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EXPONENTIALLY FITTED SPLINE APPROXIMATION METHOD FOR SOLVING SELFADJOINT SINGULAR PERTURBATION PROBLEMS

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A numerical method based on cubic spline with exponential fitting factor is given for the selfadjoint singularly perturbed two-point boundary value problems. The scheme derived in this method is second-order accurate. Numerical examples are given to support the predicted theory.

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1. Introduction. We consider the following selfadjoint singularly perturbed two-point boundary value problem:

$$\begin{gathered} Ly \equiv -\varepsilon\left(a(x)y'\right)' + b(x)y = f(x) \quad \text{on } (0,1), \\ y(0) = \eta_0, \qquad y(1) = \eta_1, \end{gathered} \tag{1.1}$$

where η_0, η_1 are given constants and ε is a small positive parameter. Further, the coefficients $f(x)$, $a(x)$, and $b(x)$ are smooth functions and satisfy

$$a(x) \geq a > 0, \qquad a'(x) \geq 0, \qquad b(x) \geq b > 0. \tag{1.2}$$

Under these conditions, the operator L admits a maximum principle [8].

The problems in which a small parameter multiplies to a highest derivative arise in various fields of science and engineering, for instance, fluid mechanics, fluid dynamics, elasticity, quantum mechanics, chemical reactor theory, hydrodynamics, and so forth.

Out of the three principal approaches to solve such problems numerically, namely, the finite-difference methods, the finite-element methods, and the spline approximation methods, the first two have been used by several authors. Niijima [6] gave uniformly second-order accurate difference schemes whereas Miller [5] gave sufficient conditions for the uniform first-order convergence of a general three-point difference scheme. Boglaev [3] and Schatz and Wahlbin [9] used finite-element techniques to solve such problems. It is known that the most classical methods fail when ε is small relative to the mesh width h that is used for discretization of the operator L.

In this paper, we have used the third approach, namely, the spline approximation method, to solve problems of type (1.1). There are two possibilities to obtain small truncation error inside the boundary layer(s). The first is to choose a fine mesh there, whereas the second one is to choose a difference formula reflecting the behaviour of the solution(s) inside the boundary layer(s). The present work deals with the second approach, whereas the first one is currently under the investigation of the authors.

We have reduced the original problem (i.e., problem (1.1)) to the normal form. In the normalized form, we replace the perturbation parameter ε affecting the highest derivative by a fitting factor $\sigma(x,\varepsilon)$. Using cubic spline, this factor is determined in such a way that the truncation error of the corresponding scheme for the boundary layer function(s), in the case of constant coefficients, should be equal to zero. This procedure is known as the exponential fitting or the introducing of artificial viscosity [2, 4]. By making use of the continuity of the first-order derivative of the spline function, the resulting spline difference scheme gives a tridiagonal system which can be solved efficiently by the well-known algorithms.

In Section 2, we give a brief description of the method. The derivation of the difference scheme has been given in Section 3. The fitting factor is determined in Section 4, whereas the second-order accuracy of the method is shown in Section 5. To demonstrate the applicability of the proposed method, four numerical examples have been solved in Section 6 and the results are presented along with their comparison with those obtained by other authors. Finally, the discussion on these numerical results, along with some comparisons with the results obtained earlier by others, is presented in Section 7.

2. Description of the method.

Rewrite (1.1) as

$$y'' + P(x)y' + Q(x)y = R(x), \tag{2.1}$$

where

$$P(x) = \frac{a'(x)}{a(x)}, \qquad Q(x) = -\frac{b(x)}{\varepsilon a(x)}, \qquad R(x) = -\frac{f(x)}{\varepsilon a(x)}. \tag{2.2}$$

Let

$$y(x) = U(x)V(x), \tag{2.3}$$

and transform (2.1) into the normal form, that is,

$$V'' + A(x)V = B(x), \tag{2.4}$$

where

$$\begin{aligned} A(x) &= Q(x) - \frac{1}{2}P'(x) - \frac{1}{4}(P(x))^2, \\ B(x) &= R(x)\exp\left(\frac{1}{2}\int P(x)dx\right), \\ U(x) &= \exp\left(-\frac{1}{2}\int P(x)dx\right), \quad x \in (0,1), \end{aligned} \tag{2.5}$$

with

$$V(0) = \frac{y(0)}{U(0)} = \alpha_0, \quad V(1) = \frac{y(1)}{U(1)} = \alpha_1, \quad \alpha_0, \alpha_1 \in \mathbb{R}. \tag{2.6}$$

Multiplying (2.4) throughout by $-\varepsilon$ (where $0 < \varepsilon \le 1$), we get

$$\begin{gathered} -\varepsilon V'' + W(x)V = Z(x), \\ V(0) = \alpha_0, \qquad V(1) = \alpha_1, \end{gathered} \tag{2.7}$$

where

$$W(x) = -\varepsilon A(x), \qquad Z(x) = -\varepsilon B(x). \tag{2.8}$$

We define the fitting comparison problem associated with (2.7) by

$$\begin{gathered} -\sigma(x,\varepsilon)V'' + W(x)V = Z(x), \\ V(0) = \alpha_0, \qquad V(1) = \alpha_1, \end{gathered} \tag{2.9}$$

where $\sigma(x,\varepsilon)$ is an exponential fitting factor which is to be determined subsequently.

The approximate solution of problem (2.9) is sought in the form of the cubic spline function $S_j(x)$, which is defined as follows: let

$$x_0 = 0, \qquad x_j = x_0 + jh, \quad j = 1(1)n, \; h = x_j - x_{j-1}, \qquad x_n = 1. \tag{2.10}$$

For the values $V(x_0), V(x_1), \ldots, V(x_n)$, there exists an interpolating cubic spline with the following properties:

(i) $S_j(x)$ coincides with a polynomial of degree 3 on each interval $[x_{j-1}, x_j]$, $j = 1(1)n$;

(ii) $S_j(x) \in C^2[0,1]$;

(iii) $S_j(x_j) = V(x_j)$, $j = 0(1)n$.

