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ON PIERCE-LIKE IDEMPOTENTS AND HOPF INVARIANTS

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Given a set K with cardinality $\|K\| = n$, a wedge decomposition of a space Y indexed by K, and a cogroup A, the homotopy group $G = [A,Y]$ is shown, by using Pierce-like idempotents, to have a direct sum decomposition indexed by $P(K) - \{\phi\}$ which is strictly functorial if G is abelian. Given a class $\rho : X \to Y$, there is a Hopf invariant HI_ρ on $[A,Y]$ which extends Hopf's definition when ρ is a comultiplication. Then $\mathrm{HI} = \mathrm{HI}_\rho$ is a functorial sum of HI_L over $L \subset K$, $\|L\| \geq 2$. Each HI_L is a functorial composition of four functors, the first depending only on A^{n+1}, the second only on d, the third only on ρ, and the fourth only on Y^n. There is a connection here with Selick and Walker's work, and with the Hilton matrix calculus, as described by Bokor (1991).

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1. Introduction. In an earlier paper [5], to which this may be regarded as a sequel, the authors introduced a definition of a Hopf invariant which generalized most (but not all) existing definitions. We recall that definition.

Working in the pointed homotopy category $\mathscr{H}$, we consider a cogroup A in $\mathscr{H}$ with comultiplication $\mu : A \to A \vee A$. We now suppose given a map $d : A \to X$ and a map $\rho : X \to Y_1 \vee Y_2$. If the projections of ρ onto Y_1 and Y_2 are α_1 and α_2, respectively, so that $\alpha_i : X \to Y_i$, then ρ is referred to as a *copairing* of α_1 and α_2, which are themselves described as *copairable* (see [5]).

We have a diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad d \quad} & X \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\rho} \\ A \vee A & \xrightarrow{\alpha_1 d \vee \alpha_2 d} & Y_1 \vee Y_2 \end{array} \tag{1.1}$$

which does not, in general, commute. However, we may embed (1.1) in the larger diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad d \quad} & X \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\rho} \\ A \vee A & \xrightarrow{\alpha_1 d \vee \alpha_2 d} & Y_1 \vee Y_2 \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j} \\ A \times A & \xrightarrow{\alpha_1 d \times \alpha_2 d} & Y_1 \times Y_2, \end{array} \tag{1.2}$$

where j is the canonical map; and, in (1.2), the bottom square and the composite square both commute. That the composite square commutes follows from the relations $j\mu = \Delta : A \to A \times A$ and $j\rho = \{\alpha_1, \alpha_2\} : X \to Y_1 \times Y_2$.

Let $Y_1 \flat Y_2$ be the homotopy fiber of $j : Y_1 \vee Y_2 \to Y_1 \times Y_2$. Then it follows from what we have said that $\rho d - (\alpha_1 d \vee \alpha_2 d)\mu$ lifts to $Y_1 \flat Y_2$. Moreover, analysis of $j : Y_1 \vee Y_2 \to Y_1 \times Y_2$ shows that the lift is *unique*. We call this lift, as an element of $[A, Y_1 \flat Y_2]$, the *Hopf invariant of* d *with respect to* ρ, and write it as $\mathrm{HI}_\rho(d)$.

In this paper, we pursue the study of HI_ρ, but allow ourselves one further freedom. Instead of starting with a copairing ρ, which may be thought of as a kind of a *counion* of *two maps*, we consider a counion of n maps, which we call an n*-counion map* $\rho : X \to Y = Y_1 \vee Y_2 \vee \cdots \vee Y_n$. The rest of the definition of $\mathrm{HI}_\rho(d)$ will be essentially the same, except that, for simplicity, we confine our attention to the case in which A is a *commutative* cogroup. Since our principal interest in defining Hopf invariants in $\mathcal{H}$ would be in the case in which A is, at least, a double suspension, we argue that this gain in simplicity is obtained at relatively low cost.

With $[A, Y]$ commutative—indeed, we write it additively, as is customary with the higher homotopy groups—we may bring to bear the notion of orthogonal idempotents on $[A, Y]$. Thus let $\pi_i : Y \to Y_i$, $\iota_i : Y_i \to Y$ be the canonical projection onto the ith summand in Y and the corresponding injection. Then $\pi_i \iota_i$ is the identity on Y_i and $e_i = \iota_i \pi_i : Y \to Y$ is an idempotent map. Moreover,

$$e_i e_j = 0, \quad i \neq j. \tag{1.3}$$

Then e_i induces an idempotent endomorphism of $[A, Y]$ which we also call e_i; and if $i \neq j$, then e_i and e_j are orthogonal idempotent endomorphisms.

