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FUZZY SUPER IRRESOLUTE FUNCTIONS

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The concept of fuzzy super irresolute function was considered and studied by Šostak's (1985). A comparison between this type and other existing ones is established. Several characterizations, properties, and their effect on some fuzzy topological spaces are studied. Also, a new class of fuzzy topological spaces under the terminology fuzzy S^*-closed spaces is introduced and investigated.

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1. Introduction and preliminaries. Šostak [10], introduced the fundamental concept of a fuzzy topological structure, as an extension of both crisp topology and Chang fuzzy topology [1], in the sense that not only the objects are fuzzified, but also the axiomatics. In [11, 12], Šostak gave some rules and showed how such an extension can be realized. Chattopadhyay et al. [2, 3] have redefined the same concept. In [8], Ramadan gave a similar definition, namely "smooth topological space." It has been developed in many directions [4, 5, 6, 7, 13].

In the present note, some counterexamples and characterizations of fuzzy super irresolute functions are examined. It is seen that fuzzy super irresolute function implies each of fuzzy irresolute [9] and fuzzy continuity [10], but not conversely. Also, properties preserved by fuzzy super irresolute functions are examined. Finally, we define a fuzzy S^*-closed space in fuzzy topological spaces in Šostak sense and characterize such a space from different angles. Our aim is to compare the introduced type of fuzzy covering property with the existing ones.

Throughout this note, let X be a nonempty set, $I = [0,1]$, and $I_\circ = (0,1]$. For $\alpha \in I$, $\underline{\alpha}(x) = \alpha$ for all $x \in X$. The following definition and results which will be needed.

Definition 1.1 [10]. A function $\tau : I^X \to I$ is called a *fuzzy topology* on X if it satisfies the following conditions:

(1) $\tau(\underline{0}) = \tau(\underline{1}) = 1$,

(2) $\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2)$ for any $\mu_1, \mu_2 \in I^X$,

(3) $\tau(\bigvee_{i\in\Gamma} \mu_i) \geq \bigwedge_{i\in\Gamma} \tau(\mu_i)$ for any $\{\mu\}_{i\in\Gamma} \subset I^X$.

The pair (X,τ) is called a *fuzzy topological space* (FTS).

Remark 1.2. Let (X,τ) be an FTS. Then, for each $\alpha \in I$, $\tau_\alpha = \{\mu \in I^X : \tau(\mu) \geq r\}$ is a Chang's fuzzy topology on X.

Theorem 1.3 [3]. *Let (X,τ) be an FTS. Then, for each $r \in I_\circ$ and $\lambda \in I^X$, an operator $C_\tau : I^X \times I_\circ \to I^X$ is defined as follows:*

$$C_\tau(\lambda,r) = \bigwedge\{\mu \in I^X : \lambda \leq \mu,\ \tau(\underline{1}-\mu) \geq r\}. \tag{1.1}$$

For $\lambda,\mu \in I^X$ and $r,s \in I_\circ$, the operator C_τ satisfies the following conditions:

(1) $C_\tau(\underline{0},r) = \underline{0}$, $\lambda \leq C_\tau(\lambda,r)$,

(2) $C_\tau(\lambda,r) \vee C_\tau(\mu,r) = C_\tau(\lambda \vee \mu,r)$,

(3) $C_\tau(\lambda,r) \leq C_\tau(\lambda,s)$ *if* $r \leq s$,

(4) $C_\tau(C_\tau(\lambda,r),r) = C_\tau(\lambda,r)$.

Theorem 1.4 [9]. *Let (X,τ) be an FTS. Then, for each $r \in I_\circ$ and $\lambda \in I^X$, an operator $I_\tau : I^X \times I_\circ \to I^X$ is defined as follows:*

$$I_\tau(\lambda,r) = \bigvee\{\mu \in I^X : \lambda \geq \mu,\ \tau(\mu) \geq r\}. \tag{1.2}$$

For $\lambda,\mu \in I^X$ and $r,s \in I_\circ$, the operator I_τ satisfies the following conditions:

(1) $I_\tau(\underline{1}-\lambda,r) = \underline{1}-C_\tau(\lambda,r)$,

(2) $I_\tau(\underline{1},r) = \underline{1}$, $\lambda \geq I_\tau(\lambda,r)$,

(3) $I_\tau(\lambda,r) \wedge I_\tau(\mu,r) = I_\tau(\lambda \wedge \mu,r)$,

(4) $I_\tau(\lambda,r) \geq I_\tau(\lambda,s)$ *if* $r \leq s$,

(5) $I_\tau(I_\tau(\lambda,r),r) = I_\tau(\lambda,r)$.