Hence, analogous to [1], the cubic spline can be given as

$$\begin{aligned} S_j(x) = {} & \frac{(x_j - x)^3}{6h} M_{j-1} + \frac{(x - x_{j-1})^3}{6h} M_j \\ & + \left(V_{j-1} - \frac{h^2 M_{j-1}}{6}\right)\left(\frac{x_j - x}{h}\right) + \left(V_j - \frac{h^2 M_j}{6}\right)\left(\frac{x - x_{j-1}}{h}\right), \end{aligned} \tag{2.11}$$

where

$$x \in [x_{j-1}, x_j], \quad h = x_j - x_{j-1}, \quad j = 1,2,\dots,n, \\ M_j = S_j''(x_j), \quad j = 0,1,\dots,n. \tag{2.12}$$

Using this spline function, we will derive the difference scheme in Section 3, which will give us the approximate solution of $V(x)$. Since $U(x)$ is known, therefore the solution to the original problem will be obtained using (2.3).

3. Derivation of the scheme. Differentiating (2.11) and denoting the approximate solution to $V(x)$ by $v(x)$, we get

$$S_j'(x) = -\frac{(x_j - x)^2}{2h} M_{j-1} + \frac{(x - x_{j-1})^2}{2h} M_j + \left(\frac{v_j - v_{j-1}}{h}\right) - \left(\frac{M_j - M_{j-1}}{6}\right) h. \tag{3.1}$$

Since $S_j(x) \in C^2[0,1]$, therefore we must have

$$S_j'(x_j) = S_{j+1}'(x_j). \tag{3.2}$$

Using (3.1), (3.2), and (2.9), we obtain the difference scheme

$$R v_j = Q Z_j, \quad j = 1,2,\dots,n-1, \tag{3.3}$$

where

$$R v_j = r_j^- v_{j-1} + r_j^c v_j + r_j^+ v_{j+1}, \tag{3.4}$$

$$Q Z_j = q_j^- Z_{j-1} + q_j^c Z_j + q_j^+ Z_{j+1}, \tag{3.5}$$

$$v_0 = \alpha_0, \qquad v_n = \alpha_1, \tag{3.6}$$

$$r_j^- = -1\left(1 - \frac{h^2 W_{j-1}}{6\sigma_j^-}\right)\frac{1}{h}, \qquad r_j^+ = -1\left(1 - \frac{h^2 W_{j+1}}{6\sigma_j^+}\right)\frac{1}{h}, \qquad r_j^c = 2\left(1 + \frac{h^2 W_j}{3\sigma_j^c}\right)\frac{1}{h} \tag{3.7}$$

$$q_j^- = \frac{h}{6\sigma_j^-}, \qquad q_j^+ = \frac{h}{6\sigma_j^+}, \qquad q_j^c = \frac{2h}{3\sigma_j^c}, \tag{3.8}$$

where $\sigma_j^- = \sigma_{j-1}$, $\sigma_j^+ = \sigma_{j+1}$, $\sigma_j^c = \sigma_j$, and σ_j is to be determined.

Remark 3.1. The scheme without using fitting factor will be given by

$$r_j^- = -1\left(1 - \frac{h^2 W_{j-1}}{6\varepsilon}\right)\frac{1}{h}, \qquad r_j^+ = -1\left(1 - \frac{h^2 W_{j+1}}{6\varepsilon}\right)\frac{1}{h}, \qquad r_j^c = 2\left(1 + \frac{h^2 W_j}{3\varepsilon}\right)\frac{1}{h}, \\ q_j^- = \frac{h}{6\varepsilon}, \qquad q_j^+ = \frac{h}{6\varepsilon}, \qquad q_j^c = \frac{2h}{3\varepsilon}. \tag{3.9}$$

4. Determination of the fitting factor. In order to get a suitable fitting factor $\sigma(x,\varepsilon)$, we will use the following lemma.

Lemma 4.1 [4]. *Let $V(x) \in C^4[0,1]$. Let $W'(0) = W'(1) = 0$. Then the solution of problem (2.7) has the form*

$$V(x) = d(x) + e(x) + g(x), \tag{4.1}$$

where

$$\begin{aligned} d(x) &= q_0 \exp\left[-x\left\{\frac{W(0)}{\varepsilon}\right\}^{1/2}\right], \\ e(x) &= q_1 \exp\left[-(1-x)\left\{\frac{W(1)}{\varepsilon}\right\}^{1/2}\right], \end{aligned} \tag{4.2}$$

q_0 and q_1 are bounded functions of ε independent of x and

$$\left|g^{(k)}(x)\right| \le N\left(1 + (\varepsilon)^{1-k/2}\right), \quad k = 0,1,2,3,4 \tag{4.3}$$

N is a constant independent of ε.

The matrix of the system (3.3) is inverse monotone if $h^2W_i/6\sigma_i \le 1$, $i = j$, $j \pm 1$. Thus, we take a fitting factor in the following way:

$$\sigma_j^- = \frac{h^2 W_{j-1}}{6}\mu(\rho), \qquad \sigma_j^+ = \frac{h^2 W_{j+1}}{6}\mu(\rho), \qquad \sigma_j^c = \frac{h^2 W_j}{6}\mu(\rho), \tag{4.4}$$

where $\mu(\rho)$ (with ρ at x_j given by $\rho_j = \sqrt{W_j/\varepsilon}$) is to be determined.

We require that the truncation error for the boundary layer functions should be equal to zero when $W(x) = W =$ constant.

From the condition $Rd_j = 0$ for $W(x) = W =$ constant, we have

$$\mu(\rho) = 1 + \frac{3}{2\sinh^2(\rho h/2)}. \tag{4.5}$$

The condition $Re_j = 0$ for $W(x) = W =$ constant will give the same $\mu(\rho)$. Therefore, we define

$$\begin{aligned} \mu(\rho) &= 1 + \frac{3}{2\sinh^2(\rho h/2)}, && \text{when } W(x) = W = \text{constant}, \\ \mu(\rho_j) &= 1 + \frac{3}{2\sinh^2(\rho_j h/2)}, && \text{when } W(x) \ne \text{constant}. \end{aligned} \tag{4.6}$$

Hence, the variable fitting factor σ_j is defined as

$$\sigma_j = \frac{h^2 W_j}{6}\mu(\rho_j). \tag{4.7}$$

5. Proof of the uniform convergence. Throughout the paper, M will denote a positive constant which may take different values in different equations (inequalities) but are always independent of h and ε.

