q) \\ \beta \sin \pi\alpha(p+q) & \beta^2 \sin 2\pi\alpha p \end{pmatrix} \begin{pmatrix} x_{p+1,q} \\ x_{p,q-1} \end{pmatrix} \\ &\quad + \frac{\chi x_{pq}}{\sin \pi\alpha(q-p)} \begin{pmatrix} \beta^{-1} \sin \pi\alpha q \\ -\sin \pi\alpha p \end{pmatrix}, \end{aligned} \tag{3.13}$$

which is, of course, also redundant. The initial values in this case are

$$x_{pp} = 0, \qquad x_{p,p+1} = (-\beta)^{-p}x_{01}, \quad \text{for } p \ge 0. \tag{3.14}$$

Remark 3.1. Without entering the details, we would like to mention that there is an alternative approach to obtaining the polynomials $\omega_{pq}(\chi)$ by considering the Fourier expansions (i.e., the expansions in $e^{n\theta i}$) of the polynomials of the second kind for the difference equations

$$\xi_{n+1}+\xi_{n-1}+2\beta\cos(2\pi\alpha n+\theta)\xi_n=\chi\xi_n,$$
$$\xi_{n+1}+\xi_{n-1}+2\beta^{-1}\cos(2\pi\alpha n+\theta)\xi_n=\chi\xi_n, \tag{3.15}$$

for $n\ge 0$ as well as $n\le 0$.

At this point, we interrupt the discussion of the combined system (3.3) and (3.4) to give an application of what has already been established (see [14]).

Theorem 3.2. *The operator $H(\alpha,1,\theta)$ has no eigenvector in $\ell^2(\mathbb{Z})$ for any θ.*

Proof. Suppose that the opposite were true. So, there exist $\theta\in\mathbb{R}$, $\chi\in\operatorname{Sp}(\alpha,1)$, and $\xi\in\ell^2(\mathbb{Z})$, $\|\xi\|=1$, such that $H(\alpha,1,\theta)\xi=\chi\xi$. Then

$$\varphi(a)=\langle\pi_\theta(a)\xi,\xi\rangle,\quad a\in\mathcal{A}, \tag{3.16}$$

is an eigenstate of $h(\alpha,1)$ for χ. We have the following properties which are true because φ is a vector state:
(i) $\lim_{|p|\to\infty}\varphi(w_{p0})=0$,
(ii) $\lim_{|p|\to\infty}\max\{|\varphi(w_{pq})|/||p|-|q||=1\}=0$,
(iii) $\{\varphi(w_{0q})\}$ does not converge to zero as $q\to\infty$ or $q\to-\infty$.
Since $h(\alpha,1)$ is a fixed point of the automorphism ρ, the state $\psi=\varphi\circ\rho$ is also an eigenstate of $h(\alpha,1)$ for χ. Let $x_{pq}=\varphi(w_{pq})-\psi(w_{pq})$. Then $\{x_{pq}\}$ is a solution of the combined system (3.3) and (3.4). Since $\varphi(w_{11})=\varphi(w_{1,-1})$ and $\rho(w_{11})=w_{1,-1}$, we also have $x_{00}=x_{11}=0$. Furthermore, $\{x_{pq}\}$ is not the trivial solution for if it were, φ would be ρ-invariant, which is impossible in the light of (i) and (iii). We conclude that $\{x_{pq}\}$ must be a linear combination of the solutions described in (1), (2), or (3). This means, however, that x_{pq} takes a constant nonvanishing value for infinitely many (p,q) with $||p|-|q||=1$; thus, contradicting (ii). □

We resume our general discussion. In [15, pages 297–298], we have defined a three-dimensional recursion along the positive diagonal $p=q$ in order to establish the following properties. Notice that if φ is an eigenstate of $h(\alpha,\beta)$ for $\chi\in\operatorname{Sp}(\alpha,\beta)$, then the double sequence $\{\varphi(w_{pq})\}$ is uniformly bounded.

Scholium 3.3. Suppose that $|\beta|\ne 1$. Then there exist at most two linearly independent solutions of the combined system (3.3) and (3.4) which are uniformly bounded. If $\chi\in\operatorname{Sp}(\alpha,\beta)$, then there exists exactly one uniformly bounded solution $\{x_{pq}\}$ with the property $x_{pq}=x_{-p,-q}$ for all $p,q\in\mathbb{Z}$.

Also in [15], the following sufficient condition for the occurrence of two pure eigenstates was given.

Scholium 3.4. Let $\Omega(\alpha,\beta)=\{\chi\in\operatorname{Sp}(\alpha,\beta)\mid\chi$ is an eigenvalue of $H(\alpha,\beta,\theta)$ for some $\theta\in\pi\alpha\mathbb{Z}\cup\pi(\alpha+1)\mathbb{Z}\}$. If $\chi\in\operatorname{Sp}(\alpha,\beta)\backslash\Omega(\alpha,\beta)$ is an eigenvalue for $H(\alpha,\beta,\theta)$, then $h(\alpha,\beta)$ has two distinct pure eigenstates for χ.

For future references, we point out that the set $\Omega(\alpha,\beta)$ is at most countable. The set $\pi\alpha\mathbb{Z}\cup\pi(\alpha+1)\mathbb{Z}$ is trivially countable, and for every θ in this set, there can be no more than countably many eigenvalues of $H(\alpha,\beta,\theta)$ because $\ell^2(\mathbb{Z})$ is a separable Hilbert space. As in [15], we call the total number of pure eigenstates of $h(\alpha,\beta)$ for an element $\chi\in\operatorname{Sp}(\alpha,\beta)$ the multiplicity of χ.

Scholium 3.5. Suppose that $|\beta|\neq 1$. If $\chi\in\operatorname{Sp}(\alpha,\beta)$ has multiplicity two and φ and ψ are the pure eigenstates of $h(\alpha,\beta)$ for χ, then $\psi=\varphi\circ\sigma$ and $\{\varphi(w_{pq})-\psi(w_{pq})\}$ is a uniformly bounded (nontrivial) solution of the combined system (3.3) and (3.4) of type (1) or (2).

To see this, we observe that by Scholium 3.3, there is only one σ-invariant eigenstate for χ. Since $h(\alpha,\beta)$ is a fixed point of σ, $\varphi\circ\sigma$ is also an eigenstate for χ. If $\varphi\circ\sigma=\varphi$, then $\psi\circ\sigma=\psi$, otherwise there would be at least three pure eigenstates for χ. Therefore, $\varphi\circ\sigma=\psi$. Let $x_{pq}=\varphi(w_{pq})-\psi(w_{pq})$. Since $\{x_{pq}\}$ is a solution of (3.3) and (3.4), we have, on the one hand,

$$x_{11}=x_{-1,1}=x_{1,-1}=x_{-1,-1}. \tag{3.17}$$

On the other hand, we have

$$\begin{aligned} x_{11}&=\varphi(w_{11})-\psi(w_{11})=\varphi(w_{11})-\varphi(\sigma(w_{11}))\\ &=\varphi(w_{11})-\varphi(\sigma(w_{+1,+1}))=\varphi(\sigma(w_{-1,-1}))-\varphi(w_{-1,-1})\\ &=\psi(w_{-1,-1})-\varphi(w_{-1,-1})=-x_{-1,-1}. \end{aligned} \tag{3.18}$$

Whence $x_{11}=0$. The same manipulations yield $x_{10}=-x_{-1,0}$, $x_{01}=-x_{0,-1}$. Thus, $\{x_{pq}\}$ is either of type (1) or (2).