Definition 1.5 [9]. Let (X,τ) be an FTS. Then, for each $r \in I_\circ$ and $\lambda \in I^X$, the following statements hold:

(1) λ is called r-fuzzy semi-open (r-FSO) if there exists $\nu \in I^X$ with $\tau(\nu) \geq r$ such that $\nu \leq \lambda \leq C_\tau(\nu,r)$; equivalently, $\lambda \leq C_\tau(I_\tau(\lambda,r),r)$;

(2) λ is called r-fuzzy semiclosed (r-FSC) if there exists $\nu \in I^X$ with $\tau(\underline{1}-\nu) \geq r$ such that $I_\tau(\nu,r) \leq \lambda \leq \nu$; equivalently, $I_\tau(C_\tau(\lambda,r),r) \leq \lambda$;

(3) λ is called r-fuzzy semiclopen (r-FSCO) if λ is r-FSO and r-FSC;

(4) λ is called r-fuzzy regular open (r-FRO) if $\lambda = I_\tau(C_\tau(\lambda,r),r)$;

(5) the r-fuzzy semi-interior of λ, denoted $\mathrm{SI}_\tau(\lambda,r)$, is defined by $\mathrm{SI}_\tau(\lambda,r) = \bigvee\{\nu \in I^X : \nu \leq \lambda,\ \nu \text{ is } r\text{-FSO}\}$;

(6) the r-fuzzy semiclosure of λ, denoted $\mathrm{SC}_\tau(\lambda,r)$, is defined by $\mathrm{SC}_\tau(\lambda,r) = \bigwedge\{\nu \in I^X : \nu \geq \lambda,\ \nu \text{ is } r\text{-FSC}\}$.

Theorem 1.6 [9]. *Let (X,τ) be an FTS. For $\lambda \in I^X$ and $r \in I_\circ$, the following statements are valid:*

(1) λ *is r-FSO if and only if* $\lambda = \mathrm{SI}_\tau(\lambda,r)$, *and λ is r-FSC if and only if* $\lambda = \mathrm{SC}_\tau(\lambda,r)$;

(2) $I_\tau(\lambda,r) \leq \mathrm{SI}_\tau(\lambda,r) \leq \lambda \leq \mathrm{SC}_\tau(\lambda,r) \leq C_\tau(\lambda,r)$;

(3) $\mathrm{SC}_\tau(\mathrm{SC}_\tau(\lambda,r),r) = \mathrm{SC}_\tau(\lambda,r)$;
(4) $C_\tau(\mathrm{SC}_\tau(\lambda,r),r) = \mathrm{SC}_\tau(C_\tau(\lambda,r),r) = C_\tau(\lambda,r)$;
(5) $\mathrm{SI}_\tau(\underline{1}-\lambda,r) = \underline{1}-\mathrm{SC}_\tau(\lambda,r)$.

Lemma 1.7. *For any fuzzy set λ in an FTS (X,τ) and $r \in I_\circ$, if $\tau(\lambda) \geq r$, then $I_\tau(C_\tau(\lambda,r),r) = \mathrm{SC}_\tau(\lambda,r)$.*

Proof. Since $\mathrm{SC}_\tau(\lambda,r)$ is r-FSC, $I_\tau(C_\tau(\mathrm{SC}_\tau(\lambda,r),r),r) \leq \mathrm{SC}_\tau(\lambda,r)$ and hence, by Theorem 1.6(4), $I_\tau(C_\tau(\lambda,r),r) \leq \mathrm{SC}_\tau(\lambda,r)$. To prove the opposite inclusion, since $\tau(\lambda) \geq r$, $r \in I_\circ$, we have $\lambda \leq I_\tau(C_\tau(\lambda,r),r)$ so that $\underline{1}-\lambda \geq \underline{1}-I_\tau(C_\tau(\lambda,r),r) = C_\tau(I_\tau(\underline{1}-\lambda,r),r)$. But $C_\tau(I_\tau(\underline{1}-\lambda,r),r)$ is r-FSO. Hence $C_\tau(I_\tau(\underline{1}-\lambda,r),r) \leq \mathrm{SI}_\tau(\underline{1}-\lambda,r)$ and so $\mathrm{SC}_\tau(\lambda,r) \leq I_\tau(C_\tau(\lambda,r),r)$. □

Definition 1.8. Let (X,τ) and (Y,η) be FTSs and let $f : X \to Y$ be a function which is called
(1) fuzzy continuous (FC) if and only if $\eta(\mu) \leq \tau(f^{-1}(\mu))$ for each $\mu \in I^Y$ [10],
(2) fuzzy open if and only if $\tau(\lambda) \leq \eta(f(\lambda))$ for each $\lambda \in I^X$ [10],
(3) fuzzy semicontinuous (FSC) if and only if $f^{-1}(\mu)$ is r-FSO set of X for each $\eta(\mu) \geq r$, $r \in I_\circ$ [9],
(4) fuzzy irresolute (FI) if and only if $f^{-1}(\mu)$ is r-FSO set of X for each μ is r-FSO set of Y, $r \in I_\circ$ [9].

2. Fuzzy super irresolute functions

Definition 2.1. Let (X,τ) and (Y,η) be FTSs and let $f : X \to Y$ be a function which is called
(1) fuzzy super irresolute (F-super I) if and only if $\tau(f^{-1}(\mu)) \geq r$ for each μ is r-FSO set of Y, $r \in I_\circ$,
(2) fuzzy completely continuous (FCC) if and only if $f^{-1}(\mu)$ is r-FRO set of X for each $\mu \in I^Y$ and $\eta(\mu) \geq r$, $r \in I_\circ$,
(3) fuzzy completely irresolute (FCI) if and only if $f^{-1}(\mu)$ is r-FRO set of X for each r-FSO set $\mu \in I^Y$ and $r \in I_\circ$.