The scheme (3.3), (3.8) can be written in the matrix form

$$A\nu = Z, \tag{5.1}$$

where A is a matrix of the system (3.3) and ν and Z are corresponding vectors.

Now, the local truncation error $\tau_j(\phi)$ of the scheme (3.3) is defined by

$$\tau_j(\phi) = R\phi_j - Q(L\phi)_j, \tag{5.2}$$

where $\phi(x)$ is an arbitrary sufficiently smooth function.

Therefore,

$$\begin{aligned} &\tau_j(V) = RV_j - Q(LV)_j = R(V_j - \nu_j) \\ &\Longrightarrow R(V_j - \nu_j) = \tau_j(V) \\ &\Longrightarrow \max_j |V_j - \nu_j| \le \|A^{-1}\| \max_j |\tau_j(V)|. \end{aligned} \tag{5.3}$$

In order to estimate the values $|V_j - \nu_j|$, we will estimate the truncation error $\tau_j(V)$ and the norm of the matrix A^{-1}.

From (4.7), it is obvious that

$$0 \le \sigma_j \le Mh^2 \Longrightarrow |\sigma_j - \varepsilon| \le Mh^2 \quad \text{for } \varepsilon \le Ch^2, \tag{5.4}$$

where C is some positive constant. Now, for the case $Ch^2 \le \varepsilon$, we see that

$$\sigma_j - \varepsilon = \frac{h^2 W_j}{6} + \varepsilon\left[\frac{(h\rho_j/2)^2}{\sinh^2(h\rho_j/2)} - 1\right] \Longrightarrow |\sigma_j - \varepsilon| \le Mh^2 \quad \text{for } Ch^2 \le \varepsilon. \tag{5.5}$$

Hence,

$$|\sigma_j - \varepsilon| \le Mh^2, \tag{5.6}$$

that is, σ_j approximates ε with the error $O(h^2)$.

Estimation of truncation error and the norm of A^{-1}. From Lemma 4.1, we have

$$\tau_j(V) = \tau_j(d) + \tau_j(e) + \tau_j(g). \tag{5.7}$$

We will estimate separately the parts of $\tau_j(V)$.

First, we consider the case in which $Ch^2 \le \varepsilon$.

We will start with $d(x)$. We calculate

$$Rd_j = r_j^- d_{j-1} + r_j^c d_j + r_j^+ d_{j+1} \tag{5.8}$$

$$\begin{aligned} Q(Ld)_j &= q_j^- Z_{j-1} + q_j^c Z_j + q_j^+ Z_{j+1} \\ &= q_j^- (-\varepsilon d''_{j-1} + W_{j-1} d_{j-1}) \\ &\quad + q_j^c (-\varepsilon d''_j + W_j d_j) + q_j^+ (-\varepsilon d''_{j+1} + W_{j+1} d_{j+1}). \end{aligned} \tag{5.9}$$

Now, from Lemma 4.1, we have

$$d(x) = q_0 \exp\left[-x \left\{ \frac{W(0)}{\varepsilon} \right\}^{1/2} \right] \tag{5.10}$$

implies

$$d_{j-1} = d_j \exp\left(h\sqrt{\frac{W_0}{\varepsilon}} \right), \qquad d_{j+1} = d_j \exp\left(-h\sqrt{\frac{W_0}{\varepsilon}} \right), \tag{5.11}$$

$$d''_{j-1} = \left(\frac{W_0}{\varepsilon} \right) d_j \exp\left(h\sqrt{\frac{W_0}{\varepsilon}} \right), \qquad d''_{j+1} = \left(\frac{W_0}{\varepsilon} \right) d_j \exp\left(-h\sqrt{\frac{W_0}{\varepsilon}} \right), \tag{5.12}$$

$$d''_j = \left(\frac{W_0}{\varepsilon} \right) d_j. \tag{5.13}$$

Putting all these expressions into (5.8) and (5.9), and since

$$\tau_j(d) = Rd_j - Q(Ld)_j, \tag{5.14}$$

we get

$$\tau_j(d) = d_{j-1}\left(-\frac{1}{h} + \frac{hW_0}{6\sigma_j^-} \right) + d_j \left(\frac{2}{h} + \frac{2hW_0}{3\sigma_j^c} \right) + d_{j+1}\left(-\frac{1}{h} + \frac{hW_0}{6\sigma_j^+} \right). \tag{5.15}$$

From (5.6), $\sigma_j = \varepsilon + O(h^2)$, and using the above expressions for d_{j-1} and d_{j+1}, we have

$$|\tau_j(d)| \le \frac{Mh^3 d_j}{\varepsilon^2}. \tag{5.16}$$

But the expression for $d(x)$ involves q_0 in the numerator, which is a bounded function of ε independent of x. Therefore, we get

$$|\tau_j(d)| \le \frac{Mh^3}{\varepsilon}. \tag{5.17}$$

Now

$$e(x) = q_1 \exp\left[-(1-x) \left\{ \frac{W(1)}{\varepsilon} \right\}^{1/2} \right] \tag{5.18}$$

implies

$$e_{j-1} = e_j \exp\left(-h\sqrt{\frac{W_1}{\varepsilon}}\right), \qquad e_{j+1} = e_j \exp\left(h\sqrt{\frac{W_1}{\varepsilon}}\right), \tag{5.19}$$

$$e''_{j-1} = \left(\frac{W_1}{\varepsilon}\right) e_j \exp\left(-h\sqrt{\frac{W_1}{\varepsilon}}\right), \qquad e''_{j+1} = \left(\frac{W_1}{\varepsilon}\right) e_j \exp\left(h\sqrt{\frac{W_1}{\varepsilon}}\right), \tag{5.20}$$

$$e''_j = \left(\frac{W_1}{\varepsilon}\right) e_j, \tag{5.21}$$

and the similar construction as was for $d(x)$ will give us

$$\left|\tau_j(e)\right| \le \frac{Mh^3}{\varepsilon}. \tag{5.22}$$

Now,

$$\tau_j(g) = Rg_j - Q(Lg)_j \tag{5.23}$$

implies

$$\tau_j(g) = -\frac{1}{h}\left(g_{j-1} - 2g_j + g_{j+1}\right) + \frac{\varepsilon h}{6}\left(\frac{g''_{j-1}}{\sigma_j^-} + \frac{4g''_j}{\sigma_j^c} + \frac{g''_{j+1}}{\sigma_j^+}\right). \tag{5.24}$$