On the other hand, it is *not* true that $e_1 + e_2 + \cdots + e_n = 1$. Indeed, if $f : A \to Y$, the mapping $f \mapsto f_0 = f - \sum_{i=1}^n e_i f$ is an endomorphism of $[A, Y]$ and, in fact, is itself idempotent. For

$$e_i f_0 = e_i f - \sum_{i=1}^n e_j e_i f = e_j f - e_j f = 0, \tag{1.4}$$

so $f_0 - \sum_{i=1}^n e_i f_0 = f_0$. We call this idempotent e_0. Then $e_0 e_j = e_j - \sum_{i=1}^n e_i e_j = 0$ and $e_j e_0 = 0$, as shown above. Thus $e_0, e_1, \ldots, e_n$ form a complete Pierce-like system of orthogonal idempotents on $[A, Y]$, and we may say that it is the nontriviality of e_0 which allows us, or requires us, to define a Hopf invariant. Notice that the commutativity of $[A, Y]$ has greatly facilitated this last discussion.

In Section 2, we define the Hopf invariant of $d : A \to X$ with respect to the n-counion map $\rho : X \to Y$ as the lift of $e_0(\rho d)$ to the homotopy fiber of the canonical map $j : Y \to Y_1 \times Y_2 \times \cdots \times Y_n$. We show in what sense the Hopf invariant—which is a homomorphism of commutative groups—is natural and we analyse it as a sum of $2^n - n - 1$ elements, each factoring through a given space determined by a summand $Y_{i_1} \vee Y_{i_2} \vee \cdots \vee Y_{i_k}$ of Y with $k \geq 2$. Notice that such an analysis is vacuous in the case $n = 2$. We also relate our Hopf invariant to the one given in [4] in connection with the calculation of relative attaching maps for Thom spaces.

In Section 3, we make an entirely different analysis of the Hopf invariant, representing it as the composition of four maps, each depending on some aspect of the original data. One of the constituent maps appears to be closely related to a very general kind of Hopf invariant defined by Walker [11].

The constructions and arguments in this paper are all carried out in the pointed homotopy category $\mathscr{H}$. However, they may be couched in category-theoretic language and executed, with minor modifications, in a general category possessing pullbacks, coproducts, and zero object (see [5]). In particular, we might, as in [5], study these ideas in the category of groups. We might also consider the dual concepts. Indeed, the dual concept is related to a paper of Selick [10].

In Section 4, we show the relationship of the idempotents of this paper, of span 2, to the matrix calculus introduced in [7] and exploited in [2].

Since we always work on $\mathscr{H}$, we do not regard it as always necessary to mention the base point explicitly nor to insist on distinguishing notationally between a map and its homotopy class.

2. The Hopf invariant and naturality.

Let the space Y be given as a wedge (coproduct)

$$Y = Y_1 \vee Y_2 \vee \cdots \vee Y_n. \tag{2.1}$$

We may describe Y as a *costructured* space, with *summands* Y_i. Thus we associate with Y the projections

$$\pi_i : Y \longrightarrow Y_i \tag{2.2}$$

and injections

$$\iota_i : Y_i \longrightarrow Y. \tag{2.3}$$

Then $\pi_i\iota_i = \mathrm{Id}_i : Y_i \to Y_i$ and $e_i = \iota_i\pi_i : Y \to Y$ is an idempotent map, $i = 1, 2, \ldots, n$. We say that the idempotents e_i are *associated* with the costructure on Y. Notice that these idempotents are *orthogonal*, that is, $e_ie_j = 0$, $i \neq j$.

Let A be a commutative cogroup in $\mathscr{H}$ (e.g., a double suspension). Then $[A, Y]$ is a commutative group and the idempotents e_i induce an orthogonal system of idempotent endomorphisms, which we also denote by e_i,

$$e_i : [A, Y] \longrightarrow [A, Y]. \tag{2.4}$$

In general, the system $\{e_i\}$ of idempotents on $[A, Y]$ is not complete; for example, if $A = S^3$ and $Y = S^2 \vee S^2$, then there is a Whitehead product element $[\mathrm{Id}_1, \mathrm{Id}_2] \in \pi_3(S^2 \vee S^2)$ of infinite order, not expressible as $e_1\alpha_1 + e_2\alpha_2$.

