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EXTENSION OF n-DIMENSIONAL EUCLIDEAN VECTOR SPACE E^n OVER $\mathbb{R}$ TO PSEUDO-FUZZY VECTOR SPACE OVER $F_p^1(1)$

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For any two points $P = (p^{(1)}, p^{(2)}, \dots, p^{(n)})$ and $Q = (q^{(1)}, q^{(2)}, \dots, q^{(n)})$ of $\mathbb{R}^n$, we define the crisp vector $\overrightarrow{PQ} = (q^{(1)} - p^{(1)}, q^{(2)} - p^{(2)}, \dots, q^{(n)} - p^{(n)}) = Q(-)P$. Then we obtain an n-dimensional vector space $E^n = \{\overrightarrow{PQ} \mid \text{ for all } P, Q \in \mathbb{R}^n\}$. Further, we extend the crisp vector into the fuzzy vector on fuzzy sets of $\mathbb{R}^n$. Let $\widetilde{D}$, $\widetilde{E}$ be any two fuzzy sets on $\mathbb{R}^n$ and define the fuzzy vector $\overrightarrow{\widetilde{E}\widetilde{D}} = \widetilde{D} \ominus \widetilde{E}$, then we have a pseudo-fuzzy vector space.

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1. Introduction. In [1, 2, 4, 5, 6], fuzzy vector space is discussed theoretically. In Katsaras and Liu [2], E denotes a vector space over K, where K is the space of real or complex numbers. A fuzzy set F in E is called a fuzzy subspace if (a) $F + F \subset F$; (b) $\lambda F \subset F$ for every scalar λ. Katsaras and Liu introduced the concept of a fuzzy subspace of a vector space. In Das [1], E denotes a vector space over a field K. Let $I = [0,1]$ and let I^E be the collection of all mappings of E into I. We say $\mu \in I^E$ is a fuzzy space of E under a triangular norm T (see [1, Definition 2.1]), or simply a T-fuzzy subspace of E if for all $x, y \in E$ and for all $a \in K$, $\mu(x + y) \geq T(\mu(x), \mu(y))$ and $\mu(ax) \geq \mu(x)$, respectively. In Lubczonok [5], a fuzzy vector space is a pair $\widetilde{E} = (E, \mu)$, where E is a vector space and $\mu : E \to [0,1]$ with the property that, for all $a, b \in \mathbb{R}$ and $x, y \in E$, we have $\mu(ax + by) \geq \mu(x) \wedge \mu(y)$. In Kumar [4], V is a vector space over F, where F is the field of real numbers. A fuzzy subset μ of V is called a fuzzy subspace if it has the following properties:

(a) $\mu(v_1 - v_2) \geq \min(\mu(v_1), \mu(v_2))$ for all $v_1, v_2 \in V$;

(b) $\mu(\alpha v) \geq \mu(v)$ for all $\alpha \in F$, $v \in V$.

There are various definitions of fuzzy vector spaces in these papers. All of them use the fuzzy set μ over a crisp vector space E, or $\mu : E \to [0,1]$, to define fuzzy vector. These are different from our work. Pick two points $P = (p_1, p_2, \dots, p_n)$, $Q = (q_1, q_2, \dots, q_n)$ in $\mathbb{R}^n$ to form a vector $\overrightarrow{PQ} = (q_1 - p_1, q_2 - p_2, \dots, q_n - p_n)$. Then extend this vector to the fuzzy vector $\overrightarrow{\widetilde{P}\widetilde{Q}} = \widetilde{Q} \ominus \widetilde{P}$ formed by fuzzy sets $\widetilde{P}$, $\widetilde{Q}$ on $\mathbb{R}^n$. This is very useful compared to the abstract one defined in [1, 2, 4, 5].

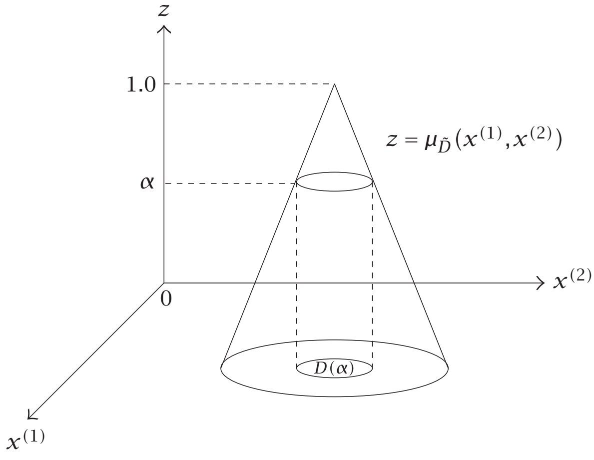

FIGURE 2.1. α-cut of fuzzy set $\tilde{D}$ on $\mathbb{R}^2$.

Section 2 is a preparing work. Section 3 is the extension of the crisp n-dimensional Euclidean vector E^n to the pseudo-fuzzy vector space. We talk in Section 4 about the length of the fuzzy vectors and fuzzy inner product. Section 5 is a more discussion.

2. Preparation. In order to consider the fuzzy vectors of fuzzy sets on $\mathbb{R}^n$, we ought to know the following. First, from Kaufmann and Gupta [3] and Zimmermann [8], we have the following definition.

DEFINITION 2.1. (a) A fuzzy set $\tilde{A}$ on $\mathbb{R} = (-\infty, \infty)$ is convex if and only if every ordinary set $A(\alpha) = \{x \mid \mu_{\tilde{A}}(x) \geq \alpha\}$ for all $\alpha \in [0,1]$ is convex. Thus $A(\alpha)$ is a closed interval in $\mathbb{R}$.

(b) A fuzzy set $\tilde{A}$ on $\mathbb{R}$ is normal if and only if $\bigvee_{x \in \mathbb{R}} \mu_{\tilde{A}}(x) = 1$.

We can extend this definition to $\mathbb{R}^n$, say if $\tilde{D}$ is a fuzzy set on $\mathbb{R}^n$ with membership function

$$\mu_{\tilde{D}}(x^{(1)}, x^{(2)}, \dots, x^{(n)}) \in [0,1] \quad \forall (x^{(1)}, x^{(2)}, \dots, x^{(n)}) \in \mathbb{R}^n, \tag{2.1}$$

then we have the following definition.

DEFINITION 2.2. The α-cut of fuzzy set $\tilde{D}$ on $\mathbb{R}^n$, $0 \leq \alpha \leq 1$, is defined by

$$D(\alpha) = \{(x^{(1)}, x^{(2)}, \dots, x^{(n)}) \mid \mu_{\tilde{D}}(x^{(1)}, x^{(2)}, \dots, x^{(n)}) \geq \alpha\}. \tag{2.2}$$

For $n = 2$, see Figure 2.1.

Definition 2.3. (a) A fuzzy set $\widetilde{D}$ on $\mathbb{R}^n$ is convex if and only if for each $\alpha \in [0,1]$, every ordinary set

$$D(\alpha) = \{(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \mid \mu_{\widetilde{D}}(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \geq \alpha\} \tag{2.3}$$

is a convex closed subset of $\mathbb{R}^n$.

(b) A fuzzy set $\widetilde{D}$ is normal if and only if

$$\bigvee_{(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \in \mathbb{R}^n} \mu_{\widetilde{D}}(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) = 1. \tag{2.4}$$

Let F_c be the family of all fuzzy sets on $\mathbb{R}^n$ satisfying Definition 2.3(a), (b).

Remark 2.4. When $\alpha = 0$, then the α-cut is

$$\{(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \mid \mu_{\widetilde{D}}(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \geq 0\}. \tag{2.5}$$

Let $D(0)$ be the smallest convex closed subset in $\mathbb{R}^n$ satisfying

$$\{(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \mid \mu_{\widetilde{D}}(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \geq 0\} \tag{2.6}$$

(see Example 4.11).

Definition 2.5 (Pu and Liu [7])**.** If the membership function of a fuzzy set a_α, $0 \leq \alpha \leq 1$, on $\mathbb{R}$ is

$$\mu_{a_\alpha} = \begin{cases} \alpha, & x = a, \\ 0, & x \neq a, \end{cases} \tag{2.7}$$

then we call a_α a level-α fuzzy point on $\mathbb{R}$.

Let $F_p^1(\alpha) = \{a_\alpha \mid \text{ for all } a \in \mathbb{R}\}$ be the family of all level-α fuzzy points on $\mathbb{R}$ satisfying (2.7).

Definition 2.6. If the membership function of a fuzzy set $(a^{(1)}, a^{(2)}, \ldots, a^{(n)})_\alpha$, $0 \leq \alpha \leq 1$, on $\mathbb{R}^n$ is

$$\mu_{(a^{(1)}, a^{(2)}, \ldots, a^{(n)})_\alpha}(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) = \begin{cases} \alpha, & \text{if } (x^{(1)}, x^{(2)}, \ldots, x^{(n)}) = (a^{(1)}, a^{(2)}, \ldots, a^{(n)}), \\ 0, & \text{elsewhere}, \end{cases} \tag{2.8}$$

then we call $(a^{(1)}, a^{(2)}, \ldots, a^{(n)})_\alpha$ a level-α fuzzy point on $\mathbb{R}^n$.

Let $F_p^n(\alpha) = \{(a^{(1)}, a^{(2)}, \ldots, a^{(n)})_\alpha \mid \text{ for all } (a^{(1)}, a^{(2)}, \ldots, a^{(n)}) \in \mathbb{R}^n\}$ be the family of all level-α fuzzy points on $\mathbb{R}^n$ satisfying (2.8).

For every $a_\alpha \in F_p^1(\alpha)$, let $a_\alpha = (a,a,\dots,a)_\alpha$, then a_α can be regarded as a special case of the level-α fuzzy point $(a^{(1)},a^{(2)},\dots,a^{(n)})_\alpha$ degenerating to $a^{(1)} = a^{(2)} = \cdots = a^{(n)} = a$. Thus

$$\mu_{(a,a,\dots,a)_\alpha}(x^{(1)},x^{(2)},\dots,x^{(n)}) = \begin{cases} \alpha, & (x^{(1)},x^{(2)},\dots,x^{(n)}) = (a,a,\dots,a), \\ 0, & (x^{(1)},x^{(2)},\dots,x^{(n)}) \neq (a,a,\dots,a) \end{cases} \\ = \mu_{a_\alpha}(x^{(1)},x^{(2)},\dots,x^{(n)}). \tag{2.9}$$

Remark 2.7. We can regard a_α as a fuzzy set on $\mathbb{R}$ as the form in (2.7) or it can also be regarded as a fuzzy set on $\mathbb{R}^n$ such as $a_\alpha = (a,a,\dots,a)_\alpha$ in (2.9) according to how we want it to be. That is, $0_1 = (0,0,\dots,0)_1$ and $a_\alpha = (a,a,\dots,a)_\alpha$, $\alpha \in [0,1]$.

From Kaufmann and Gupta [3], for $D,E \subset \mathbb{R}^n$, $k \in \mathbb{R}$, we have

(i) $D(+)E = \{(x^{(1)}+y^{(1)},x^{(2)}+y^{(2)},\dots,x^{(n)}+y^{(n)}) \mid \forall (x^{(1)},x^{(2)},\dots,x^{(n)}) \in D,\ (y^{(1)},y^{(2)},\dots,y^{(n)}) \in E\}$,

(ii) $D(-)E = \{(x^{(1)}-y^{(1)},x^{(2)}-y^{(2)},\dots,x^{(n)}-y^{(n)}) \mid \forall (x^{(1)},x^{(2)},\dots,x^{(n)}) \in D,\ (y^{(1)},y^{(2)},\dots,y^{(n)}) \in E\}$,

(iii) $k(\cdot)D = \{(kx^{(1)},kx^{(2)},\dots,kx^{(n)}) \mid \forall (x^{(1)},x^{(2)},\dots,x^{(n)}) \in D\}$,

(iv) the α-cut of $\tilde{D} \oplus \tilde{E}$ is $D(\alpha)(+)E(\alpha)$,

(v) the α-cut of $\tilde{D} \ominus \tilde{E}$ is $D(\alpha)(-)E(\alpha)$,

(vi) the α-cut of $k_1 \odot \tilde{D}$ is $k(\cdot)D(\alpha)$.

3. The extension of the crisp n-dimensional Euclidean vector space E^n to the pseudo-fuzzy vector space SFR. In crisp case, for $P = (p^{(1)},p^{(2)},\dots,p^{(n)})$, $Q = (q^{(1)},q^{(2)},\dots,q^{(n)})$, $A = (a^{(1)},a^{(2)},\dots,a^{(n)})$, $B = (b^{(1)},b^{(2)},\dots,b^{(n)}) \in \mathbb{R}^n$, and $k \in \mathbb{R}$, we can define the operations "$+,\cdot$" for the crisp vectors $\overrightarrow{PQ}$, $\overrightarrow{AB}$ in E^n, the n-dimensional vector space over $\mathbb{R}^n$, by

$$\overrightarrow{AB} = (b^{(1)}-a^{(1)},b^{(2)}-a^{(2)},\dots,b^{(n)}-a^{(n)}), \\ \overrightarrow{PQ} = (q^{(1)}-p^{(1)},q^{(2)}-p^{(2)},\dots,q^{(n)}-p^{(n)}), \tag{3.1}$$

$$\overrightarrow{AB}+\overrightarrow{PQ} = (b^{(1)}+q^{(1)}-a^{(1)}-p^{(1)},b^{(2)}+q^{(2)} \\ -a^{(2)}-p^{(2)},\dots,b^{(n)}+q^{(n)}-a^{(n)}-p^{(n)}), \\ k\cdot\overrightarrow{PQ} = (kq^{(1)}-kp^{(1)},kq^{(2)}-kp^{(2)},\dots,kq^{(n)}-kp^{(n)}). \tag{3.2}$$

Let $O = (0,0,\dots,0) \in \mathbb{R}^n$, then $\overrightarrow{OP} = (p^{(1)},p^{(2)},\dots,p^{(n)})$ and $\overrightarrow{OO} = (0,0,\dots,0) \in E^n$.

Let E^n be an n-dimensional vector space over $\mathbb{R}$. By Definition 2.6, $F_p^n(1) = \{(a^{(1)},a^{(2)},\dots,a^{(n)})_1 \mid \text{for all } (a^{(1)},a^{(2)},\dots,a^{(n)}) \in \mathbb{R}^n\}$. This is a family of all level-1 fuzzy points on $\mathbb{R}^n$.

We notice that there is a one-to-one onto mapping ρ between $(a^{(1)},a^{(2)},\dots,a^{(n)}) \in \mathbb{R}^n$ and $(a^{(1)},a^{(2)},\dots,a^{(n)})_1 \in F_p^n(1)$. That is,

$$\begin{aligned}\rho:(a^{(1)},a^{(2)},\dots,a^{(n)}) \in \mathbb{R}^n &\longleftrightarrow \rho\big((a^{(1)},a^{(2)},\dots,a^{(n)})\big)\\ &= (a^{(1)},a^{(2)},\dots,a^{(n)})_1 \in F_p^n(1),\end{aligned} \tag{3.3}$$

$$\begin{aligned}\mu_{(a^{(1)},a^{(2)},\dots,a^{(n)})_1}&(x^{(1)},x^{(2)},\dots,x^{(n)})\\ &= C_{(a^{(1)},a^{(2)},\dots,a^{(n)})}(x^{(1)},x^{(2)},\dots,x^{(n)}),\end{aligned} \tag{3.4}$$

where C_A is the characteristic function of A.

Let $\tilde{P} = (p^{(1)},p^{(2)},\dots,p^{(n)})_1$, $\tilde{Q} = (q^{(1)},q^{(2)},\dots,q^{(n)})_1 \in F_p^n(1)$. From (3.1), (3.3), we have the following definition:

$$\overrightarrow{\tilde{P}\tilde{Q}} = (q^{(1)}-p^{(1)},q^{(2)}-p^{(2)},\dots,q^{(n)}-p^{(n)})_1 = \tilde{Q} \ominus \tilde{P}. \tag{3.5}$$

We call $\overrightarrow{\tilde{P}\tilde{Q}}$ a fuzzy vector.

Let $\tilde{O} = (0,0,\dots,0)_1 \in F_p^n(1)$, then $\overrightarrow{\tilde{O}\tilde{P}} = (p^{(1)},p^{(2)},\dots,p^{(n)})_1$, $\overrightarrow{\tilde{O}\tilde{O}} = (0,0,\dots,0)_1$. Let $FE^n = \{\overrightarrow{\tilde{P}\tilde{Q}} \mid \text{ for all } \tilde{P},\tilde{Q} \in F_p^n(1)\}$ be the family of all fuzzy vectors on $F_p^n(1)$. From (3.1), (3.5), we can have the one-to-one onto mapping ρ between E^n and FE^n by

$$\begin{aligned}\overrightarrow{PQ} &= (q^{(1)}-p^{(1)},q^{(2)}-p^{(2)},\dots,q^{(n)}-p^{(n)}) \quad (\in E^n) \longleftrightarrow \rho\big(\overrightarrow{PQ}\big)\\ &= (q^{(1)}-p^{(1)},q^{(2)}-p^{(2)},\dots,q^{(n)}-p^{(n)})_1\\ &= \overrightarrow{\tilde{P}\tilde{Q}} \in FE^n.\end{aligned} \tag{3.6}$$

Since $(p^{(1)},p^{(2)},\dots,p^{(n)}) = \overrightarrow{OP}$, hence the point in $\mathbb{R}^n$ can be regarded as a vector in E^n. Also since $(p^{(1)},p^{(2)},\dots,p^{(n)})_1 = \overrightarrow{\tilde{O}\tilde{P}}$, hence the level-1 fuzzy points on $\mathbb{R}^n$ can be regarded as the fuzzy vectors in FE^n. Therefore the mapping in (3.3) is a special case of the mapping in (3.6).

The operations "$\oplus,\odot$" of the fuzzy vectors in FE^n have the following property.

Property 3.1. For $\tilde{P} = (p^{(1)},p^{(2)},\dots,p^{(n)})_1$, $\tilde{Q} = (q^{(1)},q^{(2)},\dots,q^{(n)})_1$, $\tilde{A} = (a^{(1)},a^{(2)},\dots,a^{(n)})_1$, $\tilde{B} = (b^{(1)},b^{(2)},\dots,b^{(n)})_1 \in FE^n$, and $k \neq 0 \in \mathbb{R}$, we have

$$\begin{aligned}\overrightarrow{\tilde{A}\tilde{B}} \oplus \overrightarrow{\tilde{P}\tilde{Q}} = (&b^{(1)}+q^{(1)}-a^{(1)}-p^{(1)},b^{(2)}+q^{(2)}-a^{(2)}-p^{(2)},\dots,b^{(n)}\\ &+q^{(n)}-a^{(n)}-p^{(n)})_1;\end{aligned} \tag{3.7}$$

$$k_1 \odot \overrightarrow{\tilde{P}\tilde{Q}} = (kq^{(1)}-kp^{(1)},kq^{(2)}-kp^{(2)},\dots,kq^{(n)}-kp^{(n)})_1. \tag{3.8}$$

Proof. For (3.7),

$$\begin{aligned}
&\mu_{\overrightarrow{\tilde{A}\tilde{B}}\oplus\overrightarrow{\tilde{P}\tilde{Q}}}\left(z^{(1)},z^{(2)},\dots,z^{(n)}\right)\\
&\quad=\sup_{z^{(j)}=x^{(j)}+y^{(j)},\,j=1,2,\dots,n}\Big\{\mu_{(b^{(1)}-a^{(1)},b^{(2)}-a^{(2)},\dots,b^{(n)}-a^{(n)})_1}\left(x^{(1)},x^{(2)},\dots,x^{(n)}\right)\\
&\qquad\qquad\wedge\mu_{(q^{(1)}-p^{(1)},q^{(2)}-p^{(2)},\dots,q^{(n)}-p^{(n)})_1}\left(y^{(1)},y^{(2)},\dots,y^{(n)}\right)\Big\}\\
&\quad=\sup_{(y^{(1)},y^{(2)},\dots,y^{(n)})}\Big\{\mu_{(b^{(1)}-a^{(1)},b^{(2)}-a^{(2)},\dots,b^{(n)}-a^{(n)})_1}\left(z^{(1)}-y^{(1)},\right.\\
&\qquad\qquad\left.z^{(2)}-y^{(2)},\dots,z^{(n)}-y^{(n)}\right)\\
&\qquad\qquad\wedge\mu_{(q^{(1)}-p^{(1)},q^{(2)}-p^{(2)},\dots,q^{(n)}-p^{(n)})_1}\left(y^{(1)},y^{(2)},\dots,y^{(n)}\right)\Big\}\\
&\quad=1,\quad\text{if } z^{(j)}-y^{(j)}=b^{(j)}-a^{(j)},\ y^{(j)}=q^{(j)}-p^{(j)},\ j=1,2,\dots,n,\\
&\quad=1,\quad\text{if } z^{(j)}-q^{(j)}+p^{(j)}=b^{(j)}-a^{(j)},\ j=1,2,\dots,n,\\
&\quad=\mu_{(b^{(1)}+q^{(1)}-a^{(1)}-p^{(1)},b^{(2)}+q^{(2)}-a^{(2)}-p^{(2)},\dots,b^{(n)}+q^{(n)}-a^{(n)}-p^{(n)})}\left(z^{(1)},z^{(2)},\dots,z^{(n)}\right)\\
&\qquad\qquad\forall\left(z^{(1)},z^{(2)},\dots,z^{(n)}\right)\in\mathbb{R}^n,
\end{aligned}\tag{3.9}$$

that is,

$$\begin{aligned}
\overrightarrow{\tilde{A}\tilde{B}}\oplus\overrightarrow{\tilde{P}\tilde{Q}}=\big(&b^{(1)}+q^{(1)}-a^{(1)}-p^{(1)},b^{(2)}+q^{(2)}-a^{(2)}\\
&-p^{(2)},\dots,b^{(n)}+q^{(n)}-a^{(n)}-p^{(n)}\big)_1.
\end{aligned}\tag{3.10}$$

Similarly, we have (3.8). In the case $k=0$, it follows by Property 3.7(7). From Property 3.1, (3.2), (3.6), (3.7), and (3.8), we have

$$\begin{aligned}
\rho\left(\overrightarrow{AB}+\overrightarrow{PQ}\right)&=\big(b^{(1)}+q^{(1)}-a^{(1)}-p^{(1)},b^{(2)}+q^{(2)}\\
&\qquad-a^{(2)}-p^{(2)},\dots,b^{(n)}+q^{(n)}-a^{(n)}-p^{(n)}\big)_1\\
&=\overrightarrow{\tilde{A}\tilde{B}}\oplus\overrightarrow{\tilde{P}\tilde{Q}}=\rho\left(\overrightarrow{AB}\right)\oplus\rho\left(\overrightarrow{PQ}\right),\\
\rho\left(k\cdot\overrightarrow{AB}\right)&=k_1\odot\overrightarrow{\tilde{A}\tilde{B}}=\rho(k)\odot\rho\left(\overrightarrow{\tilde{A}\tilde{B}}\right).
\end{aligned}\tag{3.11}$$

By Remark 2.7, $k=(k,k,\dots,k)$. Hence by (3.3), $\rho(k)=\rho(k,k,\dots,k)=(k,k,\dots,k)_1=k_1$. From (3.6), (3.11), since E^n is a vector space over $\mathbb{R}$, therefore FE^n satisfies the conditions to be a vector space too. We call FE^n a fuzzy vector space over $F_p^1(1)$ and call $\overrightarrow{\tilde{P}\tilde{Q}}$ $(\in FE^n)$ a fuzzy vector. □

Remark 3.2. The zero fuzzy vector $\overrightarrow{\tilde{O}\tilde{O}}=(0,0,\dots,0)_1$ in FE^n will be obtained from the zero vector $\overrightarrow{OO}=(0,0,\dots,0)$ in E^n by mapping $\overrightarrow{OO}$ to $\overrightarrow{\tilde{O}\tilde{O}}$.

Obviously, there is a one-to-one mapping between $\mathbb{R}$ and $F_p^1(1)$ such that $a\in\mathbb{R}\leftrightarrow a_1\in F_p^1(1)$. Thus, we have the following property.

Property 3.3. The fuzzy vector space FE^n over $F_p^1(1)$ is equivalent to the vector space E^n over $\mathbb{R}$ denoted by $E^n \approx FE^n$.

Since the α-cut of the fuzzy point $\widetilde{P} = (p^{(1)}, p^{(2)}, \dots, p^{(n)})_1$ in $F_p^n(1)$ is $(p^{(1)}, p^{(2)}, \dots, p^{(n)})$ for all $\alpha \in [0,1]$, hence it can be regarded as a special case in F_c, that is, we can take $F_p^n(1)$ as a subfamily of F_c, that is, $F_p^n(1) \subset F_c$. Therefore we can extend the fuzzy vector space FE^n to F_c and have the following definition similarly as in (3.5).

Definition 3.4. For $\widetilde{X}, \widetilde{Y} \in F_c$, define

$$\overrightarrow{\widetilde{X}\widetilde{Y}} = \widetilde{Y} \ominus \widetilde{X}. \tag{3.12}$$

We call $\overrightarrow{\widetilde{X}\widetilde{Y}}$ a fuzzy vector.

Let $SFR = \{\overrightarrow{\widetilde{X}\widetilde{Y}} = \widetilde{Y} \ominus \widetilde{X} \mid \text{ for all } \widetilde{X}, \widetilde{Y} \in F_c\}$.

Property 3.5. For $\overrightarrow{\widetilde{X}\widetilde{Y}}, \overrightarrow{\widetilde{W}\widetilde{Z}} \in SFR$,

$$\overrightarrow{\widetilde{X}\widetilde{Y}} = \overrightarrow{\widetilde{W}\widetilde{Z}} \quad \text{iff } \widetilde{Y} \ominus \widetilde{X} = \widetilde{Z} \ominus \widetilde{W}. \tag{3.13}$$

Proof. The proof follows from Definition 3.4 of fuzzy vector. □

Property 3.6. For $\overrightarrow{\widetilde{X}\widetilde{Y}}, \overrightarrow{\widetilde{W}\widetilde{Z}} \in SFR$, $k \in \mathbb{R}$,

(1) $\overrightarrow{\widetilde{X}\widetilde{Y}} \oplus \overrightarrow{\widetilde{W}\widetilde{Z}} = \overrightarrow{\widetilde{A}\widetilde{B}}$; here $\widetilde{A} = \widetilde{X} \oplus \widetilde{W}$, $\widetilde{B} = \widetilde{Y} \oplus \widetilde{Z}$;

(2) $k_1 \odot \overrightarrow{\widetilde{X}\widetilde{Y}} = \overrightarrow{\widetilde{C}\widetilde{D}}$; here $\widetilde{C} = k_1 \odot \widetilde{X}$, $\widetilde{D} = k_1 \odot \widetilde{Y}$.

Proof. (1) For each $\alpha \in [0,1]$, from (i), (ii), (iv), (v), the α-cuts of $\overrightarrow{\widetilde{X}\widetilde{Y}} = \widetilde{Y} \ominus \widetilde{X}$, $\overrightarrow{\widetilde{W}\widetilde{Z}} = \widetilde{Z} \ominus \widetilde{W}$ are $Y(\alpha)(-)X(\alpha)$, $Z(\alpha)(-)W(\alpha)$, respectively. Let

$$\begin{aligned} D = \{ & (x^{(1)}, x^{(2)}, \dots, x^{(n)}) \in X(\alpha),\ (y^{(1)}, y^{(2)}, \dots, y^{(n)}) \in Y(\alpha), \\ & (z^{(1)}, z^{(2)}, \dots, z^{(n)}) \in Z(\alpha),\ (w^{(1)}, w^{(2)}, \dots, w^{(n)}) \in W(\alpha)\}. \end{aligned} \tag{3.14}$$

Therefore the α-cut of $\overrightarrow{\widetilde{X}\widetilde{Y}} \oplus \overrightarrow{\widetilde{W}\widetilde{Z}}$ is

$$\begin{aligned} &(Y(\alpha)(-)X(\alpha))(+)(Z(\alpha)(-)W(\alpha)) \\ &\quad = \{(y^{(1)} - x^{(1)} + z^{(1)} - w^{(1)}, y^{(2)} - x^{(2)} + z^{(2)} - w^{(2)}, \dots, \\ &\qquad y^{(n)} - x^{(n)} + z^{(n)} - w^{(n)}) \mid D\} \\ &\quad = \{(y^{(1)} + z^{(1)}, y^{(2)} + z^{(2)}, \dots, y^{(n)} + z^{(n)}) \\ &\qquad \mid (y^{(1)}, y^{(2)}, \dots, y^{(n)}) \in Y(\alpha),\ (z^{(1)}, z^{(2)}, \dots, z^{(n)}) \in Z(\alpha)\} \\ &\qquad (-)\{(x^{(1)} + w^{(1)}, x^{(2)} + w^{(2)}, \dots, x^{(n)} + w^{(n)}) \\ &\qquad\quad \mid (x^{(1)}, x^{(2)}, \dots, x^{(n)}) \in X(\alpha),\ (w^{(1)}, w^{(2)}, \dots, w^{(n)}) \in W(\alpha)\} \end{aligned} \tag{3.15}$$

which is the α-cut of $(\widetilde{Y} \oplus \widetilde{Z}) \ominus (\widetilde{X} \oplus \widetilde{W}) = \overrightarrow{\widetilde{A}\widetilde{B}}$.

(2) The same way as in (1). □

Property 3.7. For $\overrightarrow{\tilde{X}\tilde{Y}}, \overrightarrow{\widetilde{W}\tilde{Z}}, \overrightarrow{\tilde{U}\tilde{V}} \in SFR$, $k, t \in \mathbb{R}$,

(1) $\overrightarrow{\tilde{X}\tilde{Y}} \oplus \overrightarrow{\widetilde{W}\tilde{Z}} = \overrightarrow{\widetilde{W}\tilde{Z}} \oplus \overrightarrow{\tilde{X}\tilde{Y}}$;

(2) $(\overrightarrow{\tilde{X}\tilde{Y}} \oplus \overrightarrow{\widetilde{W}\tilde{Z}}) \oplus \overrightarrow{\tilde{U}\tilde{V}} = \overrightarrow{\tilde{X}\tilde{Y}} \oplus (\overrightarrow{\widetilde{W}\tilde{Z}} \oplus \overrightarrow{\tilde{U}\tilde{V}})$;

(3) $\overrightarrow{\tilde{X}\tilde{Y}} \oplus \overrightarrow{\tilde{O}\tilde{O}} = \overrightarrow{\tilde{X}\tilde{Y}}$;

(4) $k_1 \odot (t_1 \odot \overrightarrow{\tilde{X}\tilde{Y}}) = (kt)_1 \odot \overrightarrow{\tilde{X}\tilde{Y}}$;

(5) $k_1 \odot (\overrightarrow{\tilde{X}\tilde{Y}} \oplus \overrightarrow{\widetilde{W}\tilde{Z}}) = (k_1 \odot \overrightarrow{\tilde{X}\tilde{Y}}) \oplus (k_1 \odot \overrightarrow{\widetilde{W}\tilde{X}})$;

(6) $1_1 \odot \overrightarrow{\tilde{X}\tilde{Y}} = \overrightarrow{\tilde{X}\tilde{Y}}$;

(7) $0_1 \odot \overrightarrow{\tilde{X}\tilde{Y}} = \overrightarrow{\tilde{O}\tilde{O}}$.

Proof. For each $\alpha \in [0,1]$ and from (iv), (v), and (vi),

(1) the α-cut of $\overrightarrow{\tilde{X}\tilde{Y}} \oplus \overrightarrow{\widetilde{W}\tilde{Z}} = (\tilde{Y} \ominus \tilde{X}) \oplus (\tilde{Z} - \widetilde{W})$ is

$$\begin{aligned} &\big(Y(\alpha)(-)X(\alpha)\big)(+)\big(Z(\alpha)(-)W(\alpha)\big) \\ &\quad = \big(Z(\alpha)(-)W(\alpha)\big)(+)\big(Y(\alpha)(-)X(\alpha)\big) \end{aligned} \tag{3.16}$$

which is the α-cut of $\overrightarrow{\widetilde{W}\tilde{Z}} \oplus \overrightarrow{\tilde{X}\tilde{Y}}$. Therefore (1) holds;

(2) the proof is similar to (1);

(3) since $\overrightarrow{\tilde{O}\tilde{O}} = (0,0,\dots,0)_1$, the α-cut of $\overrightarrow{\tilde{X}\tilde{Y}} \oplus \overrightarrow{\tilde{O}\tilde{O}} = (\tilde{Y} \ominus \tilde{X}) \oplus (0,0,\dots,0)_1$ is $(Y(\alpha)(-)X(\alpha))(+)(0,0,\dots,0) = Y(\alpha)(-)X(\alpha)$ which is the α-cut of $\overrightarrow{\tilde{X}\tilde{Y}}$. Therefore $\overrightarrow{\tilde{X}\tilde{Y}} \oplus \overrightarrow{\tilde{O}\tilde{O}} = \overrightarrow{\tilde{X}\tilde{Y}}$;

(4) for each $\alpha \in [0,1]$, the α-cut of $k_1 \odot (t_1 \odot \overrightarrow{\tilde{X}\tilde{Y}}) = k_1 \odot (t_1 \odot (\tilde{Y} \ominus \tilde{X}))$ is $k(\cdot)(t(\cdot)(Y(\alpha)(-)X(\alpha))) = (kt)(\cdot)(Y(\alpha)(-)X(\alpha))$ which is the α-cut of $(kt)_1(\cdot)\overrightarrow{\tilde{X}\tilde{Y}}$;

(5) the proof is similar to (4);

(6) the proof is similar to (5);

(7) from (v), (vi), for each $\alpha \in [0,1]$, the α-cut of $0_1 \odot \overrightarrow{\tilde{X}\tilde{Y}}$ is $(0,0,\dots,0)$ which is the α-cut of $\overrightarrow{\tilde{O}\tilde{O}}$.

In order to be a fuzzy vector space, it needs that the following hold:

(8) for any $\overrightarrow{\tilde{X}\tilde{Y}} \in SFR$, there exists $\overrightarrow{\widetilde{W}\tilde{Z}} \in SFR$ such that $\overrightarrow{\tilde{X}\tilde{Y}} \oplus \overrightarrow{\widetilde{W}\tilde{Z}} = \overrightarrow{\tilde{O}\tilde{O}}$,

(9) $(m+n)_1 \odot \overrightarrow{\tilde{X}\tilde{Y}} = (m_1 \odot \overrightarrow{\tilde{X}\tilde{Y}}) \oplus (n_1 \odot \overrightarrow{\tilde{X}\tilde{Y}})$ for all $\overrightarrow{\tilde{X}\tilde{Y}} \in SFR$.

Now since FE^n is a vector space, (8), (9) hold without any question. If $\tilde{X}, \tilde{Y} \in F_c$, but $\notin F_p^n(1)$ and $\overrightarrow{\tilde{X}\tilde{Y}} \neq \overrightarrow{\tilde{O}\tilde{O}}$, $\overrightarrow{\widetilde{W}\tilde{Z}} \neq \overrightarrow{\tilde{O}\tilde{O}}$. By (iv), (v), for each $\alpha \in [0,1]$ the α-cuts of $\overrightarrow{\tilde{X}\tilde{Y}}$ and $\overrightarrow{\widetilde{W}\tilde{Z}}$ are $Y(\alpha)(-)X(\alpha) \neq (0,0,\dots,0)$ and $Z(\alpha)(-)W(\alpha) \neq (0,0,\dots,0)$, respectively. The α-cut of $\overrightarrow{\tilde{X}\tilde{Y}} \oplus \overrightarrow{\widetilde{W}\tilde{Z}} = (\tilde{Y} \ominus \tilde{X}) \oplus (\tilde{Z} \ominus \widetilde{W})$ is

$$\big(Y(\alpha) - X(\alpha)\big)(+)\big(Z(\alpha)(-)W(\alpha)\big) \neq (0,0,\dots,0), \tag{3.17}$$

that is, there exists no $\overrightarrow{\widetilde{W}\tilde{Z}} \in SFR$ such that $\overrightarrow{\tilde{X}\tilde{Y}} \oplus \overrightarrow{\widetilde{W}\tilde{Z}} = \overrightarrow{\tilde{O}\tilde{O}}$. That is, (8) does not hold in this case.

For $\widetilde{X},\widetilde{Y} \in F_c$ but $\notin F_p^n(1)$, from (i), (ii), (iii), (iv), (v), and (vi), the α-cut $(0 \le \alpha \le 1)$ of $(m+n)_1 \odot \overrightarrow{\widetilde{X}\widetilde{Y}} = (m+n)_1 \odot (\widetilde{Y} \ominus \widetilde{X})$ is

$$\begin{aligned} &((m+n)Y(\alpha))(-)((m+n)X(\alpha)) \\ &\quad = \{((m+n)y^{(1)} - (m+n)x^{(1)}, (m+n)y^{(2)} - (m+n)x^{(2)}, \dots, \\ &\qquad (m+n)y^{(n)} - (m+n)x^{(n)}) \mid (x^{(1)}, x^{(2)}, \dots, x^{(n)}) \in X(\alpha), \\ &\qquad (y^{(1)}, y^{(2)}, \dots, y^{(n)}) \in Y(\alpha)\}. \end{aligned} \tag{3.18}$$

However, the α-cut of $(m_1 \odot \overrightarrow{\widetilde{X}\widetilde{Y}}) \oplus (n_1 \odot \overrightarrow{\widetilde{X}\widetilde{Y}}) = (m_1 \odot (\widetilde{Y} \ominus \widetilde{X})) \oplus (n_1 \odot (\widetilde{Y} \ominus \widetilde{X}))$ is

$$\begin{aligned} &m(Y(\alpha)(-)X(\alpha))(+)n(Y(\alpha)(-)X(\alpha)) \\ &\quad = \{(m(y^{(1)'} - x^{(1)'}) + n(y^{(1)''} - x^{(1)''}), m(y^{(2)'} - x^{(2)'}) \\ &\qquad + n(y^{(2)''} - x^{(2)''}), \dots, m(y^{(n)'} - x^{(n)'}) + n(y^{(n)''} - x^{(n)''})) \\ &\qquad \mid (x^{(1)'}, x^{(2)'}, \dots, x^{(n)'}), (x^{(1)''}, x^{(2)''}, \dots, x^{(n)''}) \in X(\alpha), \\ &\qquad (y^{(1)'}, y^{(2)'}, \dots, y^{(n)'}), (y^{(1)''}, y^{(2)''}, \dots, y^{(n)''}) \in Y(\alpha)\}. \end{aligned} \tag{3.19}$$

Therefore $(m+n)y^{(j)} - (m+n)x^{(j)} \ne m(y^{(j)'} - x^{(j)'}) + n(y^{(j)''} - x^{(j)''})$ if $(x^{(j)} \ne x^{(j)'}$ or $x^{(j)''})$ or $(y^{(j)} \ne y^{(j)'}$ or $y^{(j)''})$. Hence (9) does not hold. □

Definition 3.8. The SFR which satisfies Property 3.7(1), (2), (3), (4), (5), (6), and (7) is called pseudo-fuzzy vector space over $F_p^1(1)$, and call $\overrightarrow{\widetilde{X}\widetilde{Y}}$ $(\in SFR)$ a fuzzy vector.

Then $E^n \approx FE^n \subset SFR$. That is, we can regard SFR as an extension of E^n, but only obtain a pseudo-fuzzy vector space instead of a fuzzy vector space. Its addition $\oplus$ and multiplication $\odot$ are followed by Property 3.6.

Property 3.9. For $\overrightarrow{\widetilde{X}_j\widetilde{Y}_j} \in SFR$, $a_1^{(j)} \in F_p^1(1)$, $j = 1,2,\dots,m$,

$$\left(a_1^{(1)} \odot \overrightarrow{\widetilde{X}_1\widetilde{Y}_1}\right) \oplus \left(a_1^{(2)} \odot \overrightarrow{\widetilde{X}_2\widetilde{Y}_2}\right) \oplus \cdots \oplus \left(a_1^{(m)} \odot \overrightarrow{\widetilde{X}_m\widetilde{Y}_m}\right) = \overrightarrow{\widetilde{A}\widetilde{B}}; \tag{3.20}$$

here $\widetilde{A} = \widetilde{C}_1 \oplus \widetilde{C}_2 \oplus \cdots \oplus \widetilde{C}_m$, $\widetilde{B} = \widetilde{D}_1 \oplus \widetilde{D}_2 \oplus \cdots \oplus \widetilde{D}_m$, and $\widetilde{C}_j = a_1^{(j)} \odot \widetilde{X}_j$, $\widetilde{D}_j = a_1^{(j)} \odot \widetilde{Y}_j$, $j = 1,2,\dots,m$.

Proof. The proof follows from Property 3.6(1), (2) and mathematical induction. □

Property 3.10. For $\widetilde{Y} \in F_c$, but $\widetilde{Y} \notin F_p^n(1)$, $\widetilde{Y} \ne \widetilde{O}$, and $\widetilde{X} \in F_c$,

(1) $\widetilde{Y} \ominus \widetilde{Y} \ne \widetilde{O}$;

(2) $\overrightarrow{\widetilde{Y}\widetilde{Y}} \ne \overrightarrow{\widetilde{O}\widetilde{O}}$;

(3) $\overrightarrow{\tilde{X}\tilde{Y}} = \tilde{Y} \ominus \tilde{X} \neq \overrightarrow{\tilde{O}\tilde{O}}$;
(4) $\overrightarrow{\tilde{Y}\tilde{X}} = \tilde{X} \ominus \tilde{Y} \neq \overrightarrow{\tilde{O}\tilde{O}}$.

Proof. (1) Since $\tilde{Y} \neq \tilde{O}$, the α-cut of $\tilde{Y}$ is $Y(\alpha) \neq (0,0,\dots,0)$ for all $\alpha \in [0,1]$. By (ii), (v), the α-cut of $\tilde{Y} \ominus \tilde{Y}$ is

$$\begin{aligned} &Y(\alpha)(-)Y(\alpha) \\ &\quad = \big((s^{(1)} - t^{(1)}, s^{(2)} - t^{(2)}, \dots, s^{(n)} - t^{(n)}) \mid (s^{(1)}, s^{(2)}, \dots, s^{(n)}), \\ &\qquad (t^{(1)}, t^{(2)}, \dots, t^{(n)}) \in Y(\alpha)\big) \neq (0,0,\dots,0). \end{aligned} \tag{3.21}$$

Therefore $\tilde{Y} \ominus \tilde{Y} \neq \tilde{O}$.
(2) The proof follows by (1).
(3) The proof is similar to (1).
(4) The proof is similar to (1). □

Remark 3.11. (a) If $\tilde{Y} = (p^{(1)}, p^{(2)}, \dots, p^{(n)})_1 \in F_p^n(1)$, then $\tilde{Y} \ominus \tilde{Y} = (p^{(1)} - p^{(1)}, p^{(2)} - p^{(2)}, \dots, p^{(n)} - p^{(n)})_1 = (0,0,\dots,0)_1$. Hence $\overrightarrow{\tilde{Y}\tilde{Y}} = \overrightarrow{\tilde{O}\tilde{O}}$.
(b) It is trivial that $\overrightarrow{\tilde{O}\tilde{O}} \oplus \overrightarrow{\tilde{O}\tilde{O}} = \overrightarrow{\tilde{O}\tilde{O}}$.
(c) Let $SFV = \{\overrightarrow{\tilde{P}\tilde{X}} \mid$ for all $\tilde{P} = (p^{(1)}, p^{(2)}, \dots, p^{(n)})_1, \in F_p^n(1),\ \tilde{X} \in F_c\}$, then $E^n \approx FE^n \subset SFV \subset SFR$.

Property 3.12. For $\tilde{X} \in F_c$,
(1) $0_1 \odot \tilde{X} = \tilde{O}$;
(2) $\tilde{O} \oplus \tilde{X} = \tilde{X}$.

Proof. The proof is obvious. □

Example 3.13 ($n = 2$). A car carrying a rocket departs from point $Q = (1,2)$ passes through point $S = (5,8)$, arrives at point $W = (10,15)$, and launches the rocket from there. Suppose its target is located at $Z = (100,200)$. Chances are the rocket will not hit at Z exactly. Instead it would probably drop in the vicinity of Z. Let

$$O\big((100,200),1\big) = \big\{(x,y) \mid (x-100)^2 + (y-200)^2 \leq 1\big\}, \tag{3.22}$$

and the point hit is $\tilde{Z}$, ($\tilde{Z} \in F_c$) with membership function

$$\mu_{\tilde{Z}}(x,y) = \begin{cases} 1 - (x-100)^2 - (y-200)^2, & \text{if } (x,y) \in O\big((100,200),1\big), \\ 0, & \text{if } (x,y) \notin O\big((100,200),1\big). \end{cases} \tag{3.23}$$

The α-cut ($0 \leq \alpha \leq 1$) of $\tilde{Z}$ is

$$\begin{aligned} Z(\alpha) &= \big\{(x,y) \mid \mu_{\tilde{Z}}(x,y) \geq \alpha\big\} \\ &= \big\{(x,y) \mid (x-100)^2 + (y-200)^2 \leq 1 - \alpha\big\}. \end{aligned} \tag{3.24}$$

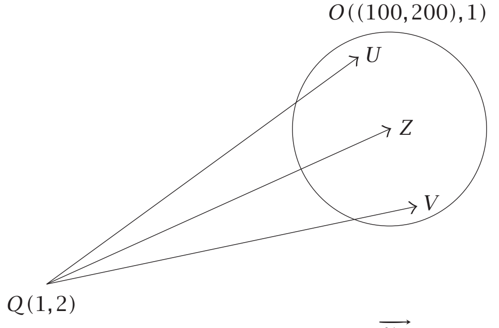

FIGURE 3.1. Fuzzy vector $\overrightarrow{\tilde{Q}\tilde{Z}}$.

If we choose the route from base to target to be $Q \to S \to W \to Z$, then we have the crisp vectors $\overrightarrow{QS} = (4,6)$, $\overrightarrow{SW} = (5,7)$, and $\overrightarrow{WZ} = (90,185)$. So the crisp vector from Q to Z is $\overrightarrow{QS} + \overrightarrow{SW} + \overrightarrow{WZ} = (99,198)$. And the route in the fuzzy sense is

$$\tilde{Q} = (1,2)_1 \longrightarrow \tilde{S} = (5,8)_1 \longrightarrow \widetilde{W} = (10,15)_1 \longrightarrow \tilde{Z} \tag{3.25}$$

with (3.23) as the membership function of $\tilde{Z}$. We then have fuzzy vectors $\overrightarrow{\tilde{Q}\tilde{S}} = (4,6)_1$, $\overrightarrow{\tilde{S}\widetilde{W}} = (5,7)_1$, and $\overrightarrow{\widetilde{W}\tilde{Z}} = \tilde{Z} \ominus \widetilde{W}$. From (3.23), the membership function of $\overrightarrow{\widetilde{W}\tilde{Z}}$ is

$$\begin{aligned} \mu_{\overrightarrow{\widetilde{W}\tilde{Z}}}(x,y) &= \mu_{\tilde{Z}}(x+10,y+15) \\ &= \begin{cases} 1-(x-90)^2-(y-185)^2, & \text{if } (x-90)^2+(y-185)^2 \le 1, \\ 0, & \text{elsewhere}. \end{cases} \end{aligned} \tag{3.26}$$

The fuzzy vector from base $\tilde{Q}$ to target $\tilde{Z}$ by Property 3.6(1) is $\overrightarrow{\tilde{Q}\tilde{S}} \oplus \overrightarrow{\tilde{S}\widetilde{W}} \oplus \overrightarrow{\widetilde{W}\tilde{Z}} = \overrightarrow{\tilde{Q}\tilde{Z}}$, with membership function

$$\begin{aligned} \mu_{\overrightarrow{\tilde{Q}\tilde{Z}}}(x,y) &= \mu_{\tilde{Z}}(x+1,y+2) \\ &= \begin{cases} 1-(x-99)^2-(y-198)^2, & \text{if } (x-99)^2+(y-198)^2 \le 1, \\ 0, & \text{elsewhere}. \end{cases} \end{aligned} \tag{3.27}$$

Let $Z = (100,200)$, $U = (99.5,200.5)$, and $V = (100.2,199.7) \in O((100,200),1)$.

As shown in Figure 3.1, the crisp vectors from Q to Z, U, V in $O((100,200),1)$ are $\overrightarrow{QZ} = (99,198)$, $\overrightarrow{QU} = (98.5,198.5)$, and $\overrightarrow{QV} = (99.2,197.7)$, respectively. The grades of membership of these crisp vectors belonging to fuzzy vector

$\overrightarrow{\tilde{Q}\tilde{Z}}$ are

$$\mu_{\overrightarrow{\tilde{Q}\tilde{Z}}}\left(\overrightarrow{QZ}\right) = \mu_{\overrightarrow{\tilde{Q}\tilde{Z}}}(99,198) = 1,$$
$$\mu_{\overrightarrow{\tilde{Q}\tilde{Z}}}\left(\overrightarrow{QU}\right) = \mu_{\overrightarrow{\tilde{Q}\tilde{Z}}}(98.5,198.5) = 0.5, \tag{3.28}$$
$$\mu_{\overrightarrow{\tilde{Q}\tilde{Z}}}\left(\overrightarrow{QV}\right) = \mu_{\overrightarrow{\tilde{Q}\tilde{Z}}}(99.2,197.7) = 0.87.$$

4. The length of fuzzy vectors in SFR and fuzzy inner product

4.1. The length of fuzzy vectors in SFR. Let $P = (p^{(1)}, p^{(2)}, \dots, p^{(n)})$, $Q = (q^{(1)}, q^{(2)}, \dots, q^{(n)}) \in \mathbb{R}^n$. The vector $\overrightarrow{PQ}$ in E^n, $\overrightarrow{PQ} = (q^{(1)} - p^{(1)}, q^{(2)} - p^{(2)}, \dots, q^{(n)} - p^{(n)}) = Q(-)P$, has length $|\overrightarrow{PQ}| = \sqrt{\sum_{j=1}^n (q^{(j)} - p^{(j)})^2}$, called the length of the vector $\overrightarrow{PQ}$. Now let $\tilde{P} = (p^{(1)}, p^{(2)}, \dots, p^{(n)})_1$ and $\tilde{Q} = (q^{(1)}, q^{(2)}, \dots, q^{(n)})_1 \in F_p^n(1)$. Since $FE^n \approx E^n$, we can define the length of fuzzy vector $\overrightarrow{\tilde{P}\tilde{Q}} = (q^{(1)} - p^{(1)}, q^{(2)} - p^{(2)}, \dots, q^{(n)} - p^{(n)})_1$ by $|\overrightarrow{\tilde{P}\tilde{Q}}| = |\overrightarrow{PQ}| = \sqrt{\sum_{j=1}^n (q^{(j)} - p^{(j)})^2}$.

Since $FE^n \subset SFR$, we may extend this thought to SFR. Similar to the fuzzy vectors in FE^n, for the vector $\overrightarrow{\tilde{X}\tilde{Y}} = \tilde{Y} \ominus \tilde{X} \in SFR$, its α-cut $(0 \le \alpha \le 1)$ is

$$\begin{aligned} Y(\alpha)(-)X(\alpha) = \{ & (y^{(1)} - x^{(1)}, y^{(2)} - x^{(2)}, \dots, y^{(n)} - x^{(n)}) \\ & \mid (x^{(1)}, x^{(2)}, \dots, x^{(n)}) \in X(\alpha), \\ & (y^{(1)}, y^{(2)}, \dots, y^{(n)}) \in Y(\alpha)\}, \end{aligned} \tag{4.1}$$

where $X(\alpha)$, $Y(\alpha)$ are the α-cuts of $\tilde{X}$, $\tilde{Y}$, respectively. For each point $P_X(\alpha) = (x^{(1)}, x^{(2)}, \dots, x^{(n)}) \in X(\alpha)$ and $P_Y(\alpha) = (y^{(1)}, y^{(2)}, \dots, y^{(n)}) \in Y(\alpha)$, the crisp vector $\overrightarrow{P_X(\alpha)P_Y(\alpha)} = (y^{(1)} - x^{(1)}, y^{(2)} - x^{(2)}, \dots, y^{(n)} - x^{(n)})$ has length

$$\left|\overrightarrow{P_X(\alpha)P_Y(\alpha)}\right| = \sqrt{\sum_{j=1}^n \left(y^{(j)} - x^{(j)}\right)^2} \tag{4.2}$$

which is the distance between two points $P_X(\alpha)$ and $P_Y(\alpha)$. For any $(x^{(1)}, x^{(2)}, \dots, x^{(n)}) \in X(\alpha)$, denoted simply by $(x^{(j)}) \in X(\alpha)$, let

$$\begin{aligned} d^*\left(Y(\alpha)(-)X(\alpha)\right) &= \sup_{(x^{(j)}) \in X(\alpha), (y^{(j)}) \in Y(\alpha)} \left|\overrightarrow{P_X(\alpha)P_Y(\alpha)}\right| \\ &= \sup_{(x^{(j)}) \in X(\alpha), (y^{(j)}) \in Y(\alpha)} \sqrt{\sum_{j=1}^n (y^{(j)} - x^{(j)})^2}. \end{aligned} \tag{4.3}$$

Since $\tilde{X}, \tilde{Y} \in F_c$, and by Definition 2.3(a), $X(\alpha)$, $Y(\alpha)$ are convex closed subsets of $\mathbb{R}^n$, so $d^*(Y(\alpha)(-)X(\alpha))$ exists. And $d^*(Y(\alpha)(-)X(\alpha))$ is the longest one among all the distances between points $P_X(\alpha)$ in $X(\alpha)$ and $P_Y(\alpha)$ in $Y(\alpha)$, which makes sense for using this as the distance between $X(\alpha)$ and $Y(\alpha)$. Therefore, we have the following definition.

Definition 4.1. For $\overrightarrow{\tilde{X}\tilde{Y}} \in SFR$, define the length of $\overrightarrow{\tilde{X}\tilde{Y}}$ to be

$$\left|\overrightarrow{\tilde{X}\tilde{Y}}\right|^* = \sup_{0\le\alpha\le1} d^*(Y(\alpha)(-)X(\alpha)). \tag{4.4}$$

Property 4.2. For $\overrightarrow{\tilde{X}\tilde{Y}} \in SFR$, let the 0-cuts ($\alpha$-cuts, $\alpha = 0$) of $\tilde{X}, \tilde{Y}$ be $X(0)$, $Y(0)$ by Remark 2.4. Then there exist $(x_m^{(1)}(0), x_m^{(2)}(0), \dots, x_m^{(n)}(0)) \in X(0)$, $(y_m^{(1)}(0), y_m^{(2)}(0), \dots, y_m^{(n)}(0)) \in Y(0)$ such that

$$\left|\overrightarrow{\tilde{X}\tilde{Y}}\right|^* = \sup_{0\le\alpha\le1} d^*(Y(\alpha)(-)X(\alpha)) = \sqrt{\sum_{j=1}^{n}\left(y_m^{(j)}(0) - x_m^{(j)}(0)\right)^2}. \tag{4.5}$$

Proof. Let the α-cuts $(0 \le \alpha \le 1)$ of $\tilde{X}$, $\tilde{Y}$ be

$$\begin{aligned} X(\alpha) &= \{(x^{(1)}, x^{(2)}, \dots, x^{(n)}) \mid \mu_{\tilde{X}}(x^{(1)}, x^{(2)}, \dots, x^{(n)}) \ge \alpha\}, \\ Y(\alpha) &= \{(y^{(1)}, y^{(2)}, \dots, y^{(n)}) \mid \mu_{\tilde{Y}}(y^{(1)}, y^{(2)}, \dots, y^{(n)}) \ge \alpha\}. \end{aligned} \tag{4.6}$$

It is obvious that $X(\alpha) \subset X(\beta)$, $Y(\alpha) \subset Y(\beta)$ if $0 \le \beta \le \alpha \le 1$.

Since

$$d^*(Y(\alpha)(-)X(\alpha)) = \sup_{(x^{(j)})\in X(\alpha),\,(y^{(j)})\in Y(\alpha)} \sqrt{\sum_{j=1}^{n}\left(y^{(j)} - x^{(j)}\right)^2}, \tag{4.7}$$

we have

$$d^*(Y\alpha)(-)X(\alpha) \le d^*(Y(\beta)(-)X(\beta)) \quad \forall 0 \le \beta \le \alpha \le 1. \tag{4.8}$$

So

$$\begin{aligned} &\sup_{0\le\alpha\le1} d^*(Y(\alpha)(-)X(\alpha)) \\ &\quad= \sup_{(x^{(j)}(0))\in X(0),\,(y^{(j)}(0))\in Y(0)} \sqrt{\sum_{j=1}^{n}\left(y^{(j)}(0) - x^{(j)}(0)\right)^2}. \end{aligned} \tag{4.9}$$

Since $\tilde{X}, \tilde{Y} \in F_c$, by the definition of F_c, we know that $X(0)$, $Y(0)$ are convex closed subsets of $\mathbb{R}^n$. Hence there exist $(x_m^{(1)}(0), x_m^{(2)}(0), \dots, x_m^{(n)}(0)) \in X(0)$, $(y_m^{(1)}(0), y_m^{(2)}(0), \dots, y_m^{(n)}(0)) \in Y(0)$ such that

$$\left|\overrightarrow{\tilde{X}\tilde{Y}}\right|^* = \sup_{0\le\alpha\le1} d^*(Y(\alpha)(-)X(\alpha)) = \sqrt{\sum_{j=1}^{n}\left(y_m^{(j)}(0) - x_m^{(j)}(0)\right)^2}. \tag{4.10}$$

□

Example 4.3. In Example 3.13, the rocket ejected at W takes the route $Q = (1,2) \to S = (5,8) \to W = (10,15)$ and aims at $Z = (100,200)$. The membership function of fuzzy target $\tilde{Z}$ is

$$\mu_{\tilde{Z}}(x,y)=\begin{cases}1-(x-100)^2-(y-200)^2, & \text{if } (x-100)^2+(y-200)^2\le 1,\\ 0, & \text{elsewhere.}\end{cases} \tag{4.11}$$

We obtain Fuzzy vectors $\overrightarrow{\tilde{Q}\tilde{S}}=(4,6)_1$, $\overrightarrow{\tilde{S}\tilde{W}}=(5,7)_1$, $\overrightarrow{\tilde{W}\tilde{Z}}$, and $\overrightarrow{\tilde{Q}\tilde{Z}}$. The former two have lengths $|\overrightarrow{\tilde{Q}\tilde{S}}|^*=\sqrt{4^2+6^2}=7.21$ and $|\overrightarrow{\tilde{S}\tilde{W}}|^*=\sqrt{5^2+7^2}=8.6$, respectively. As for the length of $\overrightarrow{\tilde{W}\tilde{Z}}$, since for each $\alpha\in[0,1]$, the α-cuts of $\tilde{W}$, $\tilde{Z}$ are $W(\alpha)=(10,15)$, $Z(\alpha)=\{(x,y)\mid(x-100)^2+(y-200)^2\le 1-\alpha\}$, respectively. The longest distance between points $P_{W(\alpha)}=(10,15)\in W(\alpha)$ and $P_{Z(\alpha)}=(x,y)\in Z(\alpha)$ is

$$\begin{aligned}d^*(Z(\alpha)(-)W(\alpha))&=\sqrt{(100-10)^2+(200-15)^2}+\sqrt{1-\alpha}\\&=\sqrt{42325}+\sqrt{1-\alpha}=205.73+\sqrt{1-\alpha}.\end{aligned} \tag{4.12}$$

Hence by Definition 4.1 $|\overrightarrow{\tilde{W}\tilde{Z}}|^*=\sup_{0\le\alpha\le 1}d^*(Z(\alpha)(-)W(\alpha))=206.73$.

Similarly, we can calculate the length of $\overrightarrow{\tilde{Q}\tilde{Z}}$: for each $\alpha\in[0,1]$, the α-cut of $\tilde{Q}$ is $Q(\alpha)=(1,2)$, so

$$\begin{aligned}d^*(Z(\alpha)(-)Q(\alpha))&=\sqrt{(100-1)^2+(200-2)^2}+\sqrt{1-\alpha}\\&=\sqrt{49005}+\sqrt{1-\alpha}=221.37+\sqrt{1-\alpha}.\end{aligned} \tag{4.13}$$

Therefore $|\overrightarrow{\tilde{Q}\tilde{Z}}|^*=\sup_{0\le\alpha\le 1}d^*(Z(\alpha)(-)Q(\alpha))=222.37$.

Remark 4.4. By Cauchy-Schwartz inequality

$$\left(a_1b_1+a_2b_2+\cdots+a_nb_n\right)^2\le\left(a_1^2+a_2^2+\cdots+a_n^2\right)\left(b_1^2+b_2^2+\cdots+b_n^2\right), \tag{4.14}$$

we have

$$\sum_{j=1}^n a_jb_j\le\left|\sum_{j=1}^n a_jb_j\right|\le\sqrt{\sum_{j=1}^n a_j^2\sum_{j=1}^n b_j^2}. \tag{4.15}$$

Therefore,

$$\sum_{j=1}^n a_j^2+\sum_{j=1}^n b_j^2+2\sum_{j=1}^n a_jb_j\le\sum_{j=1}^n a_j^2+\sum_{j=1}^n b_j^2+2\sqrt{\sum_{j=1}^n a_j^2\sum_{j=1}^n b_j^2}, \tag{4.16}$$

that is,

$$\sqrt{\sum_{j=1}^n\left(a_j+b_j\right)^2}\le\sqrt{\sum_{j=1}^n a_j^2}+\sqrt{\sum_{j=1}^n b_j^2}. \tag{4.17}$$

Property 4.5. For $\overrightarrow{\widetilde{X}\widetilde{Y}}, \overrightarrow{\widetilde{U}\widetilde{V}} \in SFR$, $k_1 \in F_p^1(1)$, $k \neq 0$,

(1) $|k_1 \odot \overrightarrow{\widetilde{X}\widetilde{Y}}|^* = |k| \cdot |\overrightarrow{\widetilde{X}\widetilde{Y}}|^*$;

(2) $|\overrightarrow{\widetilde{X}\widetilde{Y}} \oplus \overrightarrow{\widetilde{W}\widetilde{Z}}|^* \leq |\overrightarrow{\widetilde{X}\widetilde{Y}}|^* + |\overrightarrow{\widetilde{W}\widetilde{Z}}|^*$.

Proof. (1) For each $\alpha \in [0,1]$, the α-cut of $k_1 \odot \overrightarrow{\widetilde{X}\widetilde{Y}} = k_1 \odot (\widetilde{Y}(-)\widetilde{X})$ is

$$\begin{aligned} &k(\cdot)(Y(\alpha)(-)X(\alpha)) \\ &\quad = \{(ky^{(1)} - kx^{(1)}, ky^{(2)} - kx^{(2)}, \dots, ky^{(n)} - kx^{(n)}) \\ &\qquad \mid (x^{(1)}, x^{(2)}, \dots, x^{(n)}) \in X(\alpha),\ (y^{(1)}, y^{(2)}, \dots, y^{(n)}) \in Y(\alpha)\}. \end{aligned} \tag{4.18}$$

For each $\alpha \in [0,1]$,

$$\begin{aligned} d^*&(k(\cdot)(Y(\alpha)(-)X(\alpha))) \\ &= \sup_{(x^{(j)}) \in X(\alpha), (y^{(j)}) \in Y(\alpha)} \sqrt{\sum_{j=1}^{n} (ky^{(j)} - kx^{(j)})^2} \\ &= |k| d^*(Y(\alpha) - X(\alpha)). \end{aligned} \tag{4.19}$$

Therefore,

$$\begin{aligned} \left| k_1 \odot \overrightarrow{\widetilde{X}\widetilde{Y}} \right|^* &= \sup_{0 \leq \alpha \leq 1} d^*(k(\cdot)(Y(\alpha)(-)X(\alpha))) \\ &= |k| \sup_{0 \leq \alpha \leq 1} d^*(Y(\alpha)(-)X(\alpha)) = |k| \cdot \left| \overrightarrow{\widetilde{X}\widetilde{Y}} \right|^*. \end{aligned} \tag{4.20}$$

(2) From Property 3.6(1), $\overrightarrow{\widetilde{X}\widetilde{Y}} \oplus \overrightarrow{\widetilde{W}\widetilde{Z}} = \overrightarrow{\widetilde{A}\widetilde{B}} = \widetilde{B} \ominus \widetilde{A}$, where $\widetilde{A} = \widetilde{X} \oplus \widetilde{W}$, $\widetilde{B} = \widetilde{Y} \oplus \widetilde{Z}$. For each $\alpha \in [0,1]$, the α-cuts of $\widetilde{X}$, $\widetilde{Y}$, $\widetilde{W}$, and $\widetilde{Z}$ are $X(\alpha)$, $Y(\alpha)$, $W(\alpha)$, and $Z(\alpha)$, respectively, and the α-cut of $\widetilde{B} \ominus \widetilde{A}$ is

$$\begin{aligned} &(Y(\alpha)(+)Z(\alpha))(-)(X(\alpha)(+)W(\alpha)) \\ &\quad = \{(y^{(1)} + z^{(1)} - x^{(1)} - w^{(1)}, y^{(2)} + z^{(2)} - x^{(2)} - w^{(2)}, \dots, \\ &\qquad y^{(n)} + z^{(n)} - x^{(n)} - w^{(n)}) \mid (x^{(1)}, x^{(2)}, \dots, x^{(n)}) \in X(\alpha), \\ &\qquad (y^{(2)}, y^{(2)}, \dots, y^{(n)}) \in Y(\alpha),\ (z^{(1)}, z^{(2)}, \dots, z^{(n)}) \in Z(\alpha), \\ &\qquad (w^{(1)}, w^{(2)}, \dots, w^{(n)}) \in W(\alpha)\}. \end{aligned} \tag{4.21}$$

For each $\alpha \in [0,1]$, let

$$\begin{aligned} D = \{&(x^{(1)}, x^{(2)}, \dots, x^{(n)}) \in X(\alpha),\ (y^{(1)}, y^{(2)}, \dots, y^{(n)}) \in Y(\alpha), \\ &(z^{(1)}, z^{(2)}, \dots, z^{(n)}) \in Z(\alpha),\ (w^{(1)}, w^{(2)}, \dots, w^{(n)}) \in W(\alpha)\}. \end{aligned} \tag{4.22}$$

From Remark 4.4,

$$\sqrt{\sum_{j=1}^{n} (a_j + b_j)^2} \leq \sqrt{\sum_{j=1}^{n} a_j^2} + \sqrt{\sum_{j=1}^{n} b_j^2} \tag{4.23}$$

and inequality $\sup(A+B) \le \sup A + \sup B$, we have

$$\begin{aligned}
&d^*\big((Y(\alpha)(+)Z(\alpha))(-)(X(\alpha)(+)W(\alpha))\big)\\
&\quad = \sup_D \left\{\sum_{j=1}^n \left(y^{(j)}+z^{(j)}-x^{(j)}-w^{(j)}\right)^2\right\}^{1/2}\\
&\quad \le \sup_{(x^{(j)})\in X(\alpha),(y^{(j)})\in Y(\alpha)} \sqrt{\sum_{j=1}^n \left(y^{(j)}-x^{(j)}\right)^2}\\
&\qquad + \sup_{(w^{(j)})\in W(\alpha),(z^{(j)})\in Z(\alpha)} \sqrt{\sum_{j=1}^n \left(z^{(j)}-w^{(j)}\right)^2}\\
&\quad = d^*\big(Y(\alpha)(-)X(\alpha)\big)+d^*\big(Z(\alpha)(-)W(\alpha)\big).
\end{aligned} \tag{4.24}$$

By Definition 4.1, we have $|\overrightarrow{\tilde{X}\tilde{Y}} \oplus \overrightarrow{\tilde{W}\tilde{Z}}|^* \le |\overrightarrow{\tilde{X}\tilde{Y}}|^* + |\overrightarrow{\tilde{W}\tilde{Z}}|^*$. □

4.2. The fuzzy inner product and the angle between fuzzy vectors for the fuzzy vectors in SFR. Corresponding to the equation

$$\begin{aligned}
&d'\big(Y(\alpha)(-)X(\alpha),V(\alpha)(-)U(\alpha)\big)\\
&\quad = \sup_{(x^{(j)})\in X(\alpha),(y^{(j)})\in Y(\alpha),(u^{(j)})\in U(\alpha),(v^{(j)})\in V(\alpha)} \sum_{j=1}^n \left(y^{(j)}-x^{(j)}\right)\left(v^{(j)}-u^{(j)}\right),
\end{aligned} \tag{4.25}$$

we define the fuzzy inner product as follows.

Definition 4.6. For $\overrightarrow{\tilde{X}\tilde{Y}}, \overrightarrow{\tilde{U}\tilde{V}} \in SFR$, define the fuzzy inner product of them to be

$$\overrightarrow{\tilde{X}\tilde{Y}} \odot^* \overrightarrow{\tilde{U}\tilde{V}} = \sup_{0\le\alpha\le 1} d'\big(Y(\alpha)(-)X(\alpha),V(\alpha)(-)U(\alpha)\big). \tag{4.26}$$

Property 4.7. For $\overrightarrow{\tilde{X}\tilde{Y}}, \overrightarrow{\tilde{U}\tilde{V}} \in SFR$, let the 0-cuts ($\alpha$-cuts, $\alpha = 0$) of $\tilde{X}$, $\tilde{Y}$, $\tilde{U}$, and $\tilde{V}$ be $X(0)$, $Y(0)$, $U(0)$, and $V(0)$, respectively. Then there exist

$$\begin{aligned}
&\left(x_m^{(1)}(0),x_m^{(2)}(0),\dots,x_m^{(n)}(0)\right) \in X(0),\\
&\left(y_m^{(1)}(0),y_m^{(2)}(0),\dots,y_m^{(n)}(0)\right) \in Y(0),\\
&\left(u_m^{(1)}(0),u_m^{(2)}(0),\dots,u_m^{(n)}(0)\right) \in U(0),\\
&\left(v_m^{(1)}(0),v_m^{(2)}(0),\dots,v_m^{(n)}(0)\right) \in V(0),
\end{aligned} \tag{4.27}$$

such that

$$\overrightarrow{\tilde{X}\tilde{Y}} \odot^* \overrightarrow{\tilde{U}\tilde{V}} = \sum_{j=1}^n \left(y_m^{(j)}(0)-x_m^{(j)}(0)\right)\left(v_m^{(j)}(0)-u_m^{(j)}(0)\right). \tag{4.28}$$

Proof. Use the same way of the proof of Property 4.2 to prove it. □

Property 4.8. For $\overrightarrow{\tilde{X}\tilde{Y}}, \overrightarrow{\tilde{U}\tilde{V}}, \overrightarrow{\tilde{W}\tilde{Z}} \in SFR$, $k_1 \in F_p^1(1)$, $k > 0$,

(1) $\overrightarrow{\tilde{X}\tilde{Y}} \odot^* \overrightarrow{\tilde{U}\tilde{V}} = \overrightarrow{\tilde{U}\tilde{V}} \odot^* \overrightarrow{\tilde{X}\tilde{Y}}$;

(2) $\overrightarrow{\tilde{X}\tilde{Y}} \odot^* (\overrightarrow{\tilde{U}\tilde{V}} \oplus \overrightarrow{\tilde{W}\tilde{Z}}) \leq (\overrightarrow{\tilde{X}\tilde{Y}} \odot^* \overrightarrow{\tilde{U}\tilde{V}}) \oplus (\overrightarrow{\tilde{X}\tilde{Y}} \odot^* \overrightarrow{\tilde{W}\tilde{Z}})$;

(3) $k(\overrightarrow{\tilde{X}\tilde{Y}} \odot^* \overrightarrow{\tilde{U}\tilde{V}}) = (k_1 \odot \overrightarrow{\tilde{X}\tilde{Y}}) \odot^* \overrightarrow{\tilde{U}\tilde{V}} = \overrightarrow{\tilde{X}\tilde{Y}} \odot^* (k_1 \odot \overrightarrow{\tilde{U}\tilde{V}})$;

(4) $\overrightarrow{\tilde{X}\tilde{Y}} \odot^* \overrightarrow{\tilde{X}\tilde{Y}} = |\overrightarrow{\tilde{X}\tilde{Y}}|^{*2}$;

(5) $|\overrightarrow{\tilde{X}\tilde{Y}} \odot^* \overrightarrow{\tilde{U}\tilde{V}}| \leq |\overrightarrow{\tilde{X}\tilde{Y}}|^* \cdot |\overrightarrow{\tilde{U}\tilde{V}}|^*$.

Proof. (1) By Property 4.7,

$$\begin{aligned}\overrightarrow{\tilde{X}\tilde{Y}} \odot^* \overrightarrow{\tilde{U}\tilde{V}} &= \sum_{j=1}^{n} \left(y_m^{(j)}(0) - x_m^{(j)}(0)\right)\left(v_m^{(j)}(0) - u_m^{(j)}(0)\right) \\ &= \sum_{j=1}^{n} \left(v_m^{(j)}(0) - u_m^{(j)}(0)\right)\left(y_m^{(j)}(0) - x_m^{(j)}(0)\right) = \overrightarrow{\tilde{U}\tilde{V}} \odot^* \overrightarrow{\tilde{X}\tilde{Y}}.\end{aligned} \tag{4.29}$$

(2) By Definition 4.6 and Property 3.6(1), $\overrightarrow{\tilde{U}\tilde{V}} \oplus \overrightarrow{\tilde{W}\tilde{Z}} = \overrightarrow{\tilde{A}\tilde{B}} = \tilde{B} \ominus \tilde{A}$, where $\tilde{A} = \tilde{U} \oplus \tilde{W}$, $\tilde{B} = \tilde{V} \oplus \tilde{Z}$. The α-cut of $\tilde{B} \ominus \tilde{A}$ is $(V(\alpha)(+)Z(\alpha))(-)(U(\alpha)(+)W(\alpha))$. Hence for each $\alpha \in [0,1]$, set

$$\begin{aligned} H = \{&(x^{(1)}, x^{(2)}, \dots, x^{(n)}) \in X(\alpha),\ (y^{(1)}, y^{(2)}, \dots, y^{(n)}) \in Y(\alpha), \\ &(u^{(1)}, u^{(2)}, \dots, u^{(n)}) \in U(\alpha),\ (v^{(1)}, v^{(2)}, \dots, v^{(n)}) \in V(\alpha), \\ &(w^{(1)}, w^{(2)}, \dots, w^{(n)}) \in W(\alpha),\ (z^{(1)}, z^{(2)}, \dots, z^{(n)}) \in Z(\alpha)\}, \\ E = \{&(x^{(1)}, x^{(2)}, \dots, x^{(n)}) \in X(\alpha),\ (y^{(1)}, y^{(2)}, \dots, y^{(n)}) \in Y(\alpha), \\ &(u^{(1)}, u^{(2)}, \dots, u^{(n)}) \in U(\alpha),\ (v^{(1)}, v^{(2)}, \dots, v^{(n)}) \in V(\alpha)\}, \\ D = \{&(x^{(1)}, x^{(2)}, \dots, x^{(n)}) \in X(\alpha),\ (y^{(1)}, y^{(2)}, \dots, y^{(n)}) \in Y(\alpha), \\ &(w^{(1)}, w^{(2)}, \dots, w^{(n)}) \in W(\alpha),\ (z^{(1)}, z^{(2)}, \dots, z^{(n)}) \in Z(\alpha)\}. \end{aligned} \tag{4.30}$$

Then, for each $\alpha \in [0,1]$,

$$\begin{aligned} &d'\left(Y(\alpha)(-)X(\alpha), (V(\alpha)(+)Z(\alpha))(-)(U(\alpha)(+)W(\alpha))\right) \\ &\quad = \sup_H \sum_{j=1} \left(y^{(j)} - x^{(j)}\right)\left(v^{(j)} + z^{(j)} - u^{(j)} - w^{(j)}\right) \\ &\quad \leq \sup_E \sum_{j=1}^{n} \left(y^{(j)} - x^{(j)}\right)\left(v^{(j)} - u^{(j)}\right) \\ &\qquad + \sup_D \sum_{j=1}^{n} \left(y^{(j)} - x^{(j)}\right)\left(z^{(j)} - w^{(j)}\right) \\ &\quad = d'\left(Y(\alpha)(-)X(\alpha), V(\alpha)(-)U(\alpha)\right) \\ &\qquad + d'\left(Y(\alpha)(-)X(\alpha), Z(\alpha)(-)W(\alpha)\right). \end{aligned} \tag{4.31}$$

By Definition 4.6, (2) is proved.

(3) The proof follows from Property 3.6(2) and (v), (vi).

(4) The proof follows from Properties 4.2 and 4.7.

(5) For each α, where $0 \le \alpha \le 1$,

$$\begin{aligned} &(x^{(1)},x^{(2)},\dots,x^{(n)}) \in X(\alpha), \qquad (y^{(1)},y^{(2)},\dots,y^{(n)}) \in Y(\alpha),\\ &(u^{(1)},u^{(2)},\dots,u^{(n)}) \in U(\alpha), \qquad (v^{(1)},v^{(2)},\dots,v^{(n)}) \in V(\alpha), \end{aligned} \tag{4.32}$$

by Cauchy-Schwartz inequality, that is,

$$\begin{aligned} &-\sqrt{\sum_{j=1}^{n}\left(y^{(j)}-x^{(j)}\right)^2}\sqrt{\sum_{j=1}^{n}\left(v^{(j)}-u^{(j)}\right)^2}\\ &\qquad \le \sum_{j=1}^{n}\left(y^{(j)}-x^{(j)}\right)\left(v^{(j)}-u^{(j)}\right)\\ &\qquad \le \sqrt{\sum_{j=1}^{n}\left(y^{(j)}-x^{(j)}\right)^2}\sqrt{\sum_{j=1}^{n}\left(v^{(j)}-u^{(j)}\right)^2} \end{aligned} \tag{4.33}$$

since $\sup AB \le \sup A \sup B$, if $A > 0$, $B > 0$. Then, we have

$$\begin{aligned} &-d^*\left(Y(\alpha)(-)X(\alpha)\right)d^*\left(V(\alpha)(-)U(\alpha)\right)\\ &\qquad \le d'\left(Y(\alpha)(-)X(\alpha),V(\alpha)(-)U(\alpha)\right)\\ &\qquad \le d^*\left(Y(\alpha)(-)X(\alpha)\right)d^*\left(V(\alpha)(-)U(\alpha)\right), \quad \forall \alpha \in [0,1]. \end{aligned} \tag{4.34}$$

Therefore, $-|\overrightarrow{\tilde{X}\tilde{Y}}|^* \cdot |\overrightarrow{\tilde{U}\tilde{V}}|^* \le |\overrightarrow{\tilde{X}\tilde{Y}} \odot^* \overrightarrow{\tilde{U}\tilde{V}}| \le |\overrightarrow{\tilde{X}\tilde{Y}}|^* \cdot |\overrightarrow{\tilde{U}\tilde{V}}|^*$. Hence, $|\overrightarrow{\tilde{X}\tilde{Y}} \odot^* \overrightarrow{\tilde{U}\tilde{V}}| \le |\overrightarrow{\tilde{X}\tilde{Y}}|^* \cdot |\overrightarrow{\tilde{U}\tilde{V}}|^*$. □

Remark 4.9. If $|\overrightarrow{\tilde{X}\tilde{Y}}|^* > 0$ and $|\overrightarrow{\tilde{U}\tilde{V}}|^* > 0$, by Property 4.8(5),

$$-1 \le \frac{\overrightarrow{\tilde{X}\tilde{Y}} \odot^* \overrightarrow{\tilde{U}\tilde{V}}}{\left|\overrightarrow{\tilde{X}\tilde{Y}}\right|^* \cdot \left|\overrightarrow{\tilde{U}\tilde{V}}\right|^*} \le 1. \tag{4.35}$$

So we have the following definition.

Definition 4.10. For $\overrightarrow{\tilde{X}\tilde{Y}}, \overrightarrow{\tilde{U}\tilde{V}} \in SFR$, if $\overrightarrow{\tilde{X}\tilde{Y}} \ne \overrightarrow{\tilde{O}\tilde{O}}$, $\overrightarrow{\tilde{U}\tilde{V}} \ne \overrightarrow{\tilde{O}\tilde{O}}$, define the angle θ between $\overrightarrow{\tilde{X}\tilde{Y}}$ and $\overrightarrow{\tilde{U}\tilde{V}}$ by

$$\cos\theta = \frac{\overrightarrow{\tilde{X}\tilde{Y}} \odot^* \overrightarrow{\tilde{U}\tilde{V}}}{\left|\overrightarrow{\tilde{X}\tilde{Y}}\right|^* \cdot \left|\overrightarrow{\tilde{U}\tilde{V}}\right|^*}. \tag{4.36}$$

Example 4.11 $(n = 2)$**.** Eject the rocket from $(2,3)$ aiming at $(6,8)$. The rocket falls in the circle centered at $(6,8)$ with radius 2. Also eject another rocket aiming at (10,4), the rocket falls in the circle centered at (10,4) with radius 1.

Then we have the membership functions of the fuzzy sets $\tilde{X}$, $\tilde{Y}$, $\tilde{U}$, and $\tilde{V}$:

$$\mu_{\tilde{X}}(2,3) = \begin{cases} 1, & \text{if } x^{(1)} = 2,\ x^{(2)} = 3, \\ 0, & \text{elsewhere,} \end{cases}$$
$$\mu_{\tilde{Y}}(y^{(1)}, y^{(2)}) = \begin{cases} \frac{1}{4}\left[4 - (y^{(1)} - 6)^2 - (y^{(2)} - 8)^2\right], & \text{if } (y^{(1)} - 6)^2 - (y^{(2)} - 8)^2 \le 4, \\ 0, & \text{elsewhere,} \end{cases}$$
$$\mu_{\tilde{U}}(4,1) = \begin{cases} 1, & \text{if } u^{(1)} = 4,\ u^{(2)} = 1, \\ 0, & \text{elsewhere,} \end{cases}$$
$$\mu_{\tilde{V}}(v^{(1)}, v^{(2)}) = \begin{cases} 1 - (v^{(1)} - 10)^2 - (v^{(2)} - 4)^2, & \text{if } (v^{(1)} - 10)^2 - (v^{(2)} - 4)^2 \le 1, \\ 0, & \text{elsewhere.} \end{cases} \tag{4.37}$$

The α-cuts, $0 \le \alpha \le 1$, of $\tilde{X}$, $\tilde{Y}$, $\tilde{U}$, $\tilde{V}$ are

$$\begin{gathered} X(\alpha) = (2,3), \\ Y(\alpha) = \{(y^{(1)}, y^{(2)}) \mid (y^{(1)} - 6)^2 - (y^{(2)} - 8)^2 \le 4(1-\alpha)\}, \\ U(\alpha) = (4,1), \\ V(\alpha) = \{(v^{(1)}, v^{(2)}) \mid (v^{(1)} - 10)^2 - (v^{(2)} - 4)^2 \le (1-\alpha)\}, \\ X(0) = (2,3), \\ Y(0) = \{(y^{(1)}, y^{(2)}) \mid (y^{(1)} - 6)^2 - (y^{(2)} - 8)^2 \le 4\}, \\ U(0) = (4,1), \\ V(0) = \{(v^{(1)}, v^{(2)}) \mid (v^{(1)} - 10)^2 - (v^{(2)} - 4)^2 \le 1\}. \end{gathered} \tag{4.38}$$

By Figure 4.1, we have

$$\begin{gathered} x_m^{(1)}(0) = 2, \qquad x_m^{(2)}(0) = 3, \\ \frac{y_m^{(1)}(0) - 2}{4} = \frac{\sqrt{41} + 2}{\sqrt{41}}, \qquad y_m^{(1)}(0) = 6 + \frac{8}{\sqrt{41}}, \\ \frac{y_m^{(2)}(0) - 3}{5} = \frac{\sqrt{41} + 2}{\sqrt{41}}, \qquad y_m^{(2)}(0) = 8 + \frac{10}{\sqrt{41}}. \end{gathered} \tag{4.39}$$

By Property 4.2, the length is

$$|\overrightarrow{\tilde{X}\tilde{Y}}|^* = \sqrt{(y_m^{(1)}(0) - x_m^{(1)}(0))^2 + (y_m^{(2)}(0) - x_m^{(2)}(0))^2} = 8.403. \tag{4.40}$$

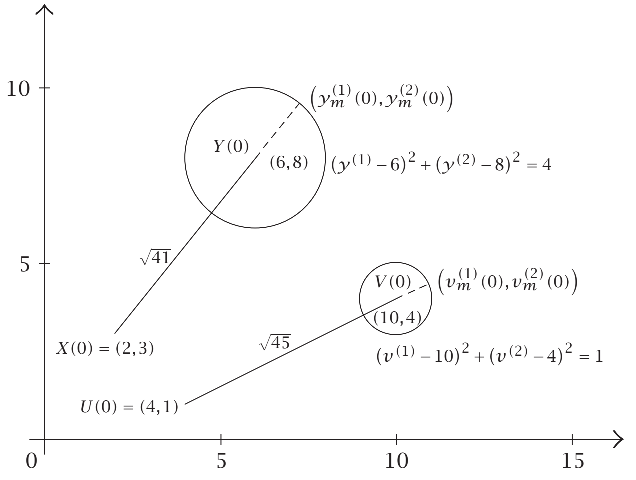

FIGURE 4.1. Fuzzy vectors $\overrightarrow{\tilde{X}\tilde{Y}}$ and $\overrightarrow{\tilde{U}\tilde{V}}$.

Similarly, we have

$$\begin{aligned} u_m^{(1)}(0) &= 4, \qquad u_m^{(2)}(0) = 1, \\ v_m^{(1)}(0) &= 10+\frac{6}{\sqrt{45}}, \qquad v_m^{(2)}(0) = 4+\frac{3}{\sqrt{45}}, \end{aligned} \tag{4.41}$$

and the length

$$\left|\overrightarrow{\tilde{U}\tilde{V}}\right|^* = \sqrt{\left(v_m^{(1)}(0)-u_m^{(1)}(0)\right)^2+\left(v_m^{(2)}(0)-u_m^{(2)}(0)\right)^2} = 7.708. \tag{4.42}$$

By Property 4.7,

$$\begin{aligned} \overrightarrow{\tilde{X}\tilde{Y}} \odot^* \overrightarrow{\tilde{U}\tilde{V}} &= \left(y_m^{(1)}(0)-x_m^{(1)}(0)\right)\left(v_m^{(1)}(0)-u_m^{(1)}(0)\right) \\ &\quad + \left(y_m^{(2)}(0)-x_m^{(2)}(0)\right)\left(v_m^{(2)}(0)-u_m^{(2)}(0)\right) \\ &= 58.81125. \end{aligned} \tag{4.43}$$

By Definition 4.10, the angle between $\overrightarrow{\tilde{X}\tilde{Y}}$ and $\overrightarrow{\tilde{U}\tilde{V}}$ has

$$\cos\theta = \frac{\overrightarrow{\tilde{X}\tilde{Y}} \odot^* \overrightarrow{\tilde{U}\tilde{V}}}{\left|\overrightarrow{\tilde{X}\tilde{Y}}\right|^* \cdot \left|\overrightarrow{\tilde{U}\tilde{V}}\right|^*} = 0.907996. \tag{4.44}$$

In crisp case, the vector $\overrightarrow{\tilde{X}\tilde{Y}}$ from $X=(2,3)$ to $Y=(6,8)$ is $(4,5)$ and the vector $\overrightarrow{\tilde{U}\tilde{V}}$ from $U=(4,1)$ to $V=(10,4)$ is $(6,3)$. Their lengths are $|\overrightarrow{\tilde{X}\tilde{Y}}| = 6.403$ and

$|\overrightarrow{\tilde{U}\tilde{V}}| = 6.708$; $\overrightarrow{\tilde{X}\tilde{Y}} \cdot \overrightarrow{\tilde{U}\tilde{V}} = 4 \cdot 6 + 5 \cdot 3 = 39$ and

$$\cos\theta = \frac{\overrightarrow{\tilde{X}\tilde{Y}} \cdot \overrightarrow{\tilde{U}\tilde{V}}}{\left|\overrightarrow{\tilde{X}\tilde{Y}}\right| \cdot \left|\overrightarrow{\tilde{U}\tilde{V}}\right|} = 0.908005. \tag{4.45}$$

Example 4.12. Let

$$\begin{aligned}
&\mu_{\tilde{X}}(x^{(1)},x^{(2)}) \\
&\quad = \begin{cases} 1-(x^{(1)}-5)^2-(x^{(2)}-10)^2, & \text{if } (x^{(1)}-5)^2+(x^{(2)}-10)^2 \le 1, \\ 0, & \text{elsewhere,} \end{cases} \\
&\mu_{\tilde{Y}}(y^{(1)},y^{(2)}) \\
&\quad = \begin{cases} \frac{1}{4}\left\{4-(y^{(1)}-14)^2-(y^{(2)}-15)^2\right\}, & \text{if } (y^{(1)}-14)^2+(y^{(2)}-15)^2 \le 4, \\ 0, & \text{elsewhere,} \end{cases} \\
&\mu_{\tilde{U}}(u^{(1)},u^{(2)}) \\
&\quad = \begin{cases} \frac{1}{4}\left\{4-(u^{(1)}-8)^2-(u^{(2)}-2)^2\right\}, & \text{if } (u^{(1)}-8)^2+(u^{(2)}-2)^2 \le 4, \\ 0, & \text{elsewhere,} \end{cases} \\
&\mu_{\tilde{V}}(v^{(1)},v^{(2)}) \\
&\quad = \begin{cases} 1-(v^{(1)}-17)^2-(v^{(2)}-7)^2, & \text{if } (v^{(1)}-17)^2+(v^{(2)}-7)^2 \le 1, \\ 0, & \text{elsewhere.} \end{cases}
\end{aligned} \tag{4.46}$$

We have the $\alpha = 0$-cuts of $\tilde{X}$, $\tilde{Y}$, $\tilde{U}$, $\tilde{V}$:

$$\begin{aligned}
X(0) &= \{(x^{(1)},x^{(2)}) \mid (x^{(1)}-5)^2+(x^{(2)}-10)^2 \le 1\}, \\
Y(0) &= \{(y^{(1)},y^{(2)}) \mid (y^{(1)}-14)^2+(y^{(2)}-15)^2 \le 4\}, \\
U(0) &= \{(u^{(1)},u^{(2)}) \mid (u^{(1)}-8)^2+(u^{(2)}-2)^2 \le 4\}, \\
V(0) &= \{(v^{(1)},v^{(2)}) \mid (v^{(1)}-17)^2+(v^{(2)}-7)^2 \le 1\}.
\end{aligned} \tag{4.47}$$

As in Example 4.11, from Figure 4.2, we have

$$\begin{aligned}
x_m^{(1)}(0) &= 5-\frac{9}{\sqrt{106}}, & x_m^{(2)}(0) &= 10-\frac{5}{\sqrt{106}}, \\
y_m^{(1)}(0) &= 14+\frac{18}{\sqrt{106}}, & y_m^{(2)}(0) &= 15+\frac{10}{\sqrt{106}}, \\
u_m^{(1)}(0) &= 8-\frac{18}{\sqrt{106}}, & u_m^{(2)}(0) &= 2-\frac{10}{\sqrt{106}}, \\
v_m^{(1)}(0) &= 17+\frac{9}{\sqrt{106}}, & v_m^{(2)}(0) &= 7+\frac{5}{\sqrt{106}}.
\end{aligned} \tag{4.48}$$

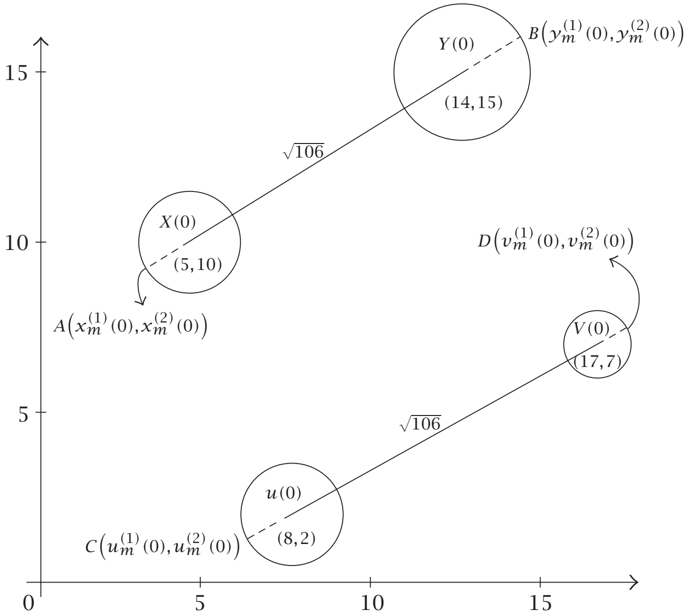

FIGURE 4.2. Fuzzy inner products of $\overrightarrow{\tilde{X}\tilde{Y}}$, $\overrightarrow{\tilde{U}\tilde{V}}$.

By Property 4.2,

$$\begin{aligned}
\left|\overrightarrow{\tilde{X}\tilde{Y}}\right|^* &= \sqrt{\left(y_m^{(1)}(0)-x_m^{(1)}(0)\right)^2+\left(y_m^{(2)}(0)-x_m^{(2)}(0)\right)^2} \\
&= \sqrt{\left(9+\frac{27}{\sqrt{106}}\right)^2+\left(5+\frac{15}{\sqrt{106}}\right)^2} \\
&= 13.29563, \\
\left|\overrightarrow{\tilde{U}\tilde{V}}\right|^* &= \sqrt{\left(v_m^{(1)}(0)-u_m^{(1)}(0)\right)^2+\left(v_m^{(2)}(0)-u_m^{(2)}(0)\right)^2} \\
&= \sqrt{\left(9+\frac{27}{\sqrt{106}}\right)^2+\left(5+\frac{15}{\sqrt{106}}\right)^2} \\
&= 13.29563
\end{aligned} \tag{4.49}$$

and by Property 4.7,

$$\begin{aligned}
&\overrightarrow{\tilde{X}\tilde{Y}} \odot^* \overrightarrow{\tilde{U}\tilde{V}} \\
&\quad = \left(y_m^{(1)}(0)-x_m^{(1)}(0)\right)\cdot\left(v_m^{(1)}(0)-u_m^{(1)}(0)\right) \\
&\qquad + \left(y_m^{(2)}(0)-x_m^{(2)}(0)\right)\cdot\left(v_m^{(2)}(0)-u_m^{(2)}(0)\right) \\
&\quad = \left(9+\frac{27}{\sqrt{106}}\right)^2+\left(5+\frac{15}{\sqrt{106}}\right)^2 = 13.29563^2.
\end{aligned} \tag{4.50}$$

By Definition 4.10, the angle between $\overrightarrow{\tilde{X}\tilde{Y}}$ and $\overrightarrow{\tilde{U}\tilde{V}}$ has

$$\cos\theta = \frac{\overrightarrow{\tilde{X}\tilde{Y}} \cdot \overrightarrow{\tilde{U}\tilde{V}}}{\left|\overrightarrow{\tilde{X}\tilde{Y}}\right| \cdot \left|\overrightarrow{\tilde{U}\tilde{V}}\right|} = 1. \tag{4.51}$$

Hence $\theta = 0$. That is, $\overrightarrow{\tilde{X}\tilde{Y}} // \overrightarrow{\tilde{U}\tilde{V}}$.

In the crisp case, the vector $\overrightarrow{\tilde{X}\tilde{Y}}$ from $X = (5,10)$ to $Y = (14,15)$ is $(9,5)$ and the vector $\overrightarrow{\tilde{U}\tilde{V}}$ from $U = (8,2)$ to $V = (17,7)$ is $(9,5)$. The lengths $|\overrightarrow{\tilde{X}\tilde{Y}}| = |\overrightarrow{\tilde{U}\tilde{V}}| = 10.2956$ and the angle between them has $\cos\phi = 1$. That is, $\overrightarrow{\tilde{X}\tilde{Y}} // \overrightarrow{\tilde{U}\tilde{V}}$.

5. Discussion

5.1. The comparison of the second definition (Definition 5.1) of the length of the fuzzy vector $\overrightarrow{\tilde{X}\tilde{Y}} \in SFR$ and the length $|\overrightarrow{\tilde{X}\tilde{Y}}|^*$ of Definition 4.1

METHOD 2

DEFINITION 5.1. (a) The length of the fuzzy vector $\overrightarrow{\tilde{X}\tilde{Y}} \in SFR$ is defined as

$$|\overrightarrow{\tilde{X}\tilde{Y}}| = \int_0^1 d^*(Y(\alpha)(-)X(\alpha))\,d\alpha. \tag{5.1}$$

(b) The inner product of $\overrightarrow{\tilde{X}\tilde{Y}}, \overrightarrow{\tilde{U}\tilde{V}} \in SFR$ is defined as

$$\overrightarrow{\tilde{X}\tilde{Y}} \odot' \overrightarrow{\tilde{U}\tilde{V}} = \int_0^1 d'(Y(\alpha)(-)X(\alpha), V(\alpha)(-)U(\alpha))\,d\alpha. \tag{5.2}$$

(c1) By Definition 4.1, since $|\overrightarrow{\tilde{X}\tilde{Y}}|^* = \sup_{0\le\beta\le1} d^*(Y(\beta) - X(\beta)) \ge d^*(Y(\alpha) - X(\alpha))$ for all $\alpha \in [0,1]$, so we have

$$|\overrightarrow{\tilde{X}\tilde{Y}}|^* \ge \int_0^1 d*(Y(\alpha)(-)X(\alpha))\,d\alpha = |\overrightarrow{\tilde{X}\tilde{Y}}|. \tag{5.3}$$

Using the same way as in Section 4, we can prove the following.

(c2) For $\overrightarrow{\tilde{X}\tilde{Y}}, \overrightarrow{\tilde{W}\tilde{Z}} \in SFR$, $k \in F_p^1(1)$, $k \ne 0$, we have

(1) $|k_1 \odot \overrightarrow{\tilde{X}\tilde{Y}}| = |k||\overrightarrow{\tilde{X}\tilde{Y}}|$;

(2) $|\overrightarrow{\tilde{X}\tilde{Y}} \oplus \overrightarrow{\tilde{W}\tilde{Z}}| \le |\overrightarrow{\tilde{X}\tilde{Y}}| + |\overrightarrow{\tilde{W}\tilde{Z}}|$.

This leads to the same results as Property 4.5.

(c3) For $\overrightarrow{\tilde{X}\tilde{Y}}, \overrightarrow{\tilde{U}\tilde{V}}, \overrightarrow{\tilde{W}\tilde{Z}} \in SFR$, $k_1 \in F_p^1(1)$, $k \ge 0$,

(1) $\overrightarrow{\tilde{X}\tilde{Y}} \odot' \overrightarrow{\tilde{U}\tilde{V}} = \overrightarrow{\tilde{U}\tilde{V}} \odot' \overrightarrow{\tilde{X}\tilde{Y}}$;

(2) $\overrightarrow{\tilde{X}\tilde{Y}} \odot' (\overrightarrow{\tilde{U}\tilde{V}} \oplus \overrightarrow{\tilde{W}\tilde{Z}}) \le (\overrightarrow{\tilde{X}\tilde{Y}} \odot' \overrightarrow{\tilde{U}\tilde{V}}) \oplus (\overrightarrow{\tilde{X}\tilde{Y}} \odot' \overrightarrow{\tilde{W}\tilde{Z}})$;

(3) $k(\overrightarrow{\tilde{X}\tilde{Y}} \odot' \overrightarrow{\tilde{U}\tilde{V}}) = (k_1 \odot \overrightarrow{\tilde{X}\tilde{Y}}) \odot' \overrightarrow{\tilde{U}\tilde{V}} = \overrightarrow{\tilde{X}\tilde{Y}} \odot' (k_1 \odot \overrightarrow{\tilde{U}\tilde{V}})$.

These are the same results as Property 4.8(1), (2), (3). However, Property 4.8(4), (5) do not hold in Method 2. Therefore, we cannot define the angle between two fuzzy vectors in SFR as in Definition 4.10.

5.2. In [5], Lubczonok defined the fuzzy vector space as follows.

Definition 5.2 (see [5, Definition 2.1]). The fuzzy vector space is a pair $\tilde{E} = (E,\mu)$, where E is a vector space and $\mu : E \to [0,1]$ with the property that for all $a,b \in \mathbb{R}$, and $x,y \in E$, $\mu(ax+by) \geq \mu(x)\wedge\mu(y)$ holds.

Then he obtained some results from this in [5].

In this paper, we obtained fuzzy vector space FE^n over $F_p^1(1)$ through n-dimensional vector space E^n over $\mathbb{R}$, then extended this to the pseudo-fuzzy vector space SFR over $F_p^1(1)$. It is strongly linked with E^n throughout this process, so it has more practical usage.

Since $E^n \approx FE^n$, we may consider the fuzzy vector space under the sense of [5]. The mapping

$$\begin{aligned} \overrightarrow{PQ} &= \left(q^{(1)}-p^{(1)},q^{(2)}-p^{(2)},\dots,q^{(n)}-p^{(n)}\right) \in E^n \longleftrightarrow \overrightarrow{\tilde{P}\tilde{Q}} \\ &= \left(q^{(1)}-p^{(1)},q^{(2)}-p^{(2)},\dots,q^{(n)}-p^{(n)}\right)_1 \in FE^n \end{aligned} \tag{5.4}$$

is one-to-one onto. In [5], let $E = E^n$ and $\mu = \nu$. For $\overrightarrow{PQ} \in E^n$, define $\nu(\overrightarrow{PQ}) = \mu_{\overrightarrow{\tilde{P}\tilde{Q}}}(q^{(1)}-p^{(1)},q^{(2)}-p^{(2)},\dots,q^{(n)}-p^{(n)}) = 1$. Let $\overrightarrow{ST} = (t^{(1)}-s^{(1)},t^{(2)}-s^{(2)},\dots,t^{(n)}-s^{(n)}) \in E^n$ and let $a,b \in \mathbb{R}$. Then

$$\begin{aligned} &a\overrightarrow{PQ}+b\overrightarrow{ST} \\ &\quad= \left(a\left(q^{(1)}-p^{(1)}\right)+b\left(t^{(1)}-s^{(1)}\right),a\left(q^{(2)}-p^{(2)}\right)+b\left(t^{(2)}-s^{(2)}\right),\dots,\right. \\ &\qquad \left. a\left(q^{(n)}-p^{(n)}\right)+b\left(t^{(n)}-s^{(n)}\right)\right) \in E^n \quad \forall a,b \in \mathbb{R} \\ &\quad\longleftrightarrow \left(a_1 \odot \overrightarrow{\tilde{P}\tilde{Q}}\right) \oplus \left(b_1 \odot \overrightarrow{\tilde{S}\tilde{T}}\right) \\ &\quad= \left(a\left(q^{(1)}-p^{(1)}\right)+b\left(t^{(1)}-s^{(1)}\right),a\left(q^{(2)}-p^{(2)}\right)+b\left(t^{(2)}-s^{(2)}\right),\dots,\right. \\ &\qquad \left. a\left(q^{(n)}-p^{(n)}\right)+b\left(t^{(n)}-s^{(n)}\right)\right)_1 \in FE^n. \end{aligned} \tag{5.5}$$

Hence by the definition of ν, we have

$$\begin{aligned} &\nu\left(a\overrightarrow{PQ}+b\overrightarrow{ST}\right) \\ &\quad= \mu_{(a_1\odot\overrightarrow{\tilde{P}\tilde{Q}}\oplus b_1\odot\overrightarrow{\tilde{S}\tilde{T}})}\left(a\left(q^{(1)}-p^{(1)}\right)+b\left(t^{(1)}-s^{(1)}\right),a\left(q^{(2)}-p^{(2)}\right)\right. \\ &\qquad \left. +b\left(t^{(2)}-s^{(2)}\right),\dots,a\left(q^{(n)}-p^{(n)}\right)+b\left(t^{(n)}-s^{(n)}\right)\right) = 1. \end{aligned} \tag{5.6}$$

Then $\nu(a\overrightarrow{PQ}+b\overrightarrow{ST}) = 1 = \nu(\overrightarrow{PQ})\wedge\nu(\overrightarrow{ST})$. Thus we get the fuzzy vector space $[E^n,\nu]$ by Definition 5.2 under the sense of [5].

In this paper, we emphasize on solving the practical problem instead of just working it out theoretically.

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