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CS-MODULES AND ANNIHILATOR CONDITIONS

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We study S-R-bimodules ${}_SM_R$ with the annihilator condition $S = l_S(A)+l_S(B)$ for any closed submodule A, and a complement B of A, in M_R. Such annihilator condition has a direct connection with the CS-condition for M_R. We make use of this to give a new characterization of CS-modules. Bimodules ${}_SM_R$ for which $r_Ml_S(A) = A$ (for every closed submodule A of M_R) are also dealt with. Such modules are called W^*-modules. We give the extra added annihilator conditions to W^*-modules to be equivalent to the continuous (quasicontinuous) modules.

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1. Introduction. Let R and S be rings and let ${}_SM_R$ be a bimodule. For any $X \le M$ and $T \le S$, write $l_S(X) = \{s \in S : sX = 0\}$ and $r_M(T) = \{m \in M : Tm = 0\}$. Let $\lambda : S \to \text{End}(M_R)$ be the canonical ring homomorphism. For each $s \in S$, we identify $\lambda(s)$ with s. A submodule A is essential in M (denoted by $A \le^e M$) if $A \cap B \ne 0$ for every nonzero submodule B of M. A submodule A is closed in M if it has no proper essential extensions in M. $A \le^{\oplus} M$ signifies that A is a direct summand of M (or simply a summand). A module M is called a CS-module if every closed submodule of M is a summand. The module M is continuous if it is a CS-module and satisfies condition (C_2): if $A \cong B \le M$ with $A \le^{\oplus} M$, then $B \le^{\oplus} M$. A generalization of condition (C_2) is (GC_2) (see [4]): if A is a submodule of M with $A \cong M$, then $A \le^{\oplus} M$. The module M is quasicontinuous if it is a CS-module and satisfies condition (C_3): if $A, B \le^{\oplus} M$ with $A \cap B = 0$, then $A \oplus B \le^{\oplus} M$. It is known that M is quasicontinuous if and only if $M = A \oplus B$ whenever A and B are complements of each other in M (see [3, Theorem 2.8]).

Camillo et al. [1] have dealt with Ikeda-Nakayama rings that are related to continuous and quasicontinuous rings.

For a bimodule ${}_SM_R$, Wisbauer et al. [4] have studied the annihilator condition $l_S(A \cap B) = l_S(A) + l_S(B)$ for any submodules A and B of M_R, and the condition $S = l_S(A) + l_S(B)$ for any submodules A and B of M_R with $A \cap B = 0$. Consequently, they obtained new characterizations of quasicontinuous modules. We adapt their ideas here to study a variation of the above annihilator condition which is connected to CS-modules, and obtain a new characterization of CS-modules in Section 2.

In Section 3, we study the bimodules ${}_SM_R$ which satisfy the following condition:

$$S = l_S(A) + l_S(B) \tag{1.1}$$

for any two relative complements A and B in M_R. Such modules are clearly quasicontinuous modules, while there are quasicontinuous modules which do not satisfy condition (1.1). For example, consider R as a commutative integral domain with field of quotients Q and let $M = Q \oplus Q$. In Lemma 3.2, we give a necessary and sufficient condition for quasicontinuous modules to satisfy condition (1.1). In the case of $S = \mathrm{End}(M_R)$, every quasicontinuous module must have condition (1.1). As a generalization of this condition, we introduce the concept of W^*-modules (bimodules ${}_SM_R$ for which $A = r_M l_S(A)$ for every closed submodule A of M_R). It is clear that any bimodule with condition (1.1) is a W^*-module, while in general the converse is not true. Proposition 3.8 indicates when a W^*-module satisfies condition (1.1).

In Section 4, we discuss the equivalence between W^*-modules and continuous (quasicontinuous) modules over an arbitrary ring S. Then we draw the consequences when S is the endomorphism ring of M_R.

2. CS-modules and annihilator conditions.

The proofs of the lemmas and propositions, presented in this section, are adaptations of the arguments in [4].