However, for any $f \in [A, Y]$, we set

$$f_0 = f - \sum_{i=1}^{n} e_i f \in [A, Y]. \tag{2.5}$$

Proposition 2.1. *The function $f \mapsto f_0$ is an idempotent endomorphism e_0 of $[A, Y]$. Moreover, the system $\{e_0, e_1, \dots, e_n\}$ is a complete system of orthogonal idempotents on $[A, Y]$.*

Proof. Write $f_0 = e_0 f$. Then $e_0(f+g) = f + g - \sum_{i=1}^{n} e_i(f+g) = f + g - \sum_{i=1}^{n}(e_i f + e_i g) = (f - \sum_{i=1}^{n} e_i f) + (g - \sum_{i=1}^{n} e_i g) = e_0 f + e_0 g$. Thus e_0 is an endomorphism. Also

$$e_0 f_0 = f_0 - \sum_{i=1}^{n} e_i f_0 = f_0, \qquad e_j f_0 = e_j f - \sum_{i=1}^{n} e_j e_i f = e_j f - e_j f = 0, \quad j = 1, 2, \dots, n. \tag{2.6}$$

Thus, $e_0 f_0 = f_0$, so e_0 is idempotent; and, by the argument above, $e_j e_0 = 0$, $j = 1, 2, \dots, n$. The formal calculation

$$e_j e_0 = e_j(1 - e_1 - \cdots - e_n) = e_j - e_j e_1 - \cdots - e_j e_n = e_j - e_j = 0 \tag{2.7}$$

would indeed have sufficed, and it shows equally well that $e_0 e_j = 0$. We conclude that $\{e_0, e_1, \dots, e_n\}$ constitutes a system of orthogonal idempotents on $[A, Y]$ such that $e_0 + e_1 + \cdots + e_n = 1$, as was to be proved. □

Now let j be the natural inclusion

$$j : Y = Y_1 \vee Y_2 \vee \cdots \vee Y_n \longrightarrow \prod Y = Y_1 \times Y_2 \times \cdots \times Y_n \tag{2.8}$$

and let $\flat(Y)$ be the homotopy fiber of j. For any $f \in [A, Y]$, consider the diagram

$$\begin{array}{ccccc} & & \flat(Y) & & \\ & & \downarrow{\scriptstyle k} & & \\ A & \xrightarrow{f_0} & Y & \xrightarrow{j} & \prod Y. \end{array} \tag{2.9}$$

Now $j = \{\pi_1, \pi_2, \dots, \pi_n\}$, so $j f_0 = \{\pi_1 f_0, \pi_2 f_0, \dots, \pi_n f_0\}$. Moreover,

$$\pi_s e_i = \begin{cases} 0, & i \neq s, \\ \pi_s, & i = s, \end{cases} \tag{2.10}$$

so that $\pi_s f_0 = \pi_s(1 - e_1 - \cdots - e_n) f = (\pi_s - \pi_s) f = 0$. Hence $j f_0 = 0$ and f_0 lifts into $\flat(Y)$.

Lemma 2.2. *The lift of f_0 into $\flat(Y)$ is unique.*

PROOF. We have the fiber sequence

$$\Omega(\flat Y) \longrightarrow \Omega Y \longrightarrow \Omega \prod Y \longrightarrow \flat Y \longrightarrow Y \longrightarrow \prod Y \tag{2.11}$$

inducing the exact sequence

$$[A, \Omega Y] \xrightarrow{j_*} \Big[A, \Omega \prod Y\Big] \longrightarrow [A, \flat Y] \xrightarrow{k_*} [A, Y] \xrightarrow{j_*} \Big[A, \prod Y\Big]. \tag{2.12}$$

Now $[A, \prod Y] = \bigoplus_i [A, Y_i]$ and j_* obviously maps $[A, Y]$ onto $[A, Y_i]$. Thus $j_* : [A, Y] \to [A, \prod Y]$ is surjective. Equally well $j_* : [A, \Omega Y] \to [A, \Omega \prod Y]$ is surjective, so $k_* : [A, \flat Y] \to [A, Y]$ is injective. □

REMARK 2.3. Clearly $j_* : [A, Y] \to [A, \prod Y]$ has a right inverse m, given by

$$m \mid [A, Y_i] = \iota_{i*}. \tag{2.13}$$

Thus

$$[A, \flat Y] \xrightarrow{k_*} [A, Y] \xrightarrow{j_*} \Big[A, \prod Y\Big] \tag{2.14}$$

is a split short exact sequence. Notice that the validity of this remark depends on the commutativity of A; so does much of the preceding reasoning.