We now give another application. It was shown in [6] that the operator $H(\alpha,\beta,\theta)$ has no eigenvalues for $|\beta|<1$. The proof of this fact was based on Oseledec's theorem. Independently, by the methods developed so far, the following weaker statement was shown to be true in [16, Theorem 3.1].

Theorem 3.6. *If $|\beta|<1$ and $\chi\in\operatorname{Sp}(\alpha,\beta)\backslash\Omega(\alpha,\beta)$, then χ is not an eigenvalue of $H(\alpha,\beta,\theta)$.*

Proof. We proceed as in the proof of Theorem 3.2. Suppose that the claim were not true. Then there exists $\chi\in\operatorname{Sp}(\alpha,\beta)\backslash\Omega(\alpha,\beta)$ and $\xi\in\ell^2(\mathbb{Z})$, $\|\xi\|=1$, such that $H(\alpha,\beta,\theta)\xi=\chi\xi$. The vector state

$$\varphi(\alpha)=\langle\pi_\theta(a)\xi,\xi\rangle,\quad a\in\mathcal{A}, \tag{3.19}$$

is an eigenstate of $h(\alpha,\beta)$ for χ, and by Scholium 3.5, the double sequence $\{x_{pq}\}$, where $x_{pq}=\varphi(w_{pq})-\varphi(w_{-p,-q})$, solves (3.3) and (3.4), and it is a linear combination of solutions of types (1) and (2). Since $|\beta|<1$ and $\{x_{pq}\}$ is uniformly bounded, a solution of type (1) is not involved in the linear combination. Thus, $\{x_{pq}\}$ is of type (2). In particular, $x_{0q} = 0$ for all $q \in \mathbb{Z}$, contradicting the property (iii) in the proof of Theorem 3.2, which is valid for all vector states. □

It may seem that the exclusion of the exceptional set $\Omega(\alpha,\beta)$ from consideration in the last theorem is a deficiency that could be overcome by a more powerful argument. However, as the reasoning leading up to the proof of Assertion 1.3 will show, this is not likely to be the case. Putting it informally, the set $\Omega(\alpha,\beta)$ is the "blind spot" of the theory. In a sense, the very existence of such an exceptional set is necessary in order for this approach to work.

4. The resolvent of perturbed operators. Suppose that α and β are fixed. For $\gamma,\delta \in \mathbb{C}\setminus\{0\}$, let

$$h_{(\gamma,\delta)} = \gamma^{-1}u + \gamma u^* + \beta(\delta^{-1}v + \delta v^*). \tag{4.1}$$

Our next objective is to study the Fourier expansion of the resolvent of these operators (according to [17]). Recall from Proposition 2.1 that the Fourier series of $(h_{(\gamma,\delta)} - z)^{-1}$ decays exponentially as the lattice parameters p and q approach infinity, at any point in the resolvent set of $h_{(\gamma,\delta)}$. We will see that there are two types of series expansions for the resolvent of $h_{(\gamma,\delta)}$; namely, those which represent the resolvent on the unbounded component of the resolvent set (we will refer to those series as being of type I) and those which represent the resolvent on the bounded components of the resolvent set (we will refer to those series as being of type II).

We are going to recast the resolvent problem for the operators $h_{(\gamma,\delta)}$ slightly, so that it parallels the induction of eigenstates in Section 3. An element $a \in \mathcal{A}$ is an inverse of $h_{(\gamma,\delta)} - \chi$ if and only if the following two conditions hold:

$$h_{(\gamma,\delta)}a + ah_{(\gamma,\delta)} = 2\chi a + 2\mathbb{I}, \tag{4.2}$$

where $\mathbb{I}$ denotes the unit in $\mathcal{A}$;

$$h_{(\gamma,\delta)}a - ah_{(\gamma,\delta)} = 0. \tag{4.3}$$

Considering the Fourier series $\sum_{p,q\in\mathbb{Z}} x_{pq}w_{pq}$ of a, condition (4.2) is equivalent with

$$\begin{aligned}\cos(\pi\alpha q)(\gamma^{-1}x_{p-1,q} + \gamma x_{p+1,q}) + \beta\cos(\pi\alpha p)(\delta^{-1}x_{p,q-1} + \delta x_{p,q+1})\\ = \chi x_{pq} + \varepsilon_{pq} \quad \text{for } p,q \in \mathbb{Z},\end{aligned} \tag{4.4}$$

where

$$\varepsilon_{pq} = \begin{cases} 1, & \text{if } p = q = 0,\\ 0, & \text{elsewhere.}\end{cases} \tag{4.5}$$

Condition (4.3) is equivalent with

$$\sin(\pi\alpha q)\left(\gamma^{-1}x_{p-1,q}-\gamma x_{p+1,q}\right)-\beta\sin(\pi\alpha p)\left(\delta^{-1}x_{p,q-1}-\delta x_{p,q+1}\right)=0. \tag{4.6}$$

A double sequence $\{x_{pq}\}$ solves the combined system (4.4) and (4.6) for parameters γ_0 and δ_0 if and only if $\{\gamma_0^p\delta_0^q x_{pq}\}$ is a solution of (4.4) and (4.6) for $\gamma=\delta=1$. So, any two systems of type (4.4) and (4.6) for distinct pairs of parameters γ and δ are equivalent. The following system covers the eigenstate problem as well as the resolvent problem.

Scholium 4.1. The combined system (4.4) and (4.6) for $\gamma=\delta=1$, but not the equation in (4.4) for $p=q=0$.

Notice that (4.6) is trivial for $p=q=0$. The system of Scholium 4.1 has exactly six linearly independent solutions. Every solution is uniquely determined by its values at the positions $(0,0)$, $(1,1)$, $(1,0)$, $(-1,0)$, $(0,1)$, and $(0,-1)$, and it can be generated by the recursions discussed in Section 3. We also record the following elementary property.

Scholium 4.2. If $\{x_{pq}\}$ is a solution of Scholium 4.1 and $y_{pq}=x_{|p|,|q|}$, then $\{y_{pq}\}$ is also a solution of Scholium 4.1.

We now describe four solutions of (4.4) and (4.6) for $\gamma=\delta=1$ which are related to (1), (2), and (3):

$$\begin{aligned} &d^{(+)}_{pq}=0 \quad \text{for } q\le|p|, \qquad d^{(+)}_{-p,q}=d^{(+)}_{pq} \quad \text{for } p,q\in\mathbb{Z},\\ &d^{(+)}_{p,p+1}=(-1)^p\beta^{-p-1} \quad \text{for } p\ge 0,\end{aligned} \tag{4.7}$$

$$\begin{aligned} &d^{(-)}_{pq}=0 \quad \text{for } q\ge-|p|, \qquad d^{(-)}_{-p,q}=d^{(-)}_{pq} \quad \text{for } p,q\in\mathbb{Z},\\ &d^{(-)}_{p,-p-1}=(-1)^p\beta^{-p-1} \quad \text{for } p\ge 0,\\ &e^{(+)}_{pq}=0 \quad \text{for } p\le|q|, \qquad e^{(+)}_{p,-q}=e^{(+)}_{pq} \quad \text{for } p,q\in\mathbb{Z},\\ &e^{(+)}_{p+1,p}=(-\beta)^p \quad \text{for } p\ge 0,\end{aligned} \tag{4.8}$$

$$\begin{aligned} &e^{(-)}_{pq}=0 \quad \text{for } p\ge-|q|, \qquad e^{(-)}_{p,-q}=e^{(-)}_{pq} \quad \text{for } p,q\in\mathbb{Z},\\ &e^{(-)}_{-p-1,p}=(-\beta)^p \quad \text{for } p\ge 0.\end{aligned} \tag{4.9}$$

The connection between these solutions and those in Section 3 is as follows:

$$\begin{aligned} &\{d^{(+)}_{pq}-d^{(-)}_{pq}\} \text{ is of type (1)},\\ &\{e^{(+)}_{pq}-e^{(-)}_{pq}\} \text{ is of type (2)},\\ &\{d^{(+)}_{pq}+d^{(-)}_{pq}-e^{(+)}_{pq}-e^{(-)}_{pq}\} \text{ is of type (3)}.\end{aligned} \tag{4.10}$$

The following test which indicates the presence of solutions of Scholium 4.1 of type (4.7) through (4.9) can be derived with the aid of the recursions discussed in Section 3.