Lemma 2.1. *Let ${}_SM_R$ be a bimodule. If for every closed submodule A of M_R there exists a complement B of A in M_R such that $S = l_S(A) + l_S(B)$, then M_R is a CS-module.*

Proof. Let A be a closed submodule of M_R. Then by assumption there exists a complement B of A in M_R such that $S = l_S(A) + l_S(B)$. Write $1_S = u + v$, where $u \in l_S(A)$ and $v \in l_S(B)$. It follows that $a = va$ for all $a \in A$, $b = ub$ for all $b \in B$, and $vB = uA = 0$. Thus $B \subseteq r_M(v) \subseteq r_M(v^2)$ and $r_M(v^2) \cap A = 0$. Since B is a complement of A in M_R, we have $B = r_M(v) = r_M(v^2)$. Similarly, $A = r_M(u) = r_M(u^2)$. Now we show that $(vu)M = 0$. Let $vum = a + b$, where $m \in M$, $a \in A$, and $b \in B$. Noting that $vu = uv$, we have that $(v^2u^2)m = (vu)(a+b) = 0$. Hence $u^2m \in r_M(v^2) = r_M(v)$, and this gives that $u^2vm = vu^2m = 0$. Then $vm \in r_M(u^2) = r_M(u)$; and thus $vum = uvm = 0$. So $(vu)M \cap (A+B) = 0$. Since $A+B$ is essential in M_R, $(vu)M = 0$. So $uM \subseteq r_M(v) = B$ and $vM \subseteq r_M(u) = A$ and hence $M = vM + uM = A + B = A \oplus B$. Therefore A is a summand of M_R. □

Remark 2.2. The converse of Lemma 2.1 is not true. For example, there are torsion-free CS-modules over commutative integral domains, which do not satisfy the given condition in Lemma 2.1.

The next lemma follows from [4, Lemma 3].

Lemma 2.3. *Let ${}_SM_R$ be a bimodule, where ${}_SM$ is faithful, and let $M_R = A \oplus B$. If the projection f of M onto A along B is given by $f(m) = sm$ for some $s \in S$, and all $m \in M$, then $S = l_S(A) + l_S(B)$.*

For any submodules A and B of M_R and any $t \in S$, define $\alpha_t : A + B \to M$, $a + b \to ta$ (see [4]).

Proposition 2.4. *Let ${}_SM_R$ be a bimodule such that ${}_SM$ is faithful. The following are equivalent:*

(1) *M_R is CS and for any $f^2 = f \in \operatorname{End}(M_R)$, there exists $s \in S$ such that $f(m) = sm$, for all $m \in M_R$;*

(2) *for every closed submodule A of M_R, there exists a complement B of A in M_R such that $S = l_S(A) + l_S(B)$;*

(3) *for every closed submodule A of M_R, there exists a complement B of A in M_R such that $S = l_S(A) \oplus l_S(B)$;*

(4) *for every closed submodule A of M_R, there exists a complement B of A in M_R such that for every $t \in S$, the diagram*

$$\begin{array}{ccccc} 0 & \longrightarrow & A+B & \longrightarrow & M \\ & & \Big\downarrow{\scriptstyle \alpha_t} & & \\ & & M & & \end{array} \tag{2.1}$$

can be extended by $\lambda(s)$, for some $s \in S$.

Proof. (1)⇒(2). Let A be a closed submodule of M_R. Since M_R is a CS-module, there exists $f^2 = f \in \operatorname{End}(M_R)$ such that $A = fM$. By (1), there exists $s \in S$ such that $f(m) = sm$, for all $m \in M_R$. Hence $(s^2 - s)M = (f^2 - f)M = 0$. Since ${}_SM$ is faithful, it follows that s is an idempotent in S. Now we have

$$l_S(A) = l_S(fM) = l_S(sM) = l_S(s) = S(1 - S). \tag{2.2}$$

Similarly, $l_S(B) = S_S$, where $B =: (1 - f)M$. Thus $S = l_S(A) + l_S(B)$.

(2)⇒(1). It is clear by Lemma 2.1 that M_R is CS. Now let $f^2 = f \in \operatorname{End}(M_R)$, and denote $A = f(M)$. By (2), there exists a complement B of A in M_R such that $S = l_S(A) + l_S(B)$. The argument of the proof of Lemma 2.1 shows that $M = A \oplus B$. Let π be the projection of M onto A along B. Then

$$l_S(A) = l_S(\pi M) = \{s \in S : s\pi = 0\} \tag{2.3}$$

(by considering s the homomorphism given by left multiplication by s) and

$$l_S(B) = l_S\big((1 - \pi)M\big) = \big\{s \in S : s(1 - \pi) = 0\big\}. \tag{2.4}$$

Let $1 = s' + s$, where $s' \in l_S(A)$ and $s \in l_S(B)$. Thus $s'\pi = 0$ and $s(1-\pi) = 0$. It follows that $0 = s(1-\pi) = (1-s')(1-\pi) = 1-\pi-s'$. Therefore $f(m) = \pi(m) = sm$ for all $m \in M$.