We write $\ell(f)$ for the lift of f_0 into $\flat(Y)$. Since e_0 and k_* are homomorphisms, it follows that

$$\ell : [A, Y] \longrightarrow [A, \flat(Y)] \tag{2.15}$$

is a homomorphism.

We now analyse $\ell(f)$. Let $Y' = Y_{i_1} \vee \cdots \vee Y_{i_k}$, $1 \le i_1 < i_2 < \cdots < i_k \le n$. We call Y' a *summand* or simply a *subspace* of the costructured space Y of span k and write $|Y'| = k$. We plainly have maps

$$\pi : Y \longrightarrow Y', \qquad \iota : Y' \longrightarrow Y, \qquad \pi : \prod Y \longrightarrow \prod Y', \qquad \iota : \prod Y' \longrightarrow \prod Y \tag{2.16}$$

and inducing maps

$$\pi_{Y'} : \flat Y \longrightarrow \flat Y', \qquad \iota_{Y'} : \flat Y' \longrightarrow \flat Y \tag{2.17}$$

such that

$$\pi_{Y'} \iota_{Y'} = \mathrm{Id}, \qquad \iota_{Y'} \pi_{Y'} = e_{Y'} : \flat Y \longrightarrow \flat Y, \tag{2.18}$$

with $e_{Y'}$ an idempotent map.

In particular, suppose that $|Y'| = 2$. Then, as Y' ranges over the subspaces of Y of span 2, we obtain a system of *orthogonal* idempotents $e_{Y'}$ on $\flat Y$. We also obtain a map

$$\pi : \flat Y \longrightarrow \prod_{|Y'|=2} \flat Y' \tag{2.19}$$

with components $\pi_{Y'}$. Thus (compare the definition of $\ell(f)$ above) for any map $\ell : A \to \flat Y$, we may lift

$$\ell - \sum_{|Y'|=2} e_{Y'}\ell \tag{2.20}$$

to the homotopy fiber of π (2.19), which we call $\flat^2 Y$. Moreover, the lift is unique so that

$$\ell = \sum_{|Y'|=2} e_{Y'}\ell + k^{(2)}\ell^{(2)}, \tag{2.21}$$

where $\ell^{(2)}$ is the lift of $\ell - \sum_{|Y'|=2} e_{Y'}\ell$ so that

$$A \xrightarrow{\ell^{(2)}} \flat^2 Y \xrightarrow{k^{(2)}} \flat Y \tag{2.22}$$

is a fiber sequence. Note that $e_{Y'}\ell$ factors through $\flat Y'$.

We now continue the process of analysing ℓ. We next have to analyse $\ell^{(2)}$. We now allow Y' to range over all subspaces of Y of span 3. We thus obtain maps

$$\pi_{Y'} : \flat^2 Y \longrightarrow \flat^2 Y', \qquad \iota_{Y'} : \flat^2 Y' \longrightarrow \flat^2 Y \tag{2.23}$$

such that

$$\pi_{Y'}\iota_{Y'} = \mathrm{Id}, \qquad \iota_{Y'}\pi_{Y'} = e_{Y'} : \flat^2 Y \longrightarrow \flat^2 Y, \tag{2.24}$$

with $e_{Y'}$ an idempotent map. As before, we obtain a map

$$\pi : \flat^2 Y \longrightarrow \prod_{|Y'|=3} \flat^2 Y' \tag{2.25}$$

with components $\pi_{Y'}$. The maps $e_{Y'}$ on $\flat^2 Y$ constitute a system of orthogonal idempotents so that the map $\ell^{(2)} : A \to \flat^2 Y$ may be written as

$$\ell^{(2)} = \sum_{|Y'|=3} e_{Y'}\ell^{(2)} + k^{(3)} + \ell^{(3)}, \tag{2.26}$$

where $\ell^{(3)}$ is the lift of $\ell^{(2)} - \sum_{|Y'|=3} e_{Y'}\ell^{(2)}$ to $\flat^3 Y$, the homotopy fiber of π of (2.25).

The process terminates when we arrive at the set of subspaces of span $(n-1)$. We will then write ℓ as a sum of $2^n - n - 1$ maps, corresponding to the subspaces of Y of span greater that or equal to 2 (including Y itself). Thus, we have proved the following proposition.