Expanding g_{j-1}, g_{j+1}, and their derivatives in terms of g_j and its derivatives, and using (5.6), we get

$$\left|\tau_j(g)\right| \le Mh^3\left|g_j^{(iv)}\right|. \tag{5.25}$$

Therefore, using Lemma 4.1, we obtain

$$\left|\tau_j(g)\right| \le \frac{Mh^3}{\varepsilon}. \tag{5.26}$$

From (5.17), (5.22), and (5.26), we have

$$\left|\tau_j(V)\right| \le \frac{Mh^3}{\varepsilon} \quad \text{when } Ch^2 \le \varepsilon. \tag{5.27}$$

Now we consider the case in which $Ch^2 \ge \varepsilon$.

We introduce the notations

$$\begin{aligned} r_j^- &= r_j^-(W_{j-1}), \qquad r_j^+ = r_j^+(W_{j+1}), \qquad r_j^c = r_j^c(W_j), \\ q_j^- &= q_j^-(W_{j-1}), \qquad q_j^+ = q_j^+(W_{j+1}), \qquad q_j^c = q_j^c(W_j). \end{aligned} \tag{5.28}$$

Since we have determined $\sigma(x,\varepsilon)$ in such a way that the truncation error for the boundary layer function(s) is equal to zero in the case of $W(x) = W =$ constant, thus $\tau_j(d) = 0$ when $W(x) = W = \text{constant}$. We will denote this expression by $\tilde{\tau}_j(d)$. Therefore,

$$\begin{aligned}\tau_j(d) &= \tau_j(d) - \tilde{\tau}_j(d)\\ &= [\{r_j^-(W_{j-1}) - r_j^-(W_0)\} - \{q_j^-(W_{j-1}) - q_j^-(W_0)\}(W_{j-1} - W_0)]d_{j-1}\\ &\quad + [\{r_j^c(W_j) - r_j^c(W_0)\} - \{q_j^c(W_j) - q_j^c(W_0)\}(W_j - W_0)]d_j\\ &\quad + [\{r_j^+(W_{j+1}) - r_j^+(W_0)\} - \{q_j^+(W_{j+1}) - q_j^+(W_0)\}(W_{j+1} - W_0)]d_{j+1}.\end{aligned} \tag{5.29}$$

Using (3.8) and (5.6), and since

$$|W_{j-1} - W_0| \le Mx_{j-1}^2, \qquad |W_j - W_0| \le Mx_j^2, \qquad |W_{j+1} - W_0| \le Mx_{j+1}^2, \tag{5.30}$$

we obtain

$$|\tau_j(d)| \le \frac{M}{h}[x_{j-1}^2 d_{j-1} + x_j^2 d_j + x_{j+1}^2 d_{j+1}]. \tag{5.31}$$

Now, using the fact that (see, e.g., Doolan et al. [4])

$$x\exp\left(-\frac{cx}{\varepsilon}\right) \le M\left(\frac{\varepsilon}{c}\right)\exp\left(-\frac{cx}{2\varepsilon}\right) \tag{5.32}$$

and d_j's involve q_0 which is a bounded function of ε, we obtain

$$x_{j-1}^2 d_{j-1} \le M\varepsilon^2, \qquad x_j^2 d_j \le M\varepsilon^2, \qquad x_{j+1}^2 d_{j+1} \le M\varepsilon^2. \tag{5.33}$$

Hence,

$$|\tau_j(d)| \le \frac{M\varepsilon^2}{h}. \tag{5.34}$$

But

$$\varepsilon < 1 \Longrightarrow \varepsilon^2 < \varepsilon, \tag{5.35}$$

therefore,

$$|\tau_j(d)| \le \frac{M\varepsilon}{h} \tag{5.36}$$

implies

$$|\tau_j(d)| \le Mh \quad (\text{since } \varepsilon \le Ch^2). \tag{5.37}$$

Similarly, we obtain

$$|\tau_j(e)| \le Mh. \tag{5.38}$$

Now, for $\tau_j(g)$, we use the form

$$\tau_j(g) = \frac{\varepsilon}{h}\left(g''_{j-1} + 4g''_j + g''_{j+1}\right) + hg''(\xi) : x_{j-1} < \xi < x_{j+1}. \tag{5.39}$$

Therefore, using Lemma 4.1, we obtain

$$\left|\tau_j(g)\right| \le Mh. \tag{5.40}$$

From (5.37), (5.38), and (5.40), we have

$$\left|\tau_j(V)\right| \le Mh. \tag{5.41}$$

Estimate of $\|A^{-1}\|$**.** Following Varah [10], we see that

$$\begin{aligned} \left| -r_j^- + r_j^c - r_j^+ \right| &= \left| \frac{1}{h} - \frac{hW_{j-1}}{6\sigma_j^-} + \frac{2}{h} + \frac{2hW_j}{3\sigma_j^c} + \frac{1}{h} - \frac{hW_{j+1}}{6\sigma_j^+} \right| \\ &= \left| \frac{4}{h} - \frac{h}{6}\left(W_{j-1} - 4W_j + W_{j+1}\right)\frac{1}{\varepsilon + O(h^2)} \right| \end{aligned} \tag{5.42}$$

using (5.6),

$$\left| -r_j^- + r_j^c - r_j^+ \right| \ge \begin{cases} \dfrac{M_1 h}{\varepsilon}, & Ch^2 \le \varepsilon, \\ \dfrac{M_1}{h}, & Ch^2 \ge \varepsilon, \end{cases} \tag{5.43}$$

implies

$$\|A^{-1}\| \le \begin{cases} \dfrac{M_2 \varepsilon}{h}, & Ch^2 \le \varepsilon, \\ M_2 h, & Ch^2 \ge \varepsilon, \end{cases} \tag{5.44}$$

where M_1 and M_2 $(= 1/M_1)$ are constants independent of h and ε.

From (5.3), (5.27), (5.41), and (5.44), we have the following theorem.

Theorem 5.1. *Let $W(x), Z(x) \in C^2[0,1]$, $W(x) \ge W > 0$, and $W'(0) = W'(1) = 0$. Let v_j, $j = 0,1,\dots,n$ be the approximate solution of (2.7), obtained using (3.3) and (3.8). Then, there is a constant M independent of ε and h such that*

$$\max_j \left| V(x_j) - v_j \right| \le Mh^2. \tag{5.45}$$

6. Test examples and numerical results.

In this section, we present some numerical results which illustrate Theorem 5.1.

Example 6.1 [11]. Consider problem (1.1) with

$$\begin{gathered} a(x)=1, \qquad b(x)=1+x(1-x) \\ f(x)=1+x(1-x)+\left[2\sqrt{\varepsilon}-x^2(1-x)\right]\exp\left[-\frac{1-x}{\sqrt{\varepsilon}}\right] \\ +\left[2\sqrt{\varepsilon}-x(1-x)^2\right]\exp\left[-\frac{x}{\sqrt{\varepsilon}}\right] \\ y(0)=0, \qquad y(1)=0. \end{gathered} \tag{6.1}$$

Its exact solution is given by

$$y(x)=1+(x-1)\exp\left[-\frac{x}{\sqrt{\varepsilon}}\right]-x\exp\left[-\frac{1-x}{\sqrt{\varepsilon}}\right]. \tag{6.2}$$

Example 6.2 [7]. Consider problem (1.1) with

$$\begin{gathered} a(x)=1, \qquad b(x)=\frac{4}{(x+1)^4}\left[1+\sqrt{\varepsilon}(x+1)\right], \\ f(x)=-\frac{4}{(x+1)^4}\left[\{1+\sqrt{\varepsilon}(x+1)+4\pi^2\varepsilon\}\cos\left(\frac{4\pi x}{x+1}\right)\right. \\ \left.-2\pi\varepsilon(x+1)\sin\left(\frac{4\pi x}{x+1}\right)+\frac{3\{1+\sqrt{\varepsilon}(x+1)\}}{1-\exp(-1/\sqrt{\varepsilon})}\right], \\ y(0)=2, \qquad y(1)=-1. \end{gathered} \tag{6.3}$$

Its exact solution is given by

$$y(x)=-\cos\left(\frac{4\pi x}{x+1}\right)+\frac{3\{\exp(-2x/\sqrt{\varepsilon}(x+1))-\exp(-1/\sqrt{\varepsilon})\}}{1-\exp(-1/\sqrt{\varepsilon})}. \tag{6.4}$$

Example 6.3 [4]. Consider problem (1.1) with

$$\begin{gathered} a(x)=1+x^2, \qquad b(x)=\frac{\cos x}{(3-x)^3}, \\ f(x)=4(3x^2-3x+1)\left[\left(x-\frac{1}{2}\right)^2+2\right], \\ y(0)=-1, \qquad y(1)=0. \end{gathered} \tag{6.5}$$

Its exact solution is not available.

Example 6.4 [9]. Consider problem (1.1) with

$$\begin{gathered} a(x)=1, \qquad b(x)=\frac{1}{\varepsilon}, \\ f(x)=\frac{1}{\varepsilon}\left(x-1-x\exp\left(-\frac{1}{\sqrt{\varepsilon}}\right)\right), \\ y(0)=0, \qquad y(1)=0. \end{gathered} \tag{6.6}$$

Its exact solution is given by

$$y(x)=x-1-x\exp\left(-\frac{1}{\sqrt{\varepsilon}}\right)+\exp\left(-\frac{x}{\sqrt{\varepsilon}}\right). \tag{6.7}$$

TABLE 6.1. Numerical results for Example 6.1 (maximum error) without using fitting factor.

ε	$n = 16$	$n = 32$	$n = 64$	$n = 128$	$n = 256$	$n = 512$
1/8	0.15E-02	0.36E-03	0.91E-04	0.23E-04	0.57E-05	0.14E-05
1/16	0.20E-02	0.49E-03	0.12E-03	0.31E-04	0.77E-05	0.19E-05
1/32	0.29E-02	0.73E-03	0.18E-03	0.45E-04	0.11E-04	0.28E-05
1/64	0.51E-02	0.13E-02	0.31E-03	0.78E-04	0.20E-04	0.49E-05
1/128	0.95E-02	0.23E-02	0.58E-03	0.15E-03	0.36E-04	0.91E-05
1/256	0.19E-01	0.45E-02	0.11E-02	0.28E-03	0.69E-04	0.17E-04
1/512	0.38E-01	0.85E-02	0.21E-02	0.53E-03	0.13E-03	0.33E-04
1/1024	0.67E-01	0.18E-01	0.42E-02	0.10E-02	0.26E-03	0.64E-04
1/2048	0.11E+00	0.36E-01	0.81E-02	0.20E-02	0.50E-03	0.13E-03

TABLE 6.2. Numerical results for Example 6.1 (maximum error) using fitting factor.

ε	$n = 16$	$n = 32$	$n = 64$	$n = 128$	$n = 256$	$n = 512$
1/8	0.32E-03	0.80E-04	0.20E-04	0.50E-05	0.12E-05	0.31E-06
1/16	0.35E-03	0.86E-04	0.21E-04	0.53E-05	0.13E-05	0.33E-06
1/32	0.40E-03	0.99E-04	0.25E-04	0.62E-05	0.15E-05	0.39E-06
1/64	0.53E-03	0.13E-03	0.33E-04	0.82E-05	0.21E-05	0.51E-06
1/128	0.83E-03	0.19E-03	0.46E-04	0.12E-04	0.29E-05	0.72E-06
1/256	0.13E-02	0.26E-03	0.66E-04	0.16E-04	0.41E-05	0.10E-05
1/512	0.18E-02	0.42E-03	0.95E-04	0.23E-04	0.58E-05	0.14E-05
1/1024	0.25E-02	0.62E-03	0.13E-03	0.33E-04	0.81E-05	0.20E-05
1/2048	0.33E-02	0.88E-03	0.21E-03	0.47E-04	0.12E-04	0.29E-05

TABLE 6.3. Numerical results for Example 6.1 (rate of convergence) using fitting factor $n = 16, 32, 64, 128, 256$.

ε	$r(0)$	$r(1)$	$r(2)$	$r(3)$	$r(4)$	Avg
1/8	0.20E+01	0.20E+01	0.20E+01	0.20E+01	0.20E+01	0.20E+01
1/16	0.20E+01	0.20E+01	0.20E+01	0.20E+01	0.20E+01	0.20E+01
1/32	0.20E+01	0.20E+01	0.20E+01	0.20E+01	0.20E+01	0.20E+01
1/64	0.20E+01	0.20E+01	0.20E+01	0.20E+01	0.20E+01	0.20E+01
1/128	0.22E+01	0.20E+01	0.20E+01	0.20E+01	0.20E+01	0.20E+01
1/256	0.23E+01	0.20E+01	0.20E+01	0.20E+01	0.20E+01	0.21E+01
1/512	0.22E+01	0.22E+01	0.20E+01	0.20E+01	0.20E+01	0.21E+01
1/1024	0.21E+01	0.23E+01	0.20E+01	0.20E+01	0.20E+01	0.21E+01
1/2048	0.21E+01	0.22E+01	0.22E+01	0.20E+01	0.20E+01	0.21E+01

Tables 6.1, 6.2, 6.5, and 6.8 contain the maximum errors at all the mesh points:

$$\max_j \left| y(x_j) - \tilde{v}(x_j) \right| \tag{6.8}$$

TABLE 6.4. Numerical results of O'Riordan and Stynes for Example 6.2 (maximum error).

n	$\varepsilon = 1$	$\varepsilon = (1/n)^{0.25}$	$\varepsilon = (1/n)^{0.5}$	$\varepsilon = (1/n)^{0.75}$	$\varepsilon = (1/n)$
8	0.42E+00	0.38E+00	0.33E+00	0.28E+00	0.25E+00
16	0.11E+00	0.95E-01	0.78E-01	0.66E-01	0.64E-01
32	0.27E-01	0.23E-01	0.18E-01	0.16E-01	0.17E-01
64	0.69E-02	0.56E-02	0.42E-02	0.40E-02	0.42E-02
128	0.17E-02	0.13E-02	0.10E-02	0.10E-02	0.13E-02
256	0.43E-03	0.31E-03	0.25E-03	0.26E-03	0.37E-03
512	0.11E-03	0.74E-04	0.63E-04	0.73E-04	0.10E-03

TABLE 6.5. Numerical results for Example 6.2 (maximum error).

n	$\varepsilon = 1$	$\varepsilon = (1/n)^{0.25}$	$\varepsilon = (1/n)^{0.5}$	$\varepsilon = (1/n)^{0.75}$	$\varepsilon = (1/n)$
8	0.20E+00	0.18E+00	0.14E+00	0.10E+00	0.10E+00
16	0.54E-01	0.47E-01	0.38E-01	0.26E-01	0.22E-01
32	0.14E-01	0.12E-01	0.95E-02	0.64E-02	0.92E-02
64	0.35E-02	0.30E-02	0.24E-02	0.16E-02	0.27E-02
128	0.86E-03	0.76E-03	0.60E-03	0.40E-03	0.66E-03
256	0.22E-03	0.19E-03	0.15E-03	0.10E-03	0.16E-03
512	0.54E-04	0.47E-04	0.37E-04	0.25E-04	0.41E-04

for different n and ε, where $\tilde{v}(x_j)$ is the approximate solution of (1.1) obtained via (2.7) and (2.3).

Table 6.6 contains maximum errors based on the double-mesh principle (Doolan et al. [4]) (as for Example 6.3, the exact solution is not available):

$$\max_{0\le j\le n} \left|\tilde{v}_j^n - \tilde{v}_{2j}^{2n}\right|, \quad n = 8, 16, 32, 64, 128, 256. \tag{6.9}$$

Tables 6.3 and 6.7 contain the numerical rate of uniform convergence for Examples 6.1 and 6.3, respectively, which is determined as in [4]:

$$r_{k,\varepsilon} = \log_2\left(\frac{z_{k,\varepsilon}}{z_{k+1,\varepsilon}}\right), \quad k = 0, 1, 2, \dots, \tag{6.10}$$

where

$$z_{k,\varepsilon} = \max_j \left|\tilde{v}_j^{h/2^k} - \tilde{v}_{2j}^{h/2^{k+1}}\right|, \quad k = 0, 1, 2, \dots, \tag{6.11}$$

and $\tilde{v}_j^{h/2^k}$ denotes the value of $\tilde{v}_j$ for the mesh length $h/2^k$.

TABLE 6.6. Numerical results for Example 6.3 (maximum error).

ε	$n=8$	$n=16$	$n=32$	$n=64$	$n=128$	$n=256$	$n=512$
1/2	0.10E-01	0.25E-02	0.63E-03	0.16E-03	0.39E-04	0.98E-05	0.24E-05
1/4	0.20E-01	0.49E-02	0.12E-02	0.31E-03	0.77E-04	0.19E-04	0.48E-05
1/8	0.39E-01	0.96E-02	0.24E-02	0.60E-03	0.15E-03	0.38E-04	0.94E-05
1/16	0.75E-01	0.19E-01	0.47E-02	0.12E-02	0.29E-03	0.73E-04	0.18E-04
1/32	0.14E+00	0.35E-01	0.88E-02	0.22E-02	0.55E-03	0.14E-03	0.34E-04
1/64	0.25E+00	0.63E-01	0.16E-01	0.40E-02	0.99E-03	0.25E-03	0.62E-04
1/128	0.42E+00	0.11E+00	0.26E-01	0.66E-02	0.16E-02	0.41E-03	0.10E-03
1/256	0.64E+00	0.16E+00	0.40E-01	0.99E-02	0.25E-02	0.62E-03	0.15E-03

TABLE 6.7. Numerical results for Example 6.3 (rate of convergence), $n = 8, 16, 32, 64, 128$.

ε	$r(0)$	$r(1)$	$r(2)$	$r(3)$	$r(4)$	Avg
1/2	0.20E+01	0.20E+01	0.20E+01	0.20E+01	0.20E+01	0.20E+01
1/4	0.20E+01	0.20E+01	0.20E+01	0.20E+01	0.20E+01	0.20E+01
1/8	0.20E+01	0.20E+01	0.20E+01	0.20E+01	0.20E+01	0.20E+01
1/16	0.20E+01	0.20E+01	0.20E+01	0.20E+01	0.20E+01	0.20E+01
1/32	0.20E+01	0.20E+01	0.20E+01	0.20E+01	0.20E+01	0.20E+01
1/64	0.20E+01	0.20E+01	0.20E+01	0.20E+01	0.20E+01	0.20E+01
1/128	0.20E+01	0.20E+01	0.20E+01	0.20E+01	0.20E+01	0.20E+01
1/256	0.20E+01	0.20E+01	0.20E+01	0.20E+01	0.20E+01	0.20E+01

TABLE 6.8. Numerical results for Example 6.4 (maximum error).

ε	Schatz's: PL* [9]		Schatz's: HC** [9]		Our results	
	$n=20$	$n=40$	$n=20$	$n=40$	$n=20$	$n=40$
5^{-1}	0.96E-03	0.24E-03	0.34E-05	0.26E-06	0.18E-14	0.60E-14
5^{-2}	0.27E-01	0.60E-02	0.83E-03	0.90E-04	0.33E-15	0.56E-15
5^{-3}	0.21E+00	0.12E+00	0.33E-01	0.94E-02	0.22E-15	0.22E-15
5^{-4}	0.26E+00	0.26E+00	0.78E-01	0.68E-01	0.22E-15	0.22E-15
5^{-5}	0.27E+00	0.27E+00	0.82E-01	0.82E-01	0.11E-15	0.11E-15
5^{-6}	0.27E+00	0.27E+00	0.82E-01	0.82E-01	0.11E-15	0.11E-15

*PL: piecewise linears
**HC: Hermite cubics.

7. Discussion. We have described a numerical method for solving selfadjoint singular perturbation problem using cubic spline with exponential fitting. It is a practical method and can easily be implemented on a computer to solve such problems. The method has been analyzed for convergence. Four

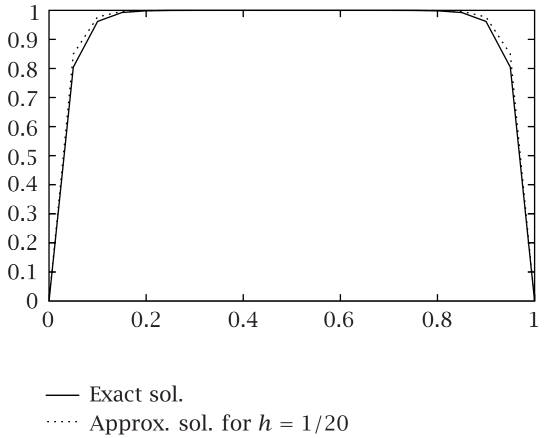

FIGURE 7.1. Exact and approximate solutions of Example 6.1 for $\varepsilon = 0.001$ without using fitting factor.

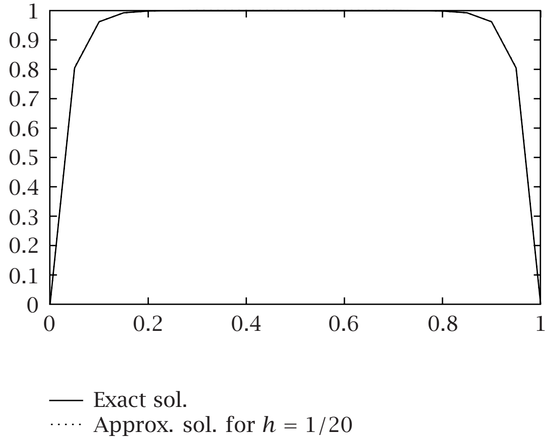

FIGURE 7.2. Exact and approximate solutions of Example 6.1 for $\varepsilon = 0.001$ using fitting factor.

examples have been solved to demonstrate the applicability of the proposed method.

For Examples 6.1 and 6.3, we have computed the rate of convergence, see Tables 6.3 and 6.7 which show the uniform second-order convergence as predicted in the theory. The same can be seen for the other examples also.

As is seen from Tables 6.1 and 6.2, the results obtained using fitting factor are better than those without using fitting factor.

Example 6.2 has been solved earlier by O'Riordan and Stynes [7]. We obtain better results than those in [7]. Using finite-element techniques, Schatz and

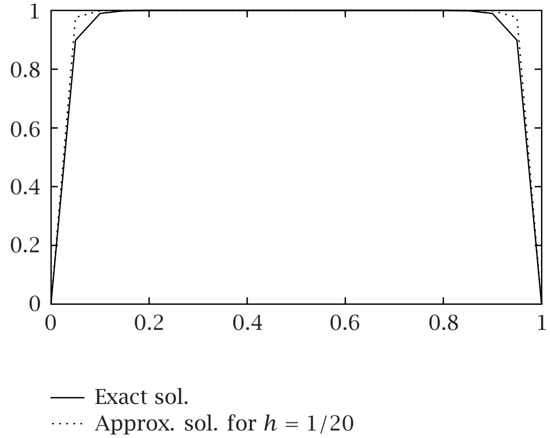

FIGURE 7.3. Exact and approximate solutions of Example 6.1 for $\varepsilon = 0.0005$ without using fitting factor.

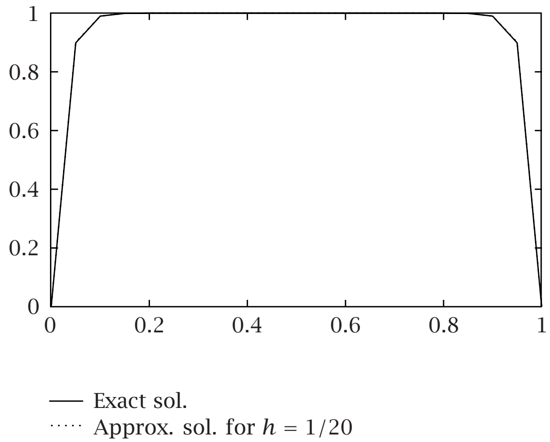

FIGURE 7.4. Exact and approximate solutions of Example 6.1 for $\varepsilon = 0.0005$ using fitting factor.