Scholium 4.3. If $\{x_{pq}\}$ is a solution of Scholium 4.1 with the property that there exist $p,q\in\mathbb{Z}$ such that $x_{pq}=x_{p+1,q}=x_{p,q+1}=x_{p+1,q+1}=0$, then $\{x_{pq}\}$ is a linear combination of the solutions (4.7) through (4.9).

We denote by $R(\gamma,\delta)$ the resolvent set of $h_{(\gamma,\delta)}$. For some $\chi\in R(\gamma,\delta)$, consider the Fourier expansion

$$\left(h_{(\gamma,\delta)}-\chi\right)^{-1}=\sum_{p,q\in\mathbb{Z}}x_{pq}w_{pq}. \tag{4.11}$$

Let $y_{pq}=\gamma^p\delta^q x_{pq}$. We say that χ is of type I if $\chi\notin \mathrm{Sp}(\alpha,\beta)$ and

$$\left(h(\alpha,\beta)-\chi\right)^{-1}=\sum_{p,q\in\mathbb{Z}}y_{pq}w_{pq}. \tag{4.12}$$

We say that χ is of type II if $\{y_{pq}\}$ equals $\{d_{pq}^{(+)}\}$, $\{d_{pq}^{(-)}\}$, $\{e_{pq}^{(+)}\}$, or $\{e_{pq}^{(-)}\}$.

Scholium 4.4. If $\{y_{pq}\}$ is a linear combination of $\{d_{pq}^{(+)}\}$, $\{d_{pq}^{(-)}\}$, $\{e_{pq}^{(+)}\}$, and $\{e_{pq}^{(-)}\}$, then χ is of type II.

To see this, suppose that the claim were not true. By assumption, in each of the four sectors of the two-dimensional lattice $\mathbb{Z}^2$, which are separated by the lines $p=q$ and $p=-q$, $\{y_{pq}\}$ is a scalar multiple of exactly one of the four double sequences in (4.7) through (4.9). It follows that in any of those four sectors S, where $\{x_{pq}\}$ does not vanish identically, we can define a solution $\{s_{pq}\}$ of (4.4) and (4.6) by carrying out the following two steps. First, let $\tilde{s}_{pq}=x_{pq}$ in S and $\tilde{s}_{pq}=0$ elsewhere. Then scale $\{\tilde{s}_{pq}\}$ with a suitable number c to obtain $\{s_{pq}\}$, that is, $s_{pq}=c\tilde{s}_{pq}$. Since $\{x_{pq}\}$ decays exponentially as $|p|,|q|\to\infty$, the same is true for $\{s_{pq}\}$. So, if χ were not of type II, then we could construct such exponentially decaying solutions of (3.3) and (3.4) for at least two distinct sectors. This would yield at least two distinct inverses of $h_{(\gamma,\delta)}-\chi$ in the C^*-algebra $\mathcal{A}$, thus contradicting the uniqueness of such an inverse.

With a little more effort, one can show the following refined statement. If χ is of type II and $(h_{(\gamma,\delta)}-\chi)^{-1}=\sum_{p,q\in\mathbb{Z}}x_{pq}w_{pq}$, then

$$\begin{aligned}
\{x_{pq}\}&=\{d_{pq}^{(+)}\} &&\text{only if } |\delta^{-1}|,|\beta^{-1}\gamma\delta^{-1}|,|\beta^{-1}\gamma^{-1}\delta^{-1}|<1,\\
\{x_{pq}\}&=\{d_{pq}^{(-)}\} &&\text{only if } |\delta|,|\beta^{-1}\gamma\delta|,|\beta^{-1}\gamma^{-1}\delta|<1,\\
\{x_{pq}\}&=\{e_{pq}^{(+)}\} &&\text{only if } |\gamma^{-1}|,|\beta\gamma^{-1}\delta|,|\beta\gamma^{-1}\delta^{-1}|<1,\\
\{x_{pq}\}&=\{e_{pq}^{(-)}\} &&\text{only if } |\gamma|,|\beta\gamma\delta|,|\beta\gamma\delta^{-1}|<1.
\end{aligned} \tag{4.13}$$

Since for no values of β, γ, and δ any two distinct conditions among those four stated in (4.13) are valid, it follows that for any operator $h_{(\gamma,\delta)}$ which has points of type II in its resolvent set, the resolvent at any two of those points always has the same form.

Scholium 4.5. If $y_{pq}=y_{|p|,|q|}$ for $p,q\in\mathbb{Z}$, then χ is of type I.

Suppose first that $|\gamma|,|\delta| \le 1$. Since by Proposition 2.1 the double sequence $\{x_{pq}\}$ decays exponentially as $|p| \to \infty$ and $|q| \to \infty$, $\{y_{pq}\}_{p,q\ge 0}$ decays exponentially as $p \to \infty$ and $q \to \infty$. Since $y_{pq} = y_{|p|,|q|}$, this entails that $\{y_{pq}\}$ decays exponentially as $|p| \to \infty$ and $|q| \to \infty$. Moreover, $\{y_{pq}\}$ solves the combined system (4.4) and (4.6) for $\gamma = \delta = 1$. In conclusion, $\sum_{p,q\in\mathbb{Z}} y_{pq} w_{pq}$ is the inverse of $h(\alpha,\beta)-\chi$. Whence, χ is of type I. A similar reasoning applies to the cases where $|\gamma| \ge 1$, $|\delta| \le 1$; $|\gamma| \le 1$, $|\delta| \ge 1$; $|\gamma| \ge 1$, $|\delta| \ge 1$.

Scholium 4.6. Every $\chi \in R(\gamma,\delta)$ is either of type I or type II.