Proposition 2.4. *The set* $\{e_{Y'} \mid Y' \subset Y, |Y'| \geq 2\}$ *is a complete system of* $2^n - n - 1$ *orthogonal idempotents on the group* $[A, \flat Y]$ *even though* $\flat Y$ *is not given as a wedge of spaces. As a corollary, any map* $\ell : A \to \flat Y$ *may be uniquely expressed as the sum of* $2^n - n - 1$ *terms* $\ell_{Y'}$*, each factoring through a subspace* $\flat^{k-1} Y'$ *of* $\flat Y$ *that corresponds to a subspace* Y' *of* Y *of span* $k \geq 2$.

Notice that the mystery of the number $2^n - n - 1$ is dissolved if one observes the corollary or the equivalent statement.

Theorem 2.5. *The set* $\{k e_{Y'} \ell \mid Y' \subset Y, |Y'| \geq 1\}$ *is a complete system of* $2^n - 1$ *orthogonal idempotents on the group* $[A, Y]$*, obtained from the fact that* Y *is a wedge of* n *subspaces. Thus any map* $f : A \to Y$ *may be uniquely expressed as the sum of* $2^n - 1$ *terms* $f_{Y'}$*, each factoring through a subspace* $\flat^{k-1} Y'$ *of* Y *that corresponds to a subspace* Y' *of* Y *of span* $k \geq 1$.

Remark 2.6. In Theorem 2.5, the idempotent $k e_{Y'} \ell$ simply means $e_{Y'}$ if $|Y'| = 1$.

We next discuss the naturality of ℓ. Let $g : A \to B$ be a homomorphism of commutative cogroups and let $Y = Y_1 \vee Y_2 \vee \cdots \vee Y_n$ and $Z = Z_1 \vee Z_2 \vee \cdots \vee Z_n$ be two costructured spaces. Finally, let $h : Y \to Z$ be a *costructure-preserving map*, that is, $h(Y_i) \subseteq Z_i$, $i = 1, 2, \ldots, n$. We then prove

Theorem 2.7. *If the diagram*

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ B & \xrightarrow{f'} & Z \end{array} \tag{2.27}$$

commutes, then so does the diagram

$$\begin{array}{ccc} A & \xrightarrow{\ell(f)} & \flat Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \flat h} \\ B & \xrightarrow{\ell(f')} & \flat Z. \end{array} \tag{2.28}$$

Proof. We first remark that $\flat$ is obviously a functor $\flat : \mathcal{H}^n \to \mathcal{H}$, so $\flat h$ is defined. Now, to prove the commutativity of (2.28), we first consider the diagram

$$\begin{array}{ccc} A & \xrightarrow{f_0} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ B & \xrightarrow{f_0'} & Z. \end{array} \tag{2.29}$$

Then $hf_0 = h(f - \sum_{i\geq 1} e_i f) = hf - \sum_{i\geq 1} he_i f$. But $he_i = h\iota_i\pi_i = \iota_i h_i \pi_i = \iota_i\pi_i h = e_i h$, where $h_i : Y_i \to Z_i$ is obtained by restricting h. Thus

$$\begin{aligned} hf_0 &= hf - \sum e_i hf \\ &= f'g - \sum e_i f'g \\ &= \Big(f' - \sum e_i f'\Big)g, \quad \text{since } g \text{ is a homomorphism} \\ &= f_0' g, \end{aligned} \tag{2.30}$$

establishing the commutativity of (2.29).

Now we have an obvious commutative diagram (used, in fact, to present $\flat$ as a functor)

$$\begin{array}{ccccc} \flat Y & \xrightarrow{k} & Y & \xrightarrow{j} & \prod Y \\ \downarrow{\scriptstyle \flat h} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \prod h} \\ \flat Z & \xrightarrow{k} & Z & \xrightarrow{j} & \prod Z, \end{array} \tag{2.31}$$

so the commutativity of (2.28) follows from (2.29) and (2.31), using the fact that k is a monomorphism.

We may further refine the naturality as follows. Let Y' be a summand of Y of span $k \geq 2$ and let $\ell_{Y'}(f)$ be the corresponding component of $\ell(f)$, regarded as a map $\ell_{Y'}(f) : A \to \flat^{k-1}Y'$. Let Z' be the *corresponding* summand of Z and define $\ell_{Z'}(f') : B \to \flat^{k-1}Z'$ similarly. Then h induces $\flat^{k-1}h : \flat^{k-1}Y' \to \flat^{k-1}Z'$ and the diagram

$$\begin{array}{ccc} A & \xrightarrow{\ell_{Y'}(f)} & \flat^{k-1}Y' \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \flat^{k-1}h} \\ B & \xrightarrow{\ell_{Z'}(f')} & \flat^{k-1}Z' \end{array} \tag{2.32}$$

commutes. We leave the details to the reader, remarking that we may reexpress this refinement by asserting the naturality of the idempotents and summation described in Proposition 2.4.

