IJMMS 2003:49, 3149–3152
PII. S0161171203211273
http://ijmms.hindawi.com

PARA-f-LIE GROUPS

ANDREW BUCKI

Received 11 November 2002

Special para-f-structures on Lie groups are studied. It is shown that every para-f-Lie group G is the quotient of the product of an almost product Lie group and a Lie group with trivial para-f-structure by a discrete subgroup.

2000 Mathematics Subject Classification: 53C15, 22E15.

1. Para-f-structures.

The notion of a para-f-structure on a differentiable manifold was introduced and studied in [2].

Let M be an n-dimensional differentiable manifold of class C^∞. The set of all vector fields on M will be denoted by $\chi(M)$ and the tangent space of M at a point $m \in M$ by $T_m M$.

Definition 1.1. Let M be an n-dimensional differentiable manifold. If φ is an endomorphism field of constant rank k on M satisfying

$$\varphi^3 - \varphi = 0, \tag{1.1}$$

then φ is called a *para-f-structure* on M and M is a para-f-manifold.

Definition 1.2. A para-f-structure φ on M is *integrable* if there exists a coordinate system in which φ has constant components

$$\begin{bmatrix} I_p & 0 & 0 \\ 0 & -I_q & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{1.2}$$

where I is the unit matrix and $p + q = k$.

Proposition 1.3. *A para-f-structure φ on M is integrable if and only if its Nijenhuis tensor field N_φ vanishes, that is,*

$$N_\varphi(X,Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X,Y] = 0, \tag{1.3}$$

where $X, Y \in \chi(M)$.

For a para-f-structure φ on M, let

$$\begin{aligned} \ker\varphi &= \bigcup_{m\in M} (\ker\varphi)_m, \\ \operatorname{im}\varphi &= \bigcup_{m\in M} (\operatorname{im}\varphi)_m \end{aligned} \tag{1.4}$$

be the kernel and image of φ, respectively, where

$$\begin{aligned} (\ker\varphi)_m &= \{X \in T_mM;\ \varphi_m(X) = 0\}, \\ (\operatorname{im}\varphi)_m &= \{Y \in T_mM;\ Y = \varphi_m(X) \text{ for some } X \in T_mM\} \end{aligned} \tag{1.5}$$

are the kernel and image of φ at any point $m \in M$, respectively.

Proposition 1.4. *If* $(\ker\varphi)_m = \{0\}$ *for a para-f-structure φ for all $m \in M$, then φ is an almost product structure on M, that is,* $\varphi^2 = \mathrm{Id}$.

Proposition 1.5. *If* $(\operatorname{im}\varphi)_m = \{0\}$ *for a para-f-structure φ for all $m \in M$, then φ is the trivial para-f-structure on M, that is, $\varphi = 0$.*

Proposition 1.6. *If φ is a para-f-structure on M, then*

$$\ker\varphi \cap \operatorname{im}\varphi = \{0\}. \tag{1.6}$$

Proof. If $Z \in \ker\varphi \cap \operatorname{im}\varphi$, then $\varphi(Z) = 0$, and there exists X such that $\varphi(X) = Z$. Hence $\varphi^2(X) = 0$, and from Definition 1.1, we get $0 = \varphi^3(X) = \varphi(X) = Z$. □

Definition 1.7. Let φ_i be a para-f-structure on a para-f-manifold M_i with $i = 1,2$. A diffeomorphism $h : M_1 \to M_2$ is called a *para-f-map* if

$$\varphi_2 \circ h_* = h_* \circ \varphi_1, \tag{1.7}$$

where h_* is the differential of h.

2. Para-f-Lie groups. In this section, the notion of a para-f-Lie group is introduced. Some properties of its Lie algebra are established. Finally, its special decomposition in terms of an almost product Lie group and a Lie group with trivial para-f-structure is proved.

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. As usual, we define

$$\begin{aligned} &L_g : G \longrightarrow G \quad (\text{left multiplication by } g \in G), \\ &R_g : G \longrightarrow G \quad (\text{right multiplication by } g \in G), \\ &\operatorname{ad}_g : G \longrightarrow G, \qquad a \longmapsto \operatorname{ad}_g(a) = gag^{-1}, \\ &\operatorname{Ad}_X : \mathfrak{g} \longrightarrow \mathfrak{g}, \qquad Y \longmapsto \operatorname{Ad}_X(Y) = [X,Y]. \end{aligned} \tag{2.1}$$

Definition 2.1. Let G be a Lie group with a para-f-structure φ. If both L_g and R_g are para-f-maps, then φ is said to be *bi-invariant*.

Definition 2.2. If G is a Lie group with an integrable bi-invariant para-f-structure φ, then G is called a *para-f-Lie* group.

Proposition 2.3. *If φ is a bi-invariant para-f-structure on a Lie group G, then*

$$\varphi[X,Y]=[\varphi(X),Y] \tag{2.2}$$

for all $X,Y\in\mathfrak{g}$.

Proof. Since $\varphi\circ(L_g)_*=(L_g)_*\circ\varphi$ and $\varphi\circ(R_g)_*=(R_g)_*\circ\varphi$, we have $\varphi\circ(\mathrm{ad}_g)_*=(\mathrm{ad}_g)_*\circ\varphi$ for all $g\in G$. If $g=\exp(tX)$, where $t\in\mathbb{R}$, then $\varphi\circ(\mathrm{ad}_{\exp(tX)})_*=(\mathrm{ad}_{\exp(tX)})_*\circ\varphi$. Hence, by a standard result in Lie groups,

$$\varphi\circ e^{\mathrm{Ad}_{tX}}=e^{\mathrm{Ad}_{tX}}\circ\varphi, \tag{2.3}$$

or, for any $Y\in\mathfrak{g}$,

$$\begin{aligned}\varphi&\left(Y+t[X,Y]+\frac{t^2}{2!}[X,[X,Y]]+\cdots\right)\\&=\varphi(Y)+t[X,\varphi(Y)]+\frac{t^2}{2!}[X,[X,\varphi(Y)]]+\cdots.\end{aligned} \tag{2.4}$$

Hence,

$$\varphi[X,Y]+\frac{t}{2!}[X,[X,Y]]+\cdots=[X,\varphi(Y)]+\frac{t}{2!}[X,[X,\varphi(Y)]]+\cdots. \tag{2.5}$$

Letting $t\to 0$ in (2.5) gives us the desired result. □

Proposition 2.4. *A bi-invariant para-f-structure φ on a Lie group G is integrable.*

Proof. From Proposition 2.3, the Nijenhuis tensor of a bi-invariant para-f-structure φ must vanish at the unity e of G. □

Corollary 2.5. *A Lie group G with a bi-invariant para-f-structure φ is a para-f-Lie group.*

Example 2.6. Let $G=\mathbf{GL}(n,\mathbb{R})$ be the group of all real nonsingular $n\times n$ matrices. Let $\varphi: G\to G$, $X\mapsto\varphi(X)=X-(1/n)\,\mathrm{trace}(X)I$, where I is the unit matrix. Then φ is a bi-invariant para-f-structure on G.

Proposition 2.7. *Let G be a para-f-Lie group with a para-f-structure φ. Then its Lie algebra $\mathfrak{g}$ is expressed as*

$$\mathfrak{g}=V_k\oplus V_i, \tag{2.6}$$

the direct sum (as a Lie algebra), where $V_k=(\ker\varphi)_e$ and $V_i=(\mathrm{im}\,\varphi)_e$ are subalgebras of $\mathfrak{g}$, and $e\in G$ is the unity of G.

Proof. From Proposition 1.6, $V_k \cap V_i = \{0\}$. Therefore, $\mathfrak{g}$ is the direct sum (as a vector space) of V_k and V_i. It is clear, from Proposition 2.3, that both V_k and V_i are Lie subalgebras of $\mathfrak{g}$. Furthermore, if $X = \varphi(Z) \in V_i$ and $Y \in V_k$, then, again applying Proposition 2.3, $[X,Y] = \varphi[Z,Y] = [Z,\varphi(Y)] = 0$. Hence, $\mathfrak{g} = V_k \oplus V_i$ as a Lie algebra. □

Theorem 2.8. *Every para-f-Lie group G is the quotient of the product of an almost product Lie group and a Lie group with trivial para-f-structure by a discrete subgroup.*

Proof. Let V_k and V_i be subalgebras (defined in Proposition 2.7) of the Lie algebra $\mathfrak{g}$ of a para-f-Lie group G. From Proposition 2.7, $\mathfrak{g}$ is the Lie algebra direct sum of V_k and V_i. Using Propositions 1.4 and 1.5, we obtain the theorem from [4]. □

Remark 2.9. Since a para-f-structure with parallelizable kernel [2] is an almost r-paracontact structure [1], some examples of almost r-paracontact structures are used in [3] to illustrate para-f-Lie groups.

References

[1] A. Bucki, *Almost r-paracontact structures of P-Sasakian type*, Tensor (N.S.) **42** (1985), no. 1, 42–54.

[2] ———, *Para-ϕ-structures with parallelizable kernel on manifolds*, Tensor (N.S.) **48** (1989), no. 1, 36–45.

[3] ———, *Lie groups with almost r-paracontact structures*, to appear in Acta Math. Hungar., 2003.

[4] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Graduate Studies in Mathematics, vol. 34, American Mathematical Society, Rhode Island, 2001.

Andrew Bucki: Department of Mathematics, Oklahoma School of Science and Mathematics, Oklahoma City, OK 73104, USA

E-mail address: abucki@crystal.math.ou.edu