(2)⇒(3). From the argument in the proof of Lemma 2.1, we have $M = A \oplus B$. Since ${}_SM$ is faithful, we have $0 = l_S(M) = l_S(A+B) = l_S(A)\hbar l_S(B)$ and hence $S = l_S(A) \oplus l_S(B)$.

(3)⇒(4). Let A be a closed submodule of M_R. By (3), there exists a complement B of A such that $S = l_S(A) \oplus l_S(B)$. Write $t = u + v$, where $u \in l_S(A)$ and $v \in l_S(B)$. Then $\alpha_t(a+b) = ta = (u+v)a = va = v(a+b) = \lambda(v)(a+b)$.

(4)⇒(2). Let A be a closed submodule of M_R. By (4), there exists a complement B of A in M_R satisfying diagram (2.1). By (4), there exists $s \in S$ such that $\lambda(s)$ extends α_t. Thus, for all $a \in A$ and $b \in B$, $ta = \alpha_t(a+b) = \lambda(s)(a+b) = s(a+b)$. It follows that $(1-s)a + (-s)b = 0$, for all $a \in A$ and $b \in B$. So $1-s \in l_S(A)$ and $-s \in l_S(B)$ and hence $1 = (1-s)-(-s) \in l_S(A) + l_S(B)$. Therefore $S = l_S(A) + l_S(B)$. □

Corollary 2.5. *The following are equivalent for a bimodule ${}_SM_R$ with $S = \mathrm{End}(M_R)$:*

(1) *M_R is a CS-module;*
(2) *for every closed submodule A of M_R, there exists a complement B of A in M_R such that $S = l_S(A) + l_S(B)$;*
(3) *for every closed submodule A of M_R, there exists a complement B of A in M_R such that $S = l_S(A) \oplus l_S(B)$;*
(4) *for every closed submodule A of M_R, there exists a complement B of A in M_R such that for every $t \in S$, diagram (2.1) can be extended by some $g : M \to M$.*

Proposition 2.6. *Let S be the center of $\mathrm{End}(M_R)$. The following are equivalent:*

(1) *for every closed submodule A of M_R, there exists a complement B of A in M_R such that $S = l_S(A) + l_S(B)$;*
(2) *M_R is CS and every idempotent of $\mathrm{End}(M_R)$ is central;*
(3) *M_R is CS and every closed submodule of M_R is fully invariant.*

Proof. (1)⇔(2) by Proposition 2.4.

(2)⇒(3). Let A be a closed submodule of M. By CS, A is a direct summand of M_R. Then $A = f(M)$ for some $f^2 = f \in \mathrm{End}(M_R)$. For any $g \in \mathrm{End}_R(M)$, since f is central by (2), $g(A) = g(f(M)) = f(g(M)) \subseteq f(M) = A$. This shows that A is a fully invariant submodule of M.

(3)⇒(2). Let $f, g \in \mathrm{End}_R(M)$ with $f^2 = f$. Therefore $f(M)$ is a closed submodule of M_R. By (3), $g(f(M)) \subseteq f(M)$ and $g((1-f)(M)) \subseteq (1-f)(M)$. It follows that $fgf = gf$ and $(1-f)g(1-f) = g(1-f)$. Thus, $g - gf = g(1-f) = (1-f)g(1-f) = g - gf - fg + fgf = g - gf - fg + gf = g - fg$. This shows that $fg = gf$. □

3. Condition (1.1) and its generalizations. The next lemma is clear.

Lemma 3.1. *The following are equivalent for a bimodule* ${}_SM_R$*:*

(1) $S = l_S(A) + l_S(B)$ *for any two relative complements* A *and* B *of* M_R*;*

(2) *for any submodules* A *and* B *of* M_R *with* $A \cap B = 0$, $S = l_S(A) + l_S(B)$.

We say that a bimodule ${}_SM_R$ has condition (1.1) if it satisfies one of the equivalent conditions of Lemma 3.1.

The next lemma follows from [4, Lemma 3].

Lemma 3.2. *Let* ${}_SM_R$ *be a bimodule such that* ${}_SM$ *is faithful. Then the following are equivalent:*

(1) M *has condition (1.1);*

(2) M *is quasicontinuous and every idempotent in* $\operatorname{End}(M_R)$ *is a left multiplication by an element of* S.

Remark 3.3 [4, Theorem 8]. In the case of $S = \operatorname{End}(M_R)$, it is clear from Lemma 3.2 that an R-module M is quasicontinuous if and only if M has condition (1.1).