We now formally introduce the Hopf invariant. A map $\rho : X \to Y$ is called an *n-counion* map or, more precisely, an n-counion of the maps $\pi_i\rho : X \to Y_i$, $i = 1, 2, \ldots, n$. If $n = 2$, this is called a *copairing* (see [5]). Now ρ induces a homomorphism $\rho : [A, X] \to [A, Y]$ and, for $d : A \to X$, we define the *Hopf invariant of d, relative to* ρ, to be $\ell(\rho d) \in [A, \flat Y]$. Plainly this defines a homomorphism

$$\mathrm{HI} = \mathrm{HI}_\rho : [A, X] \longrightarrow [A, \flat Y]. \tag{2.33}$$

□

We may now invoke Proposition 2.4, applied to the map $\ell = \ell(\rho d) = \mathrm{HI}_\rho(d)$, to conclude

Theorem 2.8. *The Hopf invariant* $\mathrm{HI}_\rho : [A,X] \to [A,\flat Y]$ *may be expressed as the sum of* $2^n - n - 1$ *homomorphisms* $\mathrm{HI}_{Y'}$, *each factoring through a group* $[A,\flat^{k-1}Y']$ *corresponding to a subspace* Y' *of* Y *of span* $k \geq 2$.

We say that the Hopf invariant is thereby expressed as a *sum of constituent subinvariants.*

The naturality of the Hopf invariant now expresses itself as follows. We suppose a given commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{d} & X_1 & \xrightarrow{\rho} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle h} \\ B & \xrightarrow{d'} & X_2 & \xrightarrow{\rho'} & Z, \end{array} \tag{2.34}$$

where g is a homomorphism of commutative cogroups and h is costructure-preserving. Then (2.33) induces a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{HI}_\rho(d)} & \flat Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \flat h} \\ B & \xrightarrow{\mathrm{HI}_{\rho'}(d')} & \flat Z. \end{array} \tag{2.35}$$

There is also a *refined* form of this naturality statement, based on (2.32), involving the constituent subinvariants. It is further of interest to interpret the *vanishing* of the Hopf invariant; thus

Theorem 2.9. *The vanishing of* $\mathrm{HI}_\rho(d)$ *is equivalent to the assertion that*

$$\rho d = e_1\rho d + e_2\rho d + \cdots + e_n\rho d. \tag{2.36}$$

Proof. $\ell(\rho d) = 0 \Leftrightarrow (\rho d)_0 = 0 \Leftrightarrow \rho d - e_1\rho d - e_2\rho d - \cdots - e_n\rho d = 0.$ Notice that $e_i\rho d = \iota_1\alpha_i d$, where ρ is a counion of $\alpha_1,\alpha_2,\ldots,\alpha_n$, with $\alpha_i : X \to Y_i$. □

A version of Theorems 2.5, 2.7, 2.8, and 2.9 appeared in [3].

The above definition of the Hopf invariant generalizes that given in the case $n = 2$ in [5], where it is further related to a number of existing definitions. Staying always with the case $n = 2$, but confining attention to *coactions* (or *cooperations*, see [6]) ρ, there is a treatment in [4] which is specialized to a stage in the version of Hilton and Milnor [1, 8] when the coaction is a comultiplication.

Example 2.10. Let $P_1,P_2,\ldots,P_n$ be spaces and let $T^{(1)} = T^{(1)}(P_1,P_2,\ldots,P_n)$ be the *fat wedge* of $\sum P_1,\sum P_2,\ldots,\sum P_n$, in the terminology of Ganea. Thus $T^{(1)}$ is the subspaces of $\sum P_1 \times \sum P_2 \times \ldots \times \sum P_n$ consisting of points with at least one coordinate at the base point. There is then a relative homeomorphism

$$C(P_1 * P_2 * \cdots * P_n), P_1 * P_2 * \cdots * P_n \longrightarrow \sum P_1 \times \sum P_2 \times \cdots \times \sum P_n, T^{(1)}, \tag{2.37}$$

where $P_1 * P_2 * \cdots * P_n$ is the (iterated) join. Let w_n be the associated attaching map, that is,

$$w_n : P_1 * P_2 * \cdots * P_n \longrightarrow T^{(1)} \tag{2.38}$$

is the restriction of the map (2.37).