Again, we assume first that $|\gamma|,|\delta| \le 1$. Let $z_{pq} = y_{|p|,|q|}$ for $p,q \in \mathbb{Z}$. Suppose first that $z_{pq} = 0$ for all $p,q \in \mathbb{Z}$. Since $\{y_{pq}\}$ is a solution of Scholium 4.1, it follows from Scholium 4.3 that $\{y_{pq}\}$ is a linear combination of $\{d^{(+)}_{pq}\}$, $\{d^{(-)}_{pq}\}$, $\{e^{(+)}_{pq}\}$, and $\{e^{(-)}_{pq}\}$. By Scholium 4.4, this entails that χ must be of type II. Now, suppose that $\{z_{pq}\}$ does not vanish identically. Then, it follows from Scholium 4.2 that $\{z_{pq}\}$ is a nontrivial solution of Scholium 4.1. Moreover, since $|\gamma|,|\delta| \le 1$, $\{y_{pq}\}_{p,q\ge 0}$ decays exponentially as $p,q \to \infty$. Therefore, $\{z_{pq}\}$ decays exponentially as $|p|,|q| \to \infty$. Since all we know is that $\{z_{pq}\}$ solves Scholium 4.1, $\{z_{pq}\}$ may or may not solve (3.3) for $p = q = 0$. If it does, then the absolutely convergent Fourier series $\sum_{p,q\in\mathbb{Z}} z_{pq} w_{pq}$ defines an element a in the C^*-algebra $\mathcal{A}$ with the property $(h(\alpha,\beta)-\chi)a = a(h(\alpha,\beta)-\chi) = 0$. In particular, if φ is any state on $\mathcal{A}$ and we define a functional φ_a by $\varphi_a(x) = \varphi(a^*xa)$, $x \in \mathcal{A}$, then $\varphi_a = c\psi$ for some eigenstate ψ of $h(\alpha,\beta)$ for χ and some constant $c \ge 0$. This gives rise to an infinite-dimensional space of uniformly bounded solutions of (3.3) and (3.4), which clearly contradicts Scholium 3.3. So, $\{z_{pq}\}$ does not solve (3.3) for $p = q = 0$. Thus, we can scale $\{z_{pq}\}$ by a suitable constant c such that $\{cz_{pq}\}$ solves (4.4) and (4.6) for $\gamma = \delta = 1$. It follows that $\chi \notin \mathrm{Sp}(\alpha,\beta)$ and the absolutely convergent Fourier series $\sum_{p,q\in\mathbb{Z}} cz_{pq} w_{pq}$ is the inverse of $h(\alpha,\beta)-\chi$. Since $|\gamma|,|\delta| \le 1$, we have for all $p,q \in \mathbb{Z}$,

$$\left|\gamma^{-p}\delta^{-q}z_{pq}\right| \le \left|\gamma^{-|p|}\delta^{-|q|}z_{|p|,|q|}\right| = \left|\gamma^{-|p|}\delta^{-|q|}y_{|p|,|q|}\right| = \left|x_{|p|,|q|}\right|. \tag{4.14}$$

It follows that $\{cz_{pq}\gamma^{-p}\delta^{-q}\}$ decays exponentially as $|p|,|q| \to \infty$, and hence the limit of the absolutely convergent Fourier series $\sum_{p,q\in\mathbb{Z}} cz_{pq}\gamma^{-p}\delta^{-q}w_{pq}$ is an inverse of $h(\gamma,\delta)-\chi$. The uniqueness of the inverse entails that

$$cz_{pq}\gamma^{-p}\delta^{-q} = x_{pq} \quad \forall p,q \in \mathbb{Z}. \tag{4.15}$$

We conclude that χ is of type I. The cases where $|\gamma| \ge 1$, $|\delta| \le 1$ or $|\gamma| \le 1$, $|\delta| \ge 1$ or $|\gamma| \ge 1$, $|\delta| \ge 1$ are treated in a similar fashion.

Scholium 4.7. All points in the same component of $R(\gamma,\delta)$ are of the same type.

Suppose that Ω is a component of $R(\gamma,\delta)$. Let Ω_{I} be the set of those points in Ω which are of type I, and let Ω_{II} be the set of those points in Ω which are of type II. By Scholium 4.6, we have $\Omega = \Omega_{\mathrm{I}} \cup \Omega_{\mathrm{II}}$. In order to prove that either $\Omega_{\mathrm{I}} = \phi$ or $\Omega_{\mathrm{II}} = \phi$, it suffices to show that both sets are relatively closed. Suppose that $\chi_1, \chi_2, \ldots$ is a sequence in Ω_{I} converging to $\chi \in \Omega$. Then

$$\lim_{n\to\infty} \tau\left(\left(h_{(\gamma,\delta)} - \chi_n\right)^{-1} w_{-p,-q}\right) = \tau\left(\left(h_{(\gamma,\delta)} - \chi\right)^{-1} w_{-p,-q}\right), \tag{4.16}$$

that is, the Fourier coefficient of $(h_{(\gamma,\delta)} - \chi_n)^{-1}$ at the position (p,q) converges to the Fourier coefficient of $(h_{(\gamma,\delta)} - \chi)^{-1}$ at the position (p,q). Since χ_n is of type I, we have

$$\tau\left(\left(h_{(\gamma,\delta)} - \chi_n\right)^{-1} w_{-p,-q}\right)\gamma^p \delta^q = \tau\left(\left(h_{(\gamma,\delta)} - \chi_n\right)^{-1} w_{-|p|,-|q|}\right)\gamma^{|p|}\delta^{|q|}, \tag{4.17}$$

whence,

$$\tau\left(\left(h_{(\gamma,\delta)} - \chi\right)^{-1} w_{-p,-q}\right)\gamma^p \delta^q = \tau\left(\left(h_{(\gamma,\delta)} - \chi\right)^{-1} w_{-|p|,-|q|}\right)\gamma^{|p|}\delta^{|q|}, \tag{4.18}$$

for all $p,q \in \mathbb{Z}$. It now follows from Scholium 4.5 that χ is in Ω_{I}. Next, suppose that $\chi_1, \chi_2, \ldots \in \Omega_{\mathrm{II}}$ converge to $\chi \in \Omega$. Then at the positions in all but one of the four sectors separated by the lines $p = q$ and $p = -q$, the Fourier coefficients of $(h_{(\gamma,\delta)} - \chi_n)^{-1}$ vanish. Since this property is preserved under limits, it follows from Scholia 4.3 and 4.4 that χ is of type II.

A component containing points of type I only will be called of type I, too. Otherwise, it will be called of type II.

Scholium 4.8. The unbounded component of $R(\gamma,\delta)$ is of type I.

The Fourier coefficients of $(h_{(\gamma,\delta)} - \chi)^{-1}$ approach zero as $|\chi| \to \infty$. However, on components of type II, the Fourier coefficients of $(h_{(\gamma,\delta)} - \chi)^{-1}$ are polynomials (see (4.10) and (4)), and thus they do not approach zero as $|\chi| \to \infty$ unless they vanish identically.

Scholium 4.9. Any $\chi \in R(\gamma,\delta) \cap \mathrm{Sp}(\alpha,\beta)$ is of type II.

If χ is in $R(\gamma,\delta) \cap \mathrm{Sp}(\alpha,\beta)$ and

$$\left(h_{(\gamma,\delta)} - \chi\right)^{-1} = \sum_{p,q\in\mathbb{Z}} x_{pq} w_{pq}, \tag{4.19}$$

then $\{x_{pq}\gamma^p\delta^q\}$ cannot be the Fourier coefficients of an inverse of $h(\alpha,\beta) - \chi$. Hence, χ must be of type II.

Scholium 4.10. Given γ and a compact subset $K \subset \mathbb{C}$, there exists a δ_0 such that for all δ with $|\delta| \geq |\delta_0|$ or $|\delta| \leq |\delta_0^{-1}|$, K is contained in a component of $R(\gamma,\delta)$ of type II.

A similar statement holds where the roles of γ and δ are interchanged. Since

$$\lim_{|\delta|\to\infty} \delta^{-1} h_{(\gamma,\delta)} = \beta v^* \tag{4.20}$$

for sufficiently large $|\delta|$, the spectrum of $\delta^{-1} h_{(\gamma,\delta)}$ is close to the spectrum of βv^*. So, for large $|\delta|$, the set $K \cup \mathrm{Sp}(\alpha,\beta)$ is contained in a single component of $R(\gamma,\delta)$. The claim now follows from Scholium 4.9.