Remark 2.2. One can show the connection between these types and other existing ones by the following diagram:

$$\begin{array}{ccccc} \mathrm{FCI} & \longrightarrow & \text{F-super I} & \longrightarrow & \mathrm{FI} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{FCC} & \longrightarrow & \mathrm{FC} & \longrightarrow & \mathrm{FSC}. \end{array} \tag{2.1}$$

The converse of the previous implications need not be true in general as shown in the following counterexample.

Counterexample 2.3. Let μ_1, μ_2, and μ_3 be fuzzy subsets of $X = \{a,b,c\}$ defined as follows:

$$\begin{array}{lll} \mu_1(a) = 0.9, & \mu_1(b) = 0.0, & \mu_1(c) = 0.1, \\ \mu_2(a) = 0.9, & \mu_2(b) = 0.7, & \mu_2(c) = 0.2, \\ \mu_3(a) = 0.9, & \mu_3(b) = 0.3, & \mu_3(c) = 0.2. \end{array} \tag{2.2}$$

Then $\tau,\eta : I^X \to I$, defined as

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda = \underline{0},\underline{1}, \\ \frac{1}{2}, & \text{if } \lambda = \mu_1, \\ \frac{1}{3}, & \text{if } \lambda = \mu_2, \\ 0, & \text{otherwise}, \end{cases} \qquad \eta(\lambda) = \begin{cases} 1, & \text{if } \lambda = \underline{0},\underline{1}, \\ \frac{1}{3}, & \text{if } \lambda = \mu_1,\mu_2, \\ \frac{1}{2}, & \text{if } \lambda = \mu_3, \\ 0, & \text{otherwise}, \end{cases} \tag{2.3}$$

are fuzzy topologies on X. Then,

(1) the identity function $\text{id}_X : (X,\tau) \to (X,\eta)$ is FI but not F-super I because μ_3 is $1/3$-FSO in (X,η) and $\tau(f^{-1}(\mu_3)) = \tau(\mu_3) = 0$;

(2) the identity function $\text{id}_X : (X,\tau) \to (X,\tau)$ is FC but not F-super I function.

Definition 2.4. An FTS (X,τ) is said to be fuzzy extremally disconnected if and only if $\tau(C_\tau(\lambda,r)) \geq r$ for every $\tau(\lambda) \geq r$ for each $\lambda \in I^X$ and $r \in I_\circ$.

Theorem 2.5. *For a function $f : X \to Y$, the following statements are true:*

(1) *if X is fuzzy extremally disconnected and f is FI, then f is F-super I;*

(2) *if Y is fuzzy extremally disconnected and f is FCI (resp., FC), then f is F-super I;*

(3) *if both X and Y are fuzzy extremally disconnected, then the concepts F-super I, FCI, FI, FCC, FSC, and FC are equivalent.*

Proof. The proof is obvious. □

Theorem 2.6. *Let (X,τ_1) and (Y,τ_2) be FTSs. Let $f : X \to Y$ be a function. The following statements are equivalent:*

(1) *a map f is F-super I;*

(2) *for each r-FSC $\mu \in I^Y$, $\tau(\underline{1} - f^{-1}(\mu)) \geq r$, $r \in I_\circ$;*

(3) *for each $\lambda \in I^X$ and $r \in I_\circ$, $f(C_{\tau_1}(\lambda,r)) \leq \text{SC}_{\tau_2}(f(\lambda),r)$;*

(4) *for each $\mu \in I^Y$ and $r \in I_\circ$, $C_{\tau_1}(f^{-1}(\mu),r) \leq f^{-1}(\text{SC}_{\tau_2}(\mu,r))$;*

(5) *for each $\mu \in I^Y$ and $r \in I_\circ$, $f^{-1}(\text{SI}_{\tau_2}(\mu,r)) \leq I_{\tau_1}(f^{-1}(\mu),r)$.*

Proof. (1)⇔(2). It is easily proved from Theorem 1.4 and from $f^{-1}(\underline{1} - \mu) = \underline{1} - f^{-1}(\mu)$.

(2)⇒(3). Suppose there exist $\lambda \in I^X$ and $r \in I_\circ$ such that

$$f(C_{\tau_1}(\lambda,r)) \nleq \text{SC}_{\tau_2}(f(\lambda),r). \tag{2.4}$$

There exist $y \in Y$ and $t \in I_\circ$ such that

$$f(C_{\tau_1}(\lambda,r))(y) > t > \mathrm{SC}_{\tau_2}(f(\lambda),r)(y). \tag{2.5}$$

If $f^{-1}(\{y\}) = \emptyset$, it is a contradiction because $f(C_{\tau_1}(\lambda,r))(y) = 0$.

If $f^{-1}(\{y\}) \neq \emptyset$, there exists $x \in f^{-1}(\{y\})$ such that

$$f(C_{\tau_1}(\lambda,r))(y) \geq C_{\tau_1}(\lambda,r)(x) > t > \mathrm{SC}_{\tau_2}(f(\lambda),r)(f(x)). \tag{2.6}$$

Since $\mathrm{SC}_{\tau_2}(f(\lambda),r)(f(x)) < t$, there exists r-FSC $\mu \in I^Y$ with $f(\lambda) \leq \mu$ such that

$$\mathrm{SC}_{\tau_2}(f(\lambda),r)(f(x)) \leq \mu(f(x)) < t. \tag{2.7}$$

Moreover, $f(\lambda) \leq \mu$ implies $\lambda \leq f^{-1}(\mu)$. From (2), $\tau(\underline{1} - f^{-1}(\mu)) \geq r$. Thus, $C_{\tau_1}(\lambda,r)(x) \leq f^{-1}(\mu)(x) = \mu(f(x)) < t$, which is a contradiction to (2.6).