Proposition 3.4. *Let* ${}_SM_R$ *be a bimodule which satisfies condition (1.1). Then* $A = r_M l_S(A)$ *for all closed submodules* A *of* M_R.

Proof. Let A be a closed submodule of M_R and B a submodule of $r_M l_S(A)$ such that $A \cap B = 0$. By Zorn's lemma, there exists a complement C of A in M_R with $B \subseteq C$. By condition (1.1), we have $S = l_S(A) + l_S(C) \subseteq l_S(A) + l_S(B)$, so $S = l_S(A) + l_S(B)$. Since $l_S(A) = l_S r_M l_S(A) \le l_S(B)$, it follows that $S = l_S(B)$ and hence $B = 0$. This shows that $A \le^e r_M l_S(A)$. Since A is a closed submodule of M_R, we have $A = r_M l_S(A)$. □

A bimodule ${}_SM_R$ is called a W^*-module if $A = r_M l_S(A)$ for every closed submodule A of M_R. It is clear by Proposition 3.4 that every bimodule ${}_SM_R$ with condition (1.1) is a W^*-module. But there are bimodules which are W^*-modules and do not satisfy condition (1.1). For example, let $S = R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$, where F is any field and let $M = {}_RR_R$. It is clear that M is W^*-module. But M_R is not quasicontinuous, and hence M does not satisfy condition (1.1).

Lemma 3.5. *The following are equivalent for a bimodule* ${}_SM_R$*:*

(1) $A \le^e r_M l_S(A)$ *for all submodules* A *of* M_R*;*

(2) ${}_SM_R$ *is a* W^**-module.*

Proof. (1)⇒(2). This implication is obvious.

(2)⇒(1). Let A be a submodule of M_R and C a maximal essential extension of A in M_R. We have by (2) that $A \le^e C = r_M l_S(C)$. Since $r_M l_S(A) \le r_M l_S(C)$, we have $A \le^e r_M l_S(A)$. □

Proposition 3.6. *If* ${}_SM_R$ *is a* W^**-module, then* $r_M(T) = 0$*, or* $r_M(T)$ *is uniform for every maximal left ideal* T *of* S.

Proof. Let T be a maximal left ideal of S. Since $T \subseteq l_S r_M(T)$, we have either $l_S r_M(T) = T$ or $l_S r_M(T) = S$. If $l_S r_M(T) = S$, then $r_M(T) = 0$. If $l_S r_M(T) = T$, let N be a nonzero submodule of $r_M(T)$. Then $T = l_S r_M(T) \subseteq l_S(N) \subseteq S$, and the maximality of T yields $T = l_S(N)$. It follows that $r_M(T) = r_M l_S(N)$. Since M is W^*-module, we have by Lemma 3.5 that $N \leq^e r_M(T)$. Therefore $r_M(T)$ is uniform. □

Corollary 3.7. *Let ${}_SM_R$ be a W^*-module, where every maximal left ideal of S is a left annihilator. Then $r_M(T)$ is uniform for every maximal left ideal T of S.*

Proof. Let T be a maximal left ideal of S. From Proposition 3.6, it is enough to show that $r_M(T) \neq 0$. Let $r_M(T) = 0$. By assumption, $T = l_S r_M(T) = l_S(0) = S$, which contradicts the maximality of T. □

Proposition 3.8. *The following are equivalent for a bimodule ${}_SM_R$:*

(1) *${}_SM_R$ is a W^*-module and $l_S(A) + l_S(B)$ is a left annihilator for any two relative complements A and B in M_R;*

(2) *${}_SM_R$ has condition (1.1).*

Proof. (1)⇒(2). Let A and B be two relative complements in M_R. Then by (1), $S = l_S(0) = l_S(A \cap B) = l_S(r_M l_S(A) \cap r_M l_S(B)) = l_S r_M(l_S(A) + l_S(B)) = l_S(A) + l_S(B)$. Therefore M has condition (1.1).

(2)⇒(1). This implication is obvious. □

4. The relation between W^*-modules and (quasi-) continuous modules.

The following is an immediate consequence of Proposition 3.8.

Proposition 4.1. *Let ${}_SM_R$ be a bimodule with $S = \mathrm{End}(M_R)$. Then the following are equivalent:*

(1) *${}_SM_R$ is a W^*-module and $l_S(A) + l_S(B)$ is a left annihilator for any two relative complements A and B of M_R;*

(2) *M_R is quasicontinuous.*

Proposition 4.2. *Let ${}_SM_R$ be a bimodule, where ${}_SM$ is faithful. Then the following are equivalent:*

(1) *${}_SM_R$ is a W^*-module, $l_S(A) + l_S(B)$ is an annihilator for any two relative complements A and B of M_R, and M_R has GC_2;*

(2) *M_R is a continuous module and every idempotent in $\mathrm{End}(M_R)$ is a left multiplication by an element of S.*

Proof. (1)⇒(2). We have by Proposition 3.8 that M_R has condition (1.1). Therefore, by Lemma 3.2, M_R is a quasicontinuous module. Let $s \in \mathrm{End}(M_R)$ be a monomorphism, with $sM \leq^e M$. By GC_2 it follows that $sM = M$. Then by [3, Lemma 3.14], M_R is a continuous module. The rest of the proof of (2) follows from Lemma 3.2.

(2)⇒(1). This implication is obvious. □

Corollary 4.3. *Let ${}_SM_R$ be a bimodule with $S = \text{End}(M_R)$. Then the following are equivalent:*

(1) *${}_SM_R$ is a W^*-module, $l_S(A)+l_S(B)$ is an annihilator for any two relative complements A and B of M_R, and M_R has GC_2;*

(2) *M_R is a continuous module.*

In particular, if M_R is of finite uniform dimension, then S is semiperfect.

Proof. It is clear that every monomorphism $f \in \text{End}(M_R)$ is an isomorphism (due to GC_2 and M of finite uniform dimension). Hence, M satisfies the assumptions in Camps and Dicks [2, Theorem 5], and so $\text{End}(M_R)$ is semilocal. Therefore by using [3, Proposition 3.5 and Lemma 3.7], idempotents of $S/J(S)$ lift to idempotents of S, and thus S is semiperfect. □

Lemma 4.4. *Let ${}_SM_R$ be a bimodule such that every finitely generated left ideal of S is a left annihilator of a subset of M_R, and every closed submodule of M_R is a right annihilator of a finite subset of S. Then M has condition (1.1).*

Proof. Let A_1 and A_2 be complements of each other in M_R. Then by assumption, we have $A_i = r_M(Y_i)$ for some finite subsets Y_i of S. Again by assumption, $SY_i = l_S(K_i)$ for some subsets K_i in M_R, where $i = 1,2$. Now $S = l_S(A_1 \cap A_2) = l_S(r_M(Y_1)\cap r_M(Y_2)) = l_Sr_M(SY_1+SY_2) = SY_1+SY_2$ (due to the assumption and since SY_1+SY_2 is finitely generated). Hence $S = l_S(K_1)+l_S(K_2) = l_Sr_Ml_S(K_1)+l_Sr_Ml_S(K_2) = l_Sr_M(Y_1)+l_Sr_M(Y_2) = l_S(A_1)+l_S(A_2)$. Therefore M satisfies condition (1.1). □

Lemma 4.5. *Let ${}_SM_R$ be a bimodule and let every idempotent in $\text{End}(M_R)$ be a left multiplication by an element of S. If M_R is a CS-module, then every closed submodule of M_R is a right annihilator of a finite subset of S.*

Proof. Let A be a closed submodule of M_R. Then by CS, there exists $f^2 = f \in \text{End}(M_R)$ such that $A = r_M(1-f) = \{m \in M : (1-s)m = 0\} = r_M(1-s)$, where $(1-s) \in S$. □

The following corollary is an immediate consequence of Lemmas 4.4 and 4.5.

Corollary 4.6. *Let ${}_SM_R$ be a bimodule, where $S = \text{End}(M_R)$. Let every finitely generated left ideal of S be a left annihilator of a subset of M. Then the following are equivalent:*

(1) *every closed submodule of M is a right annihilator of a finite subset of S;*

(2) *M is a CS-module.*

Theorem 4.7. *Let ${}_SM_R$ be a bimodule, where $S = \text{End}(M_R)$. Let every finitely generated left ideal of S be a left annihilator of a subset of M. Then the following are equivalent:*

(1) *M is a CS-module;*

(2) *M is continuous.*

Proof. By Lemmas 4.4 and 4.5, we have that M has condition (1.1) . By Remark 3.3, M is quasicontinuous. To show that M is continuous, by [3, Lemma 3.14], it is enough to show that every essential monomorphism $s \in S$ is an isomorphism. Let $s \in S$ be a monomorphism, with $sM \leq^e M$. By assumption, $S_S = l_S(X)$ for some subset X of M. It follows that $X = 0$ and hence $S_S = s$. Then s is a split monomorphism, and therefore $sM = M$. □

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