5. Proof of Assertions 1.2 and 1.3

Lemma 5.1. *If (L) holds, then the integrated density of states μ of $H(\alpha,\beta,\theta)$ is nothing but the equilibrium distribution of $\mathrm{Sp}(\alpha,\beta)$ (see Appendix A.1). Moreover,*

$$\int \log|z-s|\,d\mu(s) = \log|\beta| \quad \textit{iff } z \in \mathrm{Sp}(\alpha,\beta). \tag{5.1}$$

In particular, $\mathrm{Sp}(\alpha,\beta)$ is a regular compactum (see Appendix A.6).

Proof. Let $\gamma(\beta,z)$ be the (averaged) Lyapunov index at z. Then the Thouless formula says that (see [5] and [2, Section VI.4.3])

$$\gamma(\beta,z) = \int \log|z-s|\,d\mu(s). \tag{5.2}$$

Moreover, (see [2, Section V.4.6])

$$\gamma(\beta,z) \geq \log|\beta|. \tag{5.3}$$

By virtue of [6] (see also [2, Section V.5.4(2)]), condition (L) implies that $\gamma(\beta^{-1}, \chi/\beta) = 0$ for all $\chi \in \mathrm{Sp}(\alpha,\beta)$. Since $\gamma(\beta,\chi) = \gamma(\beta^{-1},\chi/\beta) + \log|\beta|$, we conclude that $\gamma(\beta,\chi) = \log|\beta|$ for all $\chi \in \mathrm{Sp}(\alpha,\beta)$. Since

$$\iint \log|s-t|\,d\mu(s)\,d\mu(t) \geq \log|\beta| > -\infty, \tag{5.4}$$

the set $\mathrm{Sp}(\alpha,\beta)$ has positive capacity (see Appendix A.1). Furthermore, the logarithmic potential

$$z \longmapsto \int \log|z-s|\,d\mu(s) \tag{5.5}$$

satisfies the four conditions listed in Appendix A.5. Thus, the claim follows from Appendices A.1 and A.5. □

For $z \in \mathbb{C}\setminus \mathrm{Sp}(\alpha,\beta)$, we let

$$\big(h(\alpha,\beta)-z\big)^{-1} = \sum_{p,q\in\mathbb{Z}} c_{pq}(z) w_{pq} \tag{5.6}$$

be the Fourier expansion of the resolvent of $h(\alpha,\beta)$ at the point z. The functions c_{pq} are holomorphic in $\mathbb{C}\setminus\mathrm{Sp}(\alpha,\beta)$ and, by Proposition 2.1, they decay exponentially uniformly on compact subsets of $\mathbb{C}\setminus\mathrm{Sp}(\alpha,\beta)$ as $|p|,|q|\to\infty$. For $z\in\mathbb{C}\setminus\mathrm{Sp}(\alpha,\beta)$ and $q\in\mathbb{Z}$, we define

$$\rho_q(z)=\log\left(\sum_{p\in\mathbb{Z}}|c_{pq}(z)|\right). \tag{5.7}$$

The function ρ_q is subharmonic in $\mathbb{C}\setminus\mathrm{Sp}(\alpha,\beta)$. We now define

$$\rho_{\mathrm{I}}(z)=\overline{\lim_{|q|\to\infty}}\frac{1}{|q|}\rho_q(z),\quad z\in\mathbb{C}\setminus\mathrm{Sp}(\alpha,\beta). \tag{5.8}$$

We are going to define a second function which involves the double sequence defined in (4.7). So, for each $\chi\in\mathbb{C}$, we consider the double sequence $\{d_{pq}^{(+)}(\chi)\}$ with the properties stated in (4.7), which solves the combined system (4.4) and (4.6). By (4.10) and (4), $d_{pq}^{(+)}(\chi)$ is a polynomial in χ of degree $||p|-|q||-1$ for $|p|\neq|q|$. For any $z\in\mathbb{C}$, we set

$$\rho_{\mathrm{II}}(z)=\overline{\lim_{q\to\infty}}\frac{1}{q}\log\left(\sum_{-q\le p\le q}|d_{pq}^{(+)}(z)|\right). \tag{5.9}$$

Similar functions can be defined of course involving the double sequences $\{d_{pq}^{(-)}\}$, $\{e_{pq}^{(+)}\}$, and $\{e_{pq}^{(-)}\}$.

Lemma 5.2. *The functions ρ_I and ρ_{II} defined above have the following properties:*

(i) *ρ_I is subharmonic in $\mathbb{C}\setminus\mathrm{Sp}(\alpha,\beta)$ and ρ_{II} is subharmonic in $\mathbb{C}$;*

(ii) *$\rho_I(z)<0$ and $\rho_I(z)+\rho_{II}(z)\ge 0$ for $z\in\mathbb{C}\setminus\mathrm{Sp}(\alpha,\beta)$; $\rho_{II}(z)\ge 0$ in $\mathbb{C}$;*

(iii) *$\rho_I(z)=-\log|z/\beta|+o(1)$ and $\rho_{II}(z)=\log|z/\beta|+o(1)$.*

Proof. (i) An application of Fatou's lemma shows that ρ_{I} is submean, that is, for any $z\in\mathbb{C}\setminus\mathrm{Sp}(\alpha,\beta)$, we have

$$\rho_{\mathrm{I}}(z)\le\frac{1}{2\pi}\int_0^{2\pi}\rho_{\mathrm{I}}(z+re^{it})\,dt, \tag{5.10}$$

whenever r is sufficiently small. Since $\{c_{pq}(z)\}$ decays exponentially as $|p|$, $|q|\to\infty$, we have $\rho(z)<0$ in $\mathbb{C}\setminus\mathrm{Sp}(\alpha,\beta)$. The set $\{z\in\mathbb{C}\setminus\mathrm{Sp}(\alpha,\beta)/\rho_{\mathrm{I}}(z)<-\log\delta\}$ consists of all points which are of type I for $h_{(1,\delta)}$, for $\delta\ge 1$. By Scholia 4.6 and 4.7, this set is open. In conclusion, ρ_{I} is subharmonic.

The function ρ_{II} is submean for the same reason ρ_{I} is. Also, since all points in $\mathbb{C}\setminus\mathrm{Sp}(\alpha,\beta)$ are clearly of type I for $h_{(1,1)}$, we have $\rho_{\mathrm{II}}(z)\ge 0$ in $\mathbb{C}$. It follows from Scholium 4.10 that $\rho_{\mathrm{II}}(z)<\infty$ in $\mathbb{C}$. The set $\{z\in\mathbb{C}/\rho_{\mathrm{II}}(z)<\log\delta\}$ consists of all points which are of type II for $h_{(1,\delta)}$, for $\delta\ge 1$. Again by Scholia 4.6 and 4.7, this set is open. We conclude that ρ_{II} is subharmonic.

(ii) For $z \in \mathbb{C} \setminus \mathrm{Sp}(\alpha,\beta)$, we have

$$\rho_{\mathrm{I}}(z) = \inf\left\{-\frac{\log\delta}{\delta} \geq 1,\ z \text{ is of type I for } h_{(1,\delta)}\right\}, \tag{5.11}$$

and for $z \in \mathbb{C}$,

$$\rho_{\mathrm{II}}(z) = \inf\left\{\frac{\log\delta}{\delta} \geq 1,\ z \text{ is of type II for } h_{(1,\delta)}\right\}. \tag{5.12}$$

Whence, $-\rho_{\mathrm{I}}(z) \leq \rho_{\mathrm{II}}(z)$ for $z \in \mathbb{C} \setminus \mathrm{Sp}(\alpha,\beta)$. The remaining statements in (ii) were already shown in the proof of (i).