(3)⇒(4). For all $\mu \in I^Y$, $r \in I_\circ$, put $\lambda = f^{-1}(\mu)$. From (3), we have

$$f(C_{\tau_1}(f^{-1}(\mu),r)) \leq \mathrm{SC}_{\tau_2}(f(f^{-1}(\mu)),r) \leq \mathrm{SC}_{\tau_2}(\mu,r), \tag{2.8}$$

which implies that

$$C_{\tau_1}(f^{-1}(\mu),r) \leq f^{-1}(f(C_{\tau_1}(f^{-1}(\mu),r))) \leq f^{-1}(\mathrm{SC}_{\tau_2}(\mu,r)). \tag{2.9}$$

(4)⇒(5). It is easily proved from Theorem 1.4(1).

(5)⇒(1). Let μ be r-FSO set of Y. From Theorem 1.6(1), $\mu = \mathrm{SI}_{\tau_2}(\mu,r)$. By (5),

$$f^{-1}(\mu) \leq I_{\tau_1}(f^{-1}(\mu),r). \tag{2.10}$$

On the other hand, by Theorem 1.4(2),

$$f^{-1}(\mu) \geq I_{\tau_1}(f^{-1}(\mu),r). \tag{2.11}$$

Thus, $f^{-1}(\mu) = I_{\tau_1}(f^{-1}(\mu),r)$, that is, $\tau(f^{-1}(\mu)) \geq r$. □

3. Properties preserved by F-super I functions

Definition 3.1. Let (X,τ) be an FTS and $r \in I_\circ$. Then

(1) X is called r-fuzzy compact (resp., r-fuzzy almost compact and r-fuzzy nearly compact) if and only if for each family $\{\lambda_i \in I^X : \tau(\lambda_i) \geq r,\ i \in \Gamma\}$ such that $\bigvee_{i\in\Gamma}\lambda_i = \underline{1}$, there exists a finite index set $\Gamma_\circ \subset \Gamma$ such that $\bigvee_{i\in\Gamma_\circ}\lambda_i = \underline{1}$ (resp., $\bigvee_{i\in\Gamma_\circ} C_\tau(\lambda_i,r) = \underline{1}$ and $\bigvee_{i\in\Gamma_\circ} I_\tau(C_\tau(\lambda_i,r),r) = \underline{1}$);

(2) X is called r-fuzzy semicompact (resp., r-fuzzy S-closed) if and only if for each family $\{\lambda_i \in I^X : \lambda_i \leq C_\tau(I_\tau(\lambda_i,r),r),\ i \in \Gamma\}$ such that $\bigvee_{i\in\Gamma}\lambda_i = \underline{1}$, there exists a finite index set $\Gamma_\circ \subset \Gamma$ such that $\bigvee_{i\in\Gamma_\circ}\lambda_i = \underline{1}$ (resp., $\bigvee_{i\in\Gamma_\circ} C_\tau(\lambda_i,r) = \underline{1}$).

Theorem 3.2. *Every surjective F-super I image of r-fuzzy compact space is r-fuzzy semicompact, $r \in I_\circ$.*

Proof. Let (X,τ) be r-fuzzy compact, $r \in I_\circ$, and let $f:(X,\tau)\to(Y,\eta)$ be F-super I surjective function. If $\{\lambda_i \in I^Y : \lambda_i \le C_\eta(I_\eta(\lambda_i,r),r),\ i\in\Gamma\}$ with $\bigvee_{i\in\Gamma}\lambda_i = \underline{1}$, then $\bigvee_{i\in\Gamma} f^{-1}(\lambda_i)=\underline{1}$. Since f is F-super I, $\tau(f^{-1}(\lambda_i))\ge r$. Since X is r-fuzzy compact, there exists a finite subset $\Gamma_\circ\subset\Gamma$ with $\bigvee_{i\in\Gamma_\circ} f^{-1}(\lambda_i)=\underline{1}$. From the surjectivity of f, we deduce

$$\underline{1}=f(\underline{1})=f\left(\bigvee_{i\in\Gamma_\circ} f^{-1}(\lambda_i)\right)=\bigvee_{i\in\Gamma_\circ} ff^{-1}(\lambda_i)=\bigvee_{i\in\Gamma_\circ}\lambda_i. \tag{3.1}$$

So, Y is r-fuzzy semicompact. □

Corollary 3.3. *Every surjective F-super I image of r-fuzzy compact space is r-fuzzy S-closed, $r \in I_\circ$.*

Theorem 3.4. *Every surjective F-super I image of r-fuzzy almost compact space is r-fuzzy S-closed, $r \in I_\circ$.*

Proof. The proof is similar to that of Theorem 3.2. □

Corollary 3.5. *r-fuzzy semicompactness and r-fuzzy S-closedness are preserved under an F-super I surjection function, $r\in I_\circ$.*

Proof. The Corollary is a direct consequence of Theorems 3.2 and 3.4. □

Theorem 3.6. *Let $f:X\to Y$ be FSC and F-super I surjective function. If X is r-fuzzy nearly compact, then Y is r-fuzzy S-closed, $r\in I_\circ$.*

Proof. Let (X,τ) be r-fuzzy nearly compact, and let $r\in I_\circ$, $f:(X,\tau)\to(Y,\eta)$ be FSC and F-super I surjective function. If $\{\lambda_i\in I^Y:\lambda_i\le C_\eta(I_\eta(\lambda_i,r),r),\ i\in\Gamma\}$ with $\bigvee_{i\in\Gamma}\lambda_i=\underline{1}$, then $\bigvee_{i\in\Gamma} f^{-1}(\lambda_i)=\underline{1}$. Since f is F-super I, $\tau(f^{-1}(\lambda_i))\ge r$. Since X is r-fuzzy nearly compact, there exists a finite subset $\Gamma_\circ\subset\Gamma$ with $\bigvee_{i\in\Gamma_\circ} I_\tau(C_\tau(f^{-1}(\lambda_i),r),r)=\underline{1}$. From the surjectivity of f, we deduce

$$\begin{aligned}\underline{1}=f(\underline{1})&=f\left(\bigvee_{i\in\Gamma_\circ} I_\tau(C_\tau(f^{-1}(\lambda_i),r),r)\right)\\&=\bigvee_{i\in\Gamma_\circ} f(I_\tau(C_\tau(f^{-1}(\lambda_i),r),r))\\&\le\bigvee_{i\in\Gamma_\circ} f(f^{-1}(C_\eta(\lambda_i,r)))\quad(\text{since } f \text{ is FSC [9]}).\end{aligned}\tag{3.2}$$

Thus $\bigvee_{i\in\Gamma_\circ} C_\eta(\lambda_i,r)=\underline{1}$. Hence Y is r-fuzzy S-closed. □

4. Fuzzy S^*-closed spaces: characterizations and comparisons

Definition 4.1. Let (X,τ) be an FTS and $r \in I_\circ$. Then X is called r-fuzzy S^*-closed if and only if for each family $\{\lambda_i \in I^X : \lambda_i \le C_\tau(I_\tau(\lambda_i,r),r),\ i \in \Gamma\}$ such that $\bigvee_{i\in\Gamma}\lambda_i = \underline{1}$, there exists a finite index set $\Gamma_\circ \subset \Gamma$ such that

$$\bigvee_{i\in\Gamma_\circ} \mathrm{SC}_\tau(\lambda_i,r) = \underline{1}. \tag{4.1}$$

Theorem 4.2. *For an FTS (X,τ), $r \in I_\circ$, the following statements are equivalent:*

(1) *X is r-fuzzy S^*-closed;*
(2) *for every family $\{\lambda_i \in I^X : \lambda_i$ is r-FSCO, $i \in \Gamma\}$ such that $\bigvee_{i\in\Gamma}\lambda_i = \underline{1}$, there exists a finite index set $\Gamma_\circ \subset \Gamma$ such that $\bigvee_{i\in\Gamma_\circ}\lambda_i = \underline{1}$;*
(3) *every family of r-FSCO sets having the finite intersection property has nonnull intersection;*
(4) *for every family $\{\lambda_i \in I^X : \lambda_i$ is r-FSC, $i \in \Gamma\}$ such that $\bigwedge_{i\in\Gamma}\lambda_i = \underline{1}$, there exists a finite index set $\Gamma_\circ \subset \Gamma$ such that $\bigwedge_{i\in\Gamma_\circ}\mathrm{SI}_\tau(\lambda_i,r) = \underline{1}$.*

Proof. (1)⇒(2). The proof is obvious.

(2)⇒(3). Let $\{\lambda_i\}_{i\in\Gamma}$ be a family of r-FSCO sets having the finite intersection property. If possible, let $\bigwedge_{i\in\Gamma}\lambda_i = \underline{0}$. Then $\bigvee_{i\in\Gamma}(\underline{1}-\lambda_i) = \underline{1}$, where each $(\underline{1}-\lambda_i)$ is r-FSCO. By (2), there exists a finite subset $\Gamma_\circ$ of Γ such that $\bigvee_{i\in\Gamma_\circ}\underline{1}-\lambda_i = \underline{1}$, that is, $\bigwedge_{i\in\Gamma_\circ}\lambda_i = \underline{0}$, which is a contradiction.

(3)⇒(1). Suppose that $\{\lambda_i : i \in \Gamma\}$ is a family of r-FSO sets of X with $\bigvee_{i\in\Gamma}\lambda_i = \underline{1}$, and it has no finite subfamily $\{\lambda_{i_1},\dots,\lambda_{i_n}\}$ such that $\bigvee_{j=1}^{n}\mathrm{SC}_\tau(\lambda_{i_j},r) = \underline{1}$. Then $\bigwedge_{i=1}^{n}(\underline{1}-\mathrm{SC}_\tau(\lambda_{i_j},r)) \ne \underline{0}$. Thus, $\{\underline{1}-\mathrm{SC}_\tau(\lambda_i,r) : i \in \Gamma\}$ is a family of r-FSCO sets having the finite intersection property. By (3), $\bigwedge_{i\in\Gamma}(\underline{1}-\mathrm{SC}_\tau(\lambda_i,r)) \ne \underline{0}$, and hence, $\bigvee_{i\in\Gamma}\lambda_i \ne \underline{1}$, which is a contradiction.

(1)⇒(4). If $\{\lambda_i : i \in \Gamma\}$ is a family of nonnull r-FSC sets in X, $r \in I_\circ$ with $\bigwedge_{i\in\Gamma}\lambda_i = \underline{0}$, then $\{\underline{1}-\lambda_i : i \in \Gamma\}$ is r-FSO sets in X with $\bigvee_{i\in\Gamma}\underline{1}-\lambda_i = \underline{1}$. By (1), there is a finite subset $\Gamma_\circ \subset \Gamma$ such that

$$\underline{1} = \bigvee_{i\in\Gamma_\circ}\mathrm{SC}_\tau(\underline{1}-\lambda_i,r) = \underline{1} - \bigwedge_{i\in\Gamma_\circ}\mathrm{SI}_\tau(\lambda_i,r), \tag{4.2}$$

that is, $\bigwedge_{i\in\Gamma_\circ}\mathrm{SI}_\tau(\lambda_i,r) = \underline{0}$.

(4)⇒(1). For any $\{\lambda_i \in I^X : \lambda_i$ is r-FSO, $i \in \Gamma\}$ such that $\bigvee_{i\in\Gamma}\lambda_i = \underline{1}$, $\{\underline{1}-\lambda_i,\ i \in \Gamma\}$ is a family of r-FSC sets such that $\bigwedge_{i\in\Gamma}\underline{1}-\lambda_i = \underline{0}$. We can assume, without loss of generality, that each $\underline{1}-\lambda_i \ne \underline{0}$. By (4), there is a finite subset $\Gamma_\circ \subset \Gamma$ such that $\bigwedge_{i\in\Gamma_\circ}\mathrm{SI}_\tau(\underline{1}-\lambda_i,r) = \underline{0}$, that is, $\bigvee_{i\in\Gamma_\circ}\mathrm{SC}_\tau(\lambda_i,r) = \underline{1}$, which proves the r-fuzzy S^*-closedness of X. □

Theorem 4.3. *Let (X,τ) be an FTS and $r \in I_\circ$. If X is r-fuzzy semicompact, then X is r-fuzzy S^*-closed as well.*

Proof. Since for every $\lambda \in I^X$ and $r \in I_\circ$ we have $\lambda \le \mathrm{SC}_\tau(\lambda, r)$, this immediately follows from the definitions. □

Theorem 4.4. *Let (X,τ) be an FTS and $r \in I_\circ$. If X is r-fuzzy S^*-closed, then X is r-fuzzy S-closed as well.*

Proof. Since for every $\lambda \in I^X$ and $r \in I_\circ$ we have $\mathrm{SC}_\tau(\lambda, r) \le C_\tau(\lambda, r)$, this immediately follows from the definitions. □

That the converse is false is evident from the following counterexample.

Counterexample 4.5. Let $\mathbb{N}$ denote the set of natural numbers with the fuzzy topology $\tau : I^{\mathbb{N}} \to I$ defined as

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda = \underline{0}, \underline{1}, \\ \dfrac{1}{3}, & \text{if } \lambda = \mu, \nu, \\ \dfrac{1}{2}, & \text{if } \lambda = \mu \vee \nu, \\ 0, & \text{otherwise}, \end{cases} \tag{4.3}$$

where $\mu(1) = 1$, $\mu(i) = 0$ (for $i = 2,3,4,\ldots$), and $\nu(2) = 1$, $\mu(j) = 0$ (for $j = 1,3,4,\ldots$). Let ρ_i^1 and ρ_i^2 (for $i = 3,4,5,\ldots$) be the fuzzy sets in $I^{\mathbb{N}}$ given by

$$\rho_i^1(x) = \begin{cases} 1, & \text{for } x = 1 \text{ and } i, \\ 0, & \text{otherwise}, \end{cases} \qquad \rho_i^2(x) = \begin{cases} 1, & \text{for } x = 2 \text{ and } i, \\ 0, & \text{otherwise}. \end{cases} \tag{4.4}$$

Then $\mathcal{U} = \{\rho_i^1, \rho_i^2 : i = 3,4,5,\ldots\}$ are $1/3$-FSCO sets with $\bigvee_{\rho \in \mathcal{U}} \rho = \underline{1}$ having no finite subcover. Hence $(\mathbb{N}, \tau)$ is not $1/3$-fuzzy S^*-closed, but it is easily seen that $(\mathbb{N}, \tau)$ is $1/3$-fuzzy S-closed.

Theorem 4.6. *For any fuzzy extremally disconnected FTS (X,τ) and $r \in I_\circ$, X is r-fuzzy S^*-closed if and only if X is r-fuzzy S-closed.*

Proof

Necessity. It follows from the proof of Theorem 4.4.

Sufficiency. We are going to prove that if (X,τ) is any fuzzy extremally disconnected FTS, then $C_\tau(\lambda, r) = \mathrm{SC}_\tau(\lambda, r)$ for every r-FSO set λ in (X,τ) and $r \in I_\circ$. Then our result follows from Definitions 3.1(2) and 4.1.

We always have $\mathrm{SC}_\tau(\lambda, r) \le C_\tau(\lambda, r)$ for every $\lambda \in I^X$ and $r \in I_\circ$. So, we have to prove that with our hypothesis we have $C_\tau(\lambda, r) \le \mathrm{SC}_\tau(\lambda, r)$ for every $\lambda \in I^X$ and $r \in I_\circ$.

If λ is r-FSO in (X,τ), then there exists $\nu \in I^X$ with $\tau(\nu) \ge r$ such that $\nu \le \lambda \le C_\tau(\nu, r)$. So, $C_\tau(\lambda, r) = C_\tau(\nu, r)$, where $\tau(\nu) \ge r$. Because (X,τ) is fuzzy extremally disconnected, we have that

$$C_\tau(\lambda,r) = C_\tau(\nu,r) = I_\tau(C_\tau(\nu,r),r) = I_\tau(C_\tau(\lambda,r),r). \tag{4.5}$$

By Lemma 1.7, we have $C_\tau(\lambda,r) = I_\tau(C_\tau(\lambda,r),r) \le \mathrm{SC}_\tau(\lambda,r)$. □

Remark 4.7. From Theorems 4.3 and 4.4, we have that r-fuzzy semicompactness implies r-fuzzy S-closedness, $r \in I_\circ$.

Remark 4.8. Obviously, for $r \in I_\circ$, r-fuzzy S-closed space is r-fuzzy almost compact. Hence r-fuzzy compact space need not be r-fuzzy S^*-closed. That an r-fuzzy S^*-closed space is not necessarily r-fuzzy compact is shown by the following counterexample.

Counterexample 4.9. Let X be any nonempty set and let $\tau : I^X \to I$ be defined as

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda = \underline{0}, \underline{1}, \\ \dfrac{1}{2}, & \text{if } \lambda = \underline{\alpha}, \text{ for } \dfrac{1}{2} < \alpha < 1, \\ 0, & \text{otherwise.} \end{cases} \tag{4.6}$$

Then (X,τ) is an FTS which is not $1/2$-fuzzy compact. Now for any $\underline{\alpha} \in I^X$ with $\tau(\underline{\alpha}) \ge 1/2$, $C_\tau(\underline{\alpha},1/2) = \underline{1}$ and hence $I_\tau(C_\tau(\underline{\alpha},1/2),1/2) = \underline{1}$, for all $\alpha \in (1/2,1]$. Since, by Lemma 1.7, $\mathrm{SC}_\tau(\underline{\alpha},1/2) = I_\tau(C_\tau(\underline{\alpha},1/2),1/2) = \underline{1}$, we have for any r-FSO set λ, $\mathrm{SC}_\tau(\lambda,1/2) = \underline{1}$. Hence X is r-fuzzy S^*-closed.

However, we have the following theorem.

Theorem 4.10. *For $r \in I_\circ$, every r-fuzzy S^*-closed space is r-fuzzy nearly compact, $r \in I_\circ$.*

Proof. If X is not r-fuzzy nearly compact, then there exists $\{\lambda_i \in I^X,\ i \in \Gamma\}$ with $\tau(\lambda_i) \ge r$ and $\bigvee_{i\in\Gamma} \lambda_i = \underline{1}$ such that for any finite subset $\Gamma_\circ \subset \Gamma$,

$$\bigvee_{i\in\Gamma_\circ} I_\tau(C_\tau(\lambda_i,r),r) \ne \underline{1}, \tag{4.7}$$

that is,

$$\bigvee_{i\in\Gamma_\circ} \mathrm{SC}_\tau(\lambda_i,r) \ne \underline{1} \tag{4.8}$$

(by Lemma 1.7). Thus, X is not r-fuzzy S^*-closed. □

In order to investigate for the condition under which r-fuzzy S^*-closed space is r-fuzzy compact, we set the following definition.

Definition 4.11. An FTS (X,τ) is called r-fuzzy S-regular if and only if for each r-FSO set $\mu \in I^X$, $r \in I_\circ$,

$$\mu = \bigvee \{\rho \in I^X \mid \rho \text{ is } r\text{-FSO},\ \mathrm{SC}_\tau(\rho,r) \le \mu\}. \tag{4.9}$$

An FTS (X,τ) is called fuzzy S-regular if and only if it is r-fuzzy S-regular for each $r \in I_\circ$.

Theorem 4.12. *If an FTS (X,τ) is r-fuzzy S-regular and r-fuzzy S^*-closed, $r \in I_\circ$, then it is r-fuzzy compact.*

Proof. Let $\{\lambda_i \in I^X \mid \tau(\lambda_i) \ge r,\ i \in \Gamma\}$ be a family such that $\bigvee_{i\in\Gamma}\lambda_i = \underline{1}$. Since (X,τ) is r-fuzzy S-regular, for each $\tau(\lambda_i) \ge r$, λ_i is r-FSO,

$$\lambda_i = \bigvee_{i_k \in K_i} \{\lambda_{i_k} \mid \lambda_{i_k} \text{ is } r\text{-FSO},\ \mathrm{SC}_\tau(\lambda_{i_k},r) \le \lambda_i\}. \tag{4.10}$$

Hence $\bigvee_{i\in\Gamma}(\bigvee_{i_k\in K_i}\lambda_{i_k}) = \underline{1}$. Since (X,τ) is r-fuzzy S^*-closed, there exists a finite index $J \times K_J$ such that

$$\underline{1} = \bigvee_{j\in J}\left(\bigvee_{j_k \in K_J} \mathrm{SC}_\tau(\lambda_{j_k},r)\right). \tag{4.11}$$

For each $j \in J$, since

$$\bigvee_{j_k \in K_J} \mathrm{SC}_\tau(\lambda_{j_k},r) \le \lambda_j, \tag{4.12}$$

we have $\bigvee_{j\in J}\lambda_j = \underline{1}$. Hence (X,τ) is r-fuzzy compact. □

It is evident that every FI function is FSC. That the converse is not always true is shown in [9]. Again, it is proved in [9] that $f : X \to Y$ is FI if and only if $f^{-1}(\mu)$ is r-FSC for every r-FSC set μ in Y and $r \in I_\circ$. Now we have the following theorem.

Theorem 4.13. *The FI image of r-fuzzy S^*-closed space is r-fuzzy S^*-closed, $r \in I_\circ$.*

Theorem 4.14. *If $f : (X,\tau) \to (Y,\eta)$ is FI surjective and X is r-fuzzy S^*-closed, then Y is r-fuzzy S-closed, $r \in I_\circ$.*

Proof. If $\{\lambda_i \in I^Y : \lambda_i \text{ is } r\text{-FSO},\ i \in \Gamma\}$ is a family such that $\bigvee_{i\in\Gamma}\lambda_i = \underline{1}$, then $\bigvee_{i\in\Gamma} f^{-1}(\lambda_i) = \underline{1}$. Since f is FI, then, for each $i \in \Gamma$, $f^{-1}(\lambda_i)$ is r-FSO set of X. By r-fuzzy S^*-closedness of X, there is a finite subset $\Gamma_\circ \subset \Gamma$ such that

$\bigvee_{i\in\Gamma_\circ} \mathrm{SC}_\tau(f^{-1}(\lambda_i,r)) = \underline{1}$. Now,

$$\begin{aligned}\underline{1} = f(\underline{1}) &= f\left(\bigvee_{i\in\Gamma_\circ} \mathrm{SC}_\tau\left(f^{-1}(\lambda_i),r\right)\right)\\ &\le f\left(\bigvee_{i\in\Gamma_\circ} C_\tau\left(f^{-1}(\lambda_i),r\right)\right)\\ &\le \bigvee_{i\in\Gamma_\circ} C_\eta(\lambda_i,r),\end{aligned} \tag{4.13}$$

which implies that Y is r-fuzzy S-closed. □

Theorem 4.15. *If $f:(X,\tau)\to(Y,\eta)$ is CI surjective and X is r-fuzzy nearly compact, then Y is r-fuzzy semicompact, $r\in I_\circ$.*

Proof. The proof is similar to that of Theorem 4.14. □

Definition 4.16. Let (X,τ) and (Y,η) be FTSs. A function $f:(X,\tau)\to(Y,\eta)$ is called semiweakly continuous if and only if

$$f^{-1}(\lambda) \le \mathrm{SI}_\tau\left(f^{-1}(\mathrm{SC}_\eta(\lambda,r)),r\right), \tag{4.14}$$

for each r-FSO set λ in (Y,η), $r\in I_\circ$.

Theorem 4.17. *Let (X,τ) and (Y,η) be FTSs and let $f:(X,\tau)\to(Y,\eta)$ be a semiweakly continuous function. If X is r-fuzzy semicompact, then Y is r-fuzzy S^*-closed, $r\in I_\circ$.*

Proof. If $\{\lambda_i\in I^Y : \lambda_i \text{ is } r\text{-FSO}, i\in\Gamma\}$ is a family such that $\bigvee_{i\in\Gamma}\lambda_i = \underline{1}$. From the semiweak continuity of f, we have $f^{-1}(\lambda_i) \le \mathrm{SI}_\tau(f^{-1}(\mathrm{SC}_\eta(\lambda_i,r)),r)$. So, $\mathrm{SI}_\tau(f^{-1}(\mathrm{SC}_\eta(\lambda_i,r)),r)$ is a family of r-FSO sets in (X,τ) with

$$\bigvee_{i\in\Gamma} \mathrm{SI}_\tau\left(f^{-1}(\mathrm{SC}_\eta(\lambda_i,r)),r\right) = \underline{1}. \tag{4.15}$$

By the semicompactness of X, there exists a finite subset $\Gamma_\circ\subset\Gamma$ such that $\bigvee_{i\in\Gamma_\circ}\mathrm{SI}_\tau(f^{-1}(\mathrm{SC}_\eta(\lambda_i,r)),r) = \underline{1}$. So,

$$\begin{aligned}\underline{1} = f(\underline{1}) &= f\left(\bigvee_{i\in\Gamma_\circ} \mathrm{SI}_\tau\left(f^{-1}(\mathrm{SC}_\eta(\lambda_i),r),r\right)\right)\\ &\le \bigvee_{i\in\Gamma_\circ} ff^{-1}\left(\mathrm{SC}_\eta(\lambda_i),r\right)\\ &\le \bigvee_{i\in\Gamma_\circ} \mathrm{SC}_\eta(\lambda_i,r).\end{aligned} \tag{4.16}$$

Hence, $\bigvee_{i\in\Gamma_\circ}\mathrm{SC}_\eta(\lambda_i,r) = \underline{1}$ and Y is r-fuzzy S^*-closed. □

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