Wahlbin [9] have solved Example 6.4. Table 6.8 shows (quite graphically) how badly standard methods can perform.

To further corroborate the applicability of the proposed method, graphs have been plotted for Examples 6.1 and 6.2 for values of $x \in [0,1]$ versus the computed (termed as approximate) solution obtained at different values of x for a fixed ε. For each plot, we took $n = 20$ and 40 for Examples 6.1 and 6.2, respectively. Figures 7.1 and 7.3 are the graphs without using fitting factor for Example 6.1 for $\varepsilon = 0.001$ and $\varepsilon = 0.0005$, respectively, whereas Figures 7.2 and 7.4 are the graphs which are plotted using fitting factor for the same value of n and $\varepsilon = 0.001$ and $\varepsilon = 0.0005$, respectively. Similarly Figures 7.5 and 7.7

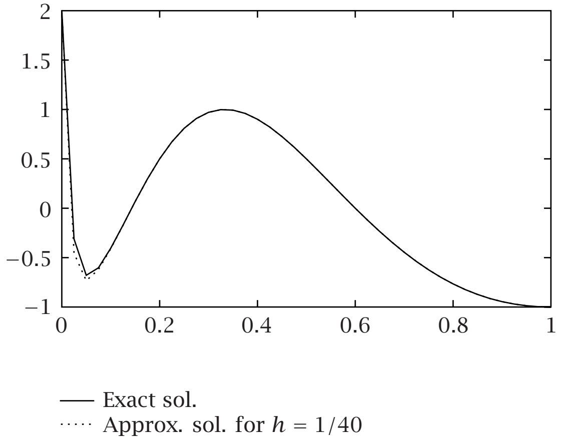

FIGURE 7.5. Exact and approximate solutions of Example 6.2 for $\varepsilon = 0.001$ without using fitting factor.

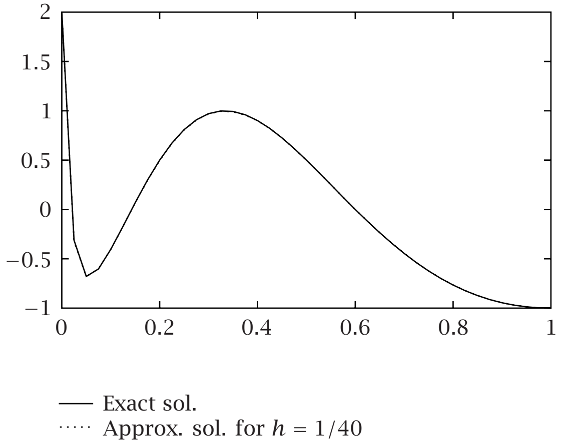

FIGURE 7.6. Exact and approximate solutions of Example 6.2 for $\varepsilon = 0.001$ using fitting factor.

are graphs without using fitting factor for Example 6.2 for $\varepsilon = 0.001$ and $\varepsilon = 0.0001$, respectively, whereas Figures 7.6 and 7.8 are graphs which are plotted using fitting factor for the same value of n and $\varepsilon = 0.001$ and $\varepsilon = 0.0001$, respectively. It can be seen from Figures 7.1, 7.3, 7.5, and 7.7 that the exact and approximate solutions without using fitting factor deviate from each other in the boundary layer regions for smaller ε. To control these fluctuations, we used fitting-factor technique and the resulting behaviour of these two solutions can be seen from Figures 7.2, 7.4, 7.6, and 7.8. The similar observation can be made for the other examples also.

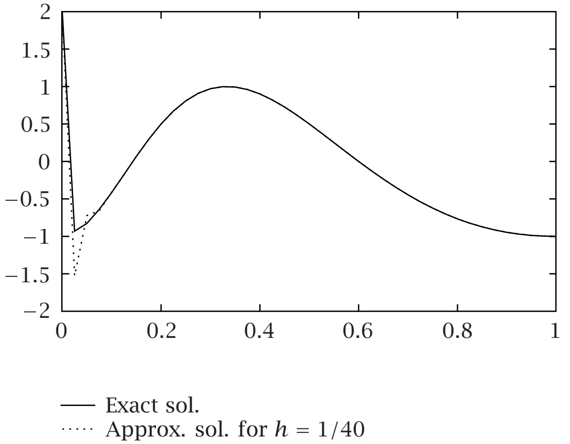

FIGURE 7.7. Exact and approximate solutions of Example 6.2 for $\varepsilon = 0.0001$ without using fitting factor.

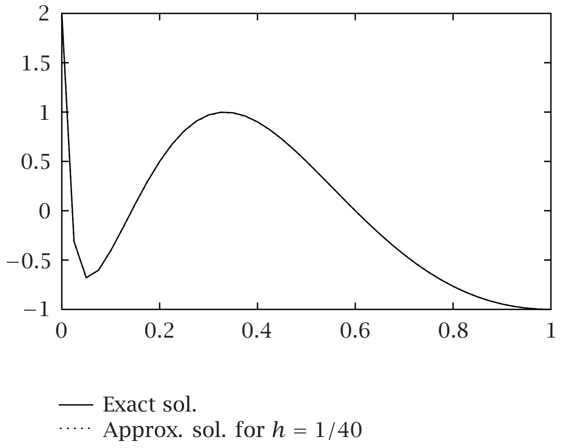

FIGURE 7.8. Exact and approximate solutions of Example 6.2 for $\varepsilon = 0.0001$ using fitting factor.

Finally, we would like to remark that we have replaced ε by $\sigma(x,\varepsilon)$ in the normalized form and not in the original selfadjoint problem (i.e., problem (1.1)), with $a(x) \neq$ constant because in that case ε is a multiple of both the second and first derivative terms which will cause implicit expressions whereas in normalized form, ε is multiplied with the second derivative term only and hence, the fitting-factor technique on the normalized form can easily be implemented. This shows the importance of reducing the original selfadjoint problem to normal form.

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