(iii) If $|z| > |\beta\delta| + \|u+u^* + \beta\delta^{-1}v^*\|$, then $|z|^{-1}\|h_{(1,\delta)}\| < 1$, and thus $h_{(1,\delta)} - z$ is invertible. This entails the first part of (iii). If

$$|z| < |\beta\delta| - \|u+u^*+\beta\delta^{-1}v^*\|, \tag{5.13}$$

then $\|(\beta\delta)^{-1}(h_{(1,\delta)} - z) - v^*\| < 1$, and thus $h_{(1,\delta)} - z$ is invertible. This entails the second part of (iii). □

Proof of Assertion 1.2. Suppose that (L) holds. By Theorems 3.2 and 3.6, we know that $|\beta| > 1$. Let $\chi \in \mathrm{Sp}(\alpha,\beta) \setminus \Omega(\alpha,\beta)$. Then χ has multiplicity two. So, there exist two distinct pure eigenstates φ and ψ of $h(\alpha,\beta)$ for χ. By Scholium 3.5, the double sequence $\{x_{pq}\} = \{\varphi(w_{pq}) - \psi(w_{pq})\}$ is a nontrivial uniformly bounded solution of the combined system (3.3) and (3.4), which is of type (1) or (2). Since $|\beta| > 1$, $\{x_{pq}\}$ cannot be of type (2) (otherwise it would not be uniformly bounded). Thus, (4.10) shows that $\{d_{pq}^{(+)}(\chi)\}$ is uniformly bounded. If $|d_{pq}^{(+)}(\chi)| \leq c$ for all $p,q \in \mathbb{Z}$ and some constant $c > 0$, then

$$\rho_{\mathrm{II}}(z) \leq \overline{\lim_{q\to\infty}} \frac{1}{q} \log\left[(2q+1)c\right] = 0. \tag{5.14}$$

Whence, $\rho_{\mathrm{II}}(\chi) = 0$. It now follows from Lemma 5.2(ii) that

$$\lim_{z\to\chi}\left(\rho_{\mathrm{I}}(z) + \rho_{\mathrm{II}}(z)\right) = 0 \quad \text{for } \chi \in \mathrm{Sp}(\alpha,\beta) \setminus \Omega(\alpha,\beta) \tag{5.15}$$

and from (iii) that

$$\lim_{|z|\to\infty}\left(\rho_{\mathrm{I}}(z) + \rho_{\mathrm{II}}(z)\right) = 0. \tag{5.16}$$

By Appendix A.2, the set $\Omega(\alpha,\beta)$, which is at most countable, is polar. Hence, Appendix A.4 yields

$$\rho_{\mathrm{I}}(z) + \rho_{\mathrm{II}}(z) = 0 \quad \text{in } \mathbb{C} \setminus \mathrm{Sp}(\alpha,\beta); \tag{5.17}$$

in particular, ρ_{II} is harmonic in $\mathbb{C} \setminus \mathrm{Sp}(\alpha,\beta)$. It now follows from Lemma 5.2(iii) and Appendix A.5 that $\rho_{\mathrm{II}} + \log|\beta|$ is the conductor potential of $\mathrm{Sp}(\alpha,\beta)$.

Whence, by Lemma 5.1,

$$\rho_{\mathrm{II}}(z)+\log|\beta| = \int \log|z-s|\,d\mu(s) \quad \text{in } \mathbb{C}. \tag{5.18}$$

In particular, ρ_{II} vanishes everywhere on $\mathrm{Sp}(\alpha,\beta)$. By the definition of the functions ρ_{I} and ρ_{II}, it is clear that z is in the spectrum of $h_{(1,\delta)}$ if and only if $\rho_{\mathrm{II}}(z) = \log|\delta|$, which settles the proof. □

Before we move on to the proof of Assertion 1.3, we would like to point out some consequence of Assertion 1.2 for the spectra of the operators $h_{(1,\delta)}$. If (L) holds, then [24, Section 4.1, Theorem 1] says that the spectrum of $h_{(1,\delta)}$ for $|\delta| \neq 1$ either consists of a finite number of mutually exterior analytic Jordan curves or consists of a finite number of Jordan curves composed of a finite number of analytic Jordan arcs, which are mutually exterior except that each of a finite number of points may belong to several Jordan curves.

Notation 5.3. For $\chi \in \mathrm{Sp}(\alpha,\beta)$, we define

$$\Delta_\chi(z) = \left| \frac{\log|\beta| - \int \log|z-s|\,d\mu(s)}{\chi - z} \right|, \quad z \neq \chi,$$
$$m_\chi = \sup\{\Delta_\chi(z) : z \neq \chi\}. \tag{5.19}$$

Lemma 5.4. *Suppose that (L) holds. Then for every $\chi \in \mathrm{Sp}(\alpha,\beta)\backslash\Omega(\alpha,\beta)$, $m_\chi < \infty$.*

Proof. Let $\chi \in \mathrm{Sp}(\alpha,\beta)\backslash\Omega(\alpha,\beta)$. Then there exist

$$\theta \in [0,2\pi)\backslash(\pi\alpha\mathbb{Z} \cup \pi(\alpha+1)\mathbb{Z}) \tag{5.20}$$

and a normalized eigenvector $\xi = \{\xi_n\}$ of $H(\alpha,\beta,\theta)$ for χ which decays exponentially as $|n| \to \infty$. So, there exist constants $r \in (0,1)$ and $b > 0$ such that

$$|\xi_n| \le b r^{|n|}, \quad n \in \mathbb{Z}. \tag{5.21}$$

Let

$$\varphi(a) = \langle \pi_\theta(a)\xi,\xi\rangle, \quad a \in \mathcal{A}. \tag{5.22}$$

Then φ is a pure eigenstate of $h(\alpha,\beta)$ for χ and the same is true for $\psi = \varphi\circ\sigma$. Specifically, we have $\varphi(w_{pq}) = e^{q\theta i}\langle w_{pq}\xi,\xi\rangle$, $\psi(w_{pq}) = e^{-q\theta i}\langle w_{-p,-q}\xi,\xi\rangle$, $q \in \mathbb{Z}$. As in the proof of Assertion 1.2, we see that the double sequence $\{\varphi(w_{pq}) - \psi(w_{pq})\}$ is uniformly bounded of type (1). Let $c = \beta(\varphi(w_{01}) - \psi(w_{01}))$. Then $c \neq 0$ and by (4.10),

$$c d_{pq}^{(+)}(\chi) = e^{q\theta i}\langle w_{pq}\xi,\xi\rangle - e^{-q\theta i}\langle w_{-p,-q}\xi,\xi\rangle \quad \text{for } q \ge 0. \tag{5.23}$$

We have

$$\begin{aligned}|\langle w_{pq}\xi,\xi\rangle| &\le \sum_{n\in\mathbb{Z}}|\xi_{n+p}\xi_n| \le b^2\sum_{n\in\mathbb{Z}} r^{|n+p|}r^{|n|} \\ &\le b^2\sum_{n\in\mathbb{Z}} r^{\|p|-|n\|+|n|} \le b(2|p|+1)r^{|p|}+b^2\sum_{|n|>|p|} r^{|n|} \\ &= b^2\Big(2|p|+1+\frac{2r}{1-r}r^{|p|}\Big).\end{aligned} \tag{5.24}$$

Let

$$a = 4c^{-1}b^2\sum_{p=0}^{\infty}\Big(2p+1+\frac{2r}{1-r}\Big)r^p. \tag{5.25}$$

Then we have for every $\delta\in(0,1)$,

$$\begin{aligned}\sum_{p,q\in\mathbb{Z}}\delta^q|d_{pq}^{(+)}(\chi)| &= \sum_{q\in\mathbb{Z},q\ge 0}\delta^q|d_{pq}^{(+)}(\chi)| \\ &\le \sum_{q=1}^{\infty}\delta^q\sum_{k=0}^{\infty}\big(|d_{k,q+k}^{(+)}(\chi)|+|d_{-k,q+k}^{(+)}(\chi)|\big) \\ &= \sum_{q=1}^{\infty}\delta^q\sum_{k=0}^{\infty}2|d_{k,q+k}^{(+)}(\chi)| \\ &\le \sum_{q=1}^{\infty}\delta^q\sum_{k=0}^{\infty}2c^{-1}\big(|\langle w_{k,q+k}\xi,\xi\rangle|+|\langle w_{-k,-q-k}\xi,\xi\rangle|\big) \\ &\le \sum_{q=1}^{\infty}\delta^q a = \frac{a\delta}{1-\delta}.\end{aligned} \tag{5.26}$$

This shows that the series $\sum_{p,q\in\mathbb{Z}}\delta^q d_{pq}^{(+)}(\chi)w_{pq}$ converges absolutely and since $\{\delta^q d_{pq}^{(+)}(\chi)\}$ solves (4.4) and (4.6) for $\gamma=1$, its limit is an inverse of $h_{(1,\delta)}-\chi$. Moreover,

$$\|(h_{(1,\delta)}-\chi)^{-1}\| \le \sum_{p,q\in\mathbb{Z}}\delta^q|d_{pq}^{(+)}(\chi)|\,\|w_{pq}\| \le \frac{a\delta}{1-\delta}. \tag{5.27}$$

Whence,

$$\sup_{\delta\in(0,1)}(1-\delta)\rho\big((h_{(1,\delta)}-\chi)^{-1}\big)<\infty, \tag{5.28}$$

where $\rho(y)$ denotes the spectral radius of y. By virtue of Assertion 1.2, this implies that

$$\sup_{z\neq\chi}(1-\beta^{-1})\exp\Big(\int\log|z-s|\,d\mu(s)\Big)\cdot|z-\chi|^{-1}<\infty, \tag{5.29}$$

which in turn yields $m_\chi<\infty$. □

Proof of Assertion 1.3. Suppose that $K \subset \mathrm{Sp}(\alpha,\beta)$ is an open and closed subset which is a Cantor set. For every $t \in \mathbb{R}\backslash K$ and $n \in \mathbb{N}$, we define

$$M(t,n) = \{\chi \in K/\Delta_\chi(t) > n\}, \tag{5.30}$$

which is an open subset of K. Whence,

$$M_n = \bigcup_{t\in\mathbb{R}\backslash K} M(t,n) \tag{5.31}$$

is open in K. Assertion 1.2 and Appendix B.2 in conjunction with the mean value property entail that for any n, the set M_n contains the (finite) boundary points of maximal intervals of $\mathbb{R}\backslash K$. Since, by assumption, K is a Cantor set, this entails that M_n is dense in K for every n. Whence, $\bigcap_{n=1}^{\infty} M_n = \{\chi \in K/m_\chi = \infty\}$ is a dense G_δ-subset of K, which is a perfect compactum. On account of the Baire category theorem, we conclude that $m_\chi = \infty$ for uncountably many $\chi \in K$. Assuming that (L) holds, this contradicts Lemma 5.4 combined with the fact that the set $\Omega(\alpha,\beta)$ is at most countable. □

Remark 5.5. In [19], it is shown that the conclusion of Assertion 1.2 always holds for irrational numbers α which are sufficiently well approximable by rationals in terms of a Diophantine condition and for $|\beta| \geq 1$. By virtue of duality, this translates into a similar statement for $0 < |\beta| < 1$.

Appendices

A. Subharmonic functions and potential theory. In the sequel, we present without proofs the material from classical potential theory which has been used in the paper.

A.1. Let $K \subset \mathbb{C}$ be a compact subset, let $\mathcal{P}(K)$ be the set of all probability measures on K, and let

$$\eta(K) = \sup\left\{\iint \log|s-t|\,d\nu(s)\,d\nu(t)/\nu \in \mathcal{P}(K)\right\}. \tag{A.1}$$

Then $-\eta(K)$ is called the Robin constant of K, and $\gamma(K) = e^{\eta(K)}$ is the (logarithmic) capacity of K. (If $\eta(K) = -\infty$, then $\gamma(K) = 0$.) If $\gamma(K) > 0$, then there exists exactly one measure $\mu \in \mathcal{P}(K)$, called the equilibrium distribution of K, such that

$$\eta(K) = \iint \log|s-t|\,d\mu(s)\,d\mu(t) \tag{A.2}$$

(see [22, page 55, Section III.32]). The logarithmic potential

$$u_K(z) = \int \log|z-s|\,d\mu(s) \tag{A.3}$$

is called the conductor potential of K.

A.2. If $M \subset \mathbb{C}$ is an arbitrary set, then its capacity $\gamma(M)$ is defined to be equal to $\sup\{\gamma(K) \mid K \subset M,\ K \text{ compact}\}$. A subset $M \subset \mathbb{C}$ is called polar if M is the union of countably many compact subsets of $\mathbb{C}$ and $\gamma(M) = 0$. Any finite or countable subset of $\mathbb{C}$ is polar (see [22, Section III.8]).

A.3. Fundamental theorem. Let $K \subset \mathbb{C}$ be compact of positive capacity. Then $u_K(z) \geq \eta(K)$ in $\mathbb{C}$ and $u_K(z) = \eta(K)$ on $K \backslash N$, where $N \subset K$ is polar (see [22, Section III.12]).

A.4. Phragmén-Lindelöf principle. Suppose that $G \subset \mathbb{C}$ is an open subset with boundary K. Let $N \subset K$ be a polar set. Suppose that u is a subharmonic function on G (i.e., $-\infty \leq u(z) < \infty$ in G, u is upper semicontinuous, and u is submean, which means that for any $z \in G$ and sufficiently small $r > 0$, $u(z) \leq \int_0^{2\pi} u(z + re^{it})\,dt$), bounded above, and c is a real constant such that for all $\xi \in K \backslash N$,

$$\overline{\lim_{z \to \xi}}\, u(z) \leq c \tag{A.4}$$

as z approaches ξ from inside G. If $K = N$, then u is constant and otherwise $u(z) < c$ in G or $u(z) = c$ in G (see [8, Section 5.16]).

A.5. Let $K \subset \mathbb{C}$ be a compact set of positive capacity. Suppose that u is a subharmonic function in $\mathbb{C}$, c is a real constant, and $N \subset K$ is a polar set such that the following properties hold:

(i) $u(z) \geq c$ in $\mathbb{C} \backslash K$,
(ii) u is harmonic in $\mathbb{C} \backslash K$,
(iii) $u(z) = c$ for all $z \in K \backslash N$,
(iv) $u(z) = \log|z| + o(1)$ as $|z| \to \infty$.

Then u is the conductor potential of K. (This is a simple consequence of Appendix A.4; see also [8, Section 5.17].)

A.6. Let $K \subset \mathbb{C}$ be compact of positive capacity. Then K is said to be regular if its conductor potential u_K is constant on K. If K is regular, then u_K is continuous in $\mathbb{C}$ (see [22, Section III.13]). The set K is regular if and only if the Dirichlet problem is solvable in the component containing ∞ of $(\mathbb{C} \cup \{\infty\}) \backslash K$ for any continuous function on K (see [22, Section III.38]).

B. Logarithmic potentials and conformal mappings. In this appendix we prove a property of conductor potentials for regular compact subsets of $\mathbb{R}$ which is used in the proof of Assertion 1.3.

Let μ be a probability measure on $\mathbb{R}$ with compact support K. On the upper half plane $\Gamma = \{z \mid \operatorname{Im} z > 0\}$, we define an analytic function

$$f_\mu(z) = \exp\left(-\int \log(z - s)\,d\mu(s)\right). \tag{B.1}$$

This function has properties akin to starlike functions (see [13]):

(1) f_μ is univalent,
(2) $tf_\mu(\Gamma) \subset f_\mu(\Gamma)$ for every $t \in (0,1]$,
(3) $f_\mu(\Gamma) \subset -\Gamma$,
(4) $\lim_{|z|\to\infty} f_\mu(z) = 0$ and $\lim_{|z|\to\infty} |zf_\mu(z)| = 1$.

We also note that f_μ has an analytic univalent extension on $(\mathbb{C}\cup\{\infty\})\setminus[m,M]$, where m is the smallest number in K and M is the largest number in K.

B.1. Suppose that K has a positive capacity c. Let $B = \{z \in -\Gamma \mid |z| = c^{-1}\} \cup [-c^{-1}, c^{-1}]$. Then the following statements are equivalent:

(I) K is a regular compactum and μ is its equilibrium distribution,
(II) the boundary of $\overline{f_\mu(\Gamma)}$ is equal to B and has a continuous extension on $\overline{\Gamma}$,
(III) the boundary of $\overline{f_\mu(\Gamma)}$ is equal to B, and for every $r \in (0, c^{-1})$, the boundary $\partial f_\mu(\Gamma)$ of $f_\mu(\Gamma)$ intersects the circle $\{z \mid |z| = r\}$ at finitely many points only.

Proof. (I)⇒(II). The logarithmic potential

$$z \longmapsto \int \log|z-s|\,d\mu(s) \tag{B.2}$$

takes its minimum value at all points of K; it is continuous everywhere (Appendix A.6) and, since μ is a diffuse measure, its conjugate function in the upper half plane is continuous on $\overline{\Gamma}$.

(II)⇒(III). If f_μ has a continuous extension on $\overline{\Gamma}$, then $\partial f_u(\Gamma)$ is locally connected (see [13, Theorem 9.8]), which, by virtue of property (2), entails that $\partial f_\mu(\Gamma)$ intersects $\{z \mid |z| = r\}$ at finitely many points only, for every $r \in (0, c^{-1})$.

(III)⇒(I). Property (III) implies that $\partial f_\mu(\Gamma)$ is locally connected. Whence, f_μ has a continuous extension on $\overline{\Gamma}$ (see [13, Theorem 9.8]). It follows that $-\log|f_\mu|$ is the restriction of a continuous subharmonic function g in $\mathbb{C}$ which takes its minimum value at all points of K. Since $g(z) = \int \log|z-s|\,d\mu(s)$ in $\mathbb{C}\setminus K$, g is the logarithmic potential associated with μ (Appendix A.4). Whence, K is regular and μ is its equilibrium distribution. □

B.2. Suppose that K is regular and μ its equilibrium distribution. Furthermore, suppose that $\mathcal{I} \subset \mathbb{R}\setminus K$ is a maximal open interval and a a finite boundary point of $\mathcal{I}$. Then $|\int (x-s)^{-1}d\mu(s)| \to \infty$ as $x \in \mathcal{I}$ approaches a.

Proof. Suppose that a is a left boundary point of $\mathcal{I}$. The case of a right boundary point can be dealt with in a similar fashion. Since $\int (x-s)^{-1}d\mu(s)$ decreases as $x \in \mathcal{I}$ increases, $\lim_{x\to a^+} \int (x-s)^{-1}d\mu(s)$ is either finite or positive infinite. Suppose that the limit is a finite number b. We claim that

$$\int (z-s)^{-1}d\mu(s) \longrightarrow b \quad \text{as } z \longrightarrow a \text{ in } A_\theta = \{z \in \mathbb{C} \mid -\theta \le \arg(z-a) \le \theta\} \tag{B.3}$$

for every $\theta \in (0,\pi)$. For convenience, we assume that $a = 0$. Then, $|z-s| \geq |s||\sin\theta|$ for $z \in A_\theta$ and $s \leq 0$, and hence

$$\begin{aligned} \int_{-\infty}^{\infty} |z-s|^{-1} d\mu(s) &= \int_{-\infty}^{0} |z-s|^{-1} d\mu(s) + \int_{0}^{\infty} |z-s|^{-1} d\mu(s), \\ \int_{-\infty}^{0} |z-s|^{-1} d\mu(s) &\leq \frac{1}{|\sin\theta|} \int_{-\infty}^{0} \frac{1}{|s|} d\mu(s). \end{aligned} \tag{B.4}$$

Since $\int_{-\infty}^{0}(1/|s|)d\mu(s)$ is finite by assumption, then it follows that the integral $\int |z-s|^{-1} d\mu(s)$ is uniformly bounded for $z \in A_\theta$, but close to a. By Lebesgue's dominated convergence theorem, we infer that the limit in question exists and it is equal to b. Since the function $x \mapsto \int (x-s)^{-1} d\mu(s)$ is decreasing in $\mathcal{I}$ and it takes the value 0 exactly once in case $\mathcal{I}$ is a finite interval (see [23, Section 7.2, Corollary 3]) and it vanishes at infinity only in case $\mathcal{I}$ is an infinite interval, we conclude that $b \neq 0$. Moreover,

$$f'_\mu(z) \longrightarrow -b \cdot w \neq 0 \quad \text{as } z \longrightarrow a \text{ in } \Gamma \cap A_\theta, \tag{B.5}$$

where $f_\mu(z) \to w$ as $z \to a$ in Γ. Let $\eta_k : [0,1] \to \mathbb{C}$ be differentiable injective curves $(k = 1,2)$ such that $\eta_k([0,1)) \subset \Gamma \cap A_\theta$ for some $\theta \in (0,\pi)$ and $\eta_k(1) = a$. Let φ be the angle between η_1 and η_2 at a. Then (B.5) entails that the images $\tilde{\eta}_1$ and $\tilde{\eta}_2$ of η_1 and η_2 with respect to f_μ form the same angle φ at the point $w \in \partial\overline{f_\mu(\Gamma)} = B$ (See Appendix B.1). Since the curves $\tilde{\eta}_1$ and $\tilde{\eta}_2$ eventually evolve exclusively on only one side of the line through 0 and w, as they approach the point w, the angle between $\tilde{\eta}_1$ and $\tilde{\eta}_2$ can never exceed the value $\pi/2$. However, since we can arrange φ to be any number in $(0,\pi)$, we have reached a contradiction. □

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