In the same way, we may consider, for each $Q_j = P_1 * P_2 * \cdots * \hat{P}_j * \cdots * P_n$, the attaching map

$$w_{n-1,j} : Q_j \longrightarrow T^{(1)}(P_1, P_2, \ldots, \hat{P}_j, \ldots, P_n), \quad j = 1, 2, \ldots, n. \tag{2.39}$$

Now the union of the spaces $T^{(1)}(P_1, P_2, \ldots, \hat{P}_j, \ldots, P_n)$ is $T^{(2)}(P_1, P_2, \ldots, P_n)$ "the next fattest wedge," that is, the subspace of $\sum P_1 \times \sum P_2 \times \cdots \times \sum P_n$ consisting of points with at least two coordinates at the base point. Moreover, the maps $w_{n-1,j}$ combine to produce a map

$$w_{n-1} : Q_1 \vee Q_2 \vee \cdots \vee Q_n \longrightarrow T^{(2)} \tag{2.40}$$

whose mapping cone is precisely $T^{(1)}$.

In general, given a map $f : A \to B$ with mapping cone C_f, there is, as explained in [6], a cooperation or coaction $\rho : C_f \to \sum A \vee C_f$ of $\sum A$ on C_f. In our case, with $f = w_{n-1}$ in (2.40), ρ becomes a map

$$\rho : T^{(1)} \longrightarrow \sum Q_1 \vee \sum Q_2 \vee \cdots \vee \sum Q_n \vee T^{(1)} \tag{2.41}$$

and thus an $(n+1)$-counion map. The map (2.41) may be fed into our definition to produce essentially the Hopf invariant of [4] and to motivate our definition in this paper.

3. A canonical factorization of the Hopf invariant. In this section we describe a canonical expression for $\mathrm{HI}_\rho(d)$ as a composition of four maps, each depending on a particular ingredient of the definition of the Hopf invariant.

Given $f : A \to Y = Y_1 \vee Y_2 \vee \cdots \vee Y_n$, where A is a commutative cogroup, we may express f_0 as the composition

$$A \xrightarrow{\mu^{(n+1)}} \bigvee_{n+1} A \xrightarrow{\langle f, -e_1 f, \ldots, -e_n f \rangle} Y, \tag{3.1}$$

where $\mu : A \to A \vee A$ is the comultiplication. Further factorizing the right-hand map in (3.1) and slightly modifying the relevant factors yield the composition

$$A \xrightarrow{\sigma} \bigvee_{n+1} A \xrightarrow{\vee f} \bigvee_{n+1} Y \xrightarrow{\langle 1,e_1,\ldots,e_n\rangle} Y \tag{3.2}$$

of f_0, where σ is the sum of the identity 1, mapping A to the first summand in $\bigvee_{n+1} A$, and maps -1, mapping A into the second, third, $\ldots$, $(n+1)$th summand of $\bigvee_{n+1} A$. We could write $\sigma = 1_1 - 1_2 - \cdots - 1_{n+1}$.

Now let j, as before, be the canonical map from $\bigvee_i B_i$ to $\prod_i B_i$. We may then embed (3.2) in the commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{\sigma} & \bigvee_{n+1} A & \xrightarrow{\vee f} & \bigvee_{n+1} Y & \xrightarrow{\langle 1,e_1,\ldots,e_n\rangle} & Y \\ & & \Big\downarrow{\scriptstyle\langle\Delta,j\rangle} & & \Big\downarrow{\scriptstyle\langle\Delta,j\rangle} & & \Big\downarrow{\scriptstyle j} \\ & & \prod_n A & \xrightarrow{\prod f} & \prod_n Y & \xrightarrow{\prod \pi_i} & \prod Y_i. \end{array} \tag{3.3}$$

Here $\Delta : A \to \prod A$ and $\Delta : Y \to \prod Y$ are diagonal maps. The commutativity of the left-hand square in (3.3) is obvious. As for the right-hand square, we observe that $(\prod \pi_i)\Delta y_k = (*,\ldots,*,y_k,*,\ldots,*) = jy_k$ if $y_k \in Y_k$; and that, if $y \in (m+1)$th copy of Y in $\bigvee_{n+1} Y$, $m \geq 1$, then

$$\begin{aligned} \left(\prod \pi_i\right) jy &= \prod \pi_i(*,\ldots,*,y,*,\ldots,*), \\ &= (*,\ldots,*,\pi_m y,*,\ldots,*) \\ &= je_m y, \end{aligned} \tag{3.4}$$

where y appears in the mth factor Y in $\prod_n Y$. Further, it is clear that $\langle\Delta, j\rangle\sigma = 0 : A \to \prod_n A$ for $j(1_2 + 1_3 + \cdots + 1_{n+1}) = \Delta$. Thus σ lifts to $\kappa : A \to F_A$, where F_A is the homotopy fiber of $\langle\Delta, j\rangle$, and we may embed (3.3) in the commutative diagram

$$\begin{array}{ccccccc} & & F_A & \xrightarrow{F_f} & F_Y & \xrightarrow{\lambda} & \flat Y \\ & {\scriptstyle\kappa}\nearrow & \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\ A & \xrightarrow{\sigma} & \bigvee_{n+1} A & \xrightarrow{\vee f} & \bigvee_{n+1} Y & \xrightarrow{\langle 1,e_1,\ldots,e_n\rangle} & Y \\ & & \Big\downarrow{\scriptstyle\langle\Delta,j\rangle} & & \Big\downarrow{\scriptstyle\langle\Delta,j\rangle} & & \Big\downarrow{\scriptstyle j} \\ & & \prod_n A & \xrightarrow{\prod f} & \prod_n Y & \xrightarrow{\prod \pi_i} & \prod Y_i \end{array} \tag{3.5}$$

and obviously

$$\lambda(F_f)\kappa = \ell(f). \tag{3.6}$$

We now revert to the Hopf invariant. We only have to replace f in (3.5) by $f = \rho d$,

$$A \xrightarrow{d} X \xrightarrow{\rho} Y. \tag{3.7}$$

Thus it is a matter of factoring the $(F_f, \vee f, \prod f)$-column of (3.5) in the obvious way to obtain

$$\begin{array}{ccccccccc}
 & \overset{\kappa}{\nearrow} & F_A & \xrightarrow{F_d} & F_X & \xrightarrow{F\rho} & F_Y & \xrightarrow{\lambda} & \flat Y \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
A & \xrightarrow{\sigma} & \bigvee_{n+1} A & \xrightarrow{\vee d} & \bigvee_{n+1} X & \xrightarrow{\vee \rho} & \bigvee_{n+1} Y & \xrightarrow{\langle 1, e_1, \dots, e_n \rangle} & Y \\
 & & \downarrow{\scriptstyle \langle \Delta, j \rangle} & & \uparrow{\scriptstyle \langle \Delta, j \rangle} & & \downarrow{\scriptstyle \langle \Delta, j \rangle} & & \downarrow{\scriptstyle j} \\
 & & \prod_n A & \xrightarrow{\prod d} & \prod_n X & \xrightarrow{\prod \rho} & \prod_n Y & \xrightarrow{\prod \pi_i} & \prod Y_i
\end{array} \tag{3.8}$$

and the factorization

$$\mathrm{HI}_\rho(d) = \lambda (F\rho)(F_d)\kappa. \tag{3.9}$$

This is the canonical factorization of the title of this section.

If the counion of size n is assumed to be given, then we notice that, in (3.9),

(i) κ depends only on A;

(ii) F_d depends only on d;

(iii) F_ρ depends only on ρ;

(iv) λ depends only on Y, with its costructure.

Without going into details, we make the obvious remark that a similar factorization is available for each of the $2^n - n - 1$ constituent subinvariants of the Hopf invariant in the sense of Theorem 2.8.

Remarks. (i) The factor F_ρ of the Hopf invariant $\mathrm{HI}_\rho(d)$ in (3.9) is related to Walker's version of the Hopf invariant [11]. Walker starts with a pair of maps

$$B \xleftarrow{f} A \xrightarrow{g} C \tag{3.10}$$

and constructs the double mapping cylinder $Z = Z_{f,g}$. There is thus a commutative square

$$\begin{array}{ccc}
A & \xrightarrow{g} & C \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \iota_C} \\
B & \xrightarrow{\iota_B} & Z.
\end{array} \tag{3.11}$$

Let P be the homotopy pullback of i_B and i_C in (3.11), creating a commutative diagram

$$\begin{array}{ccccc} & & B & & \\ & \nearrow & \downarrow & \nwarrow & \\ P & & Z & & A \\ & \searrow & \uparrow & \swarrow & \\ & & C. & & \end{array} \tag{3.12}$$

We now pinch A to a point in (3.11). We think of B and C in (3.11) as having been replaced by the mapping cylinders of f and g, respectively, so that (3.11) becomes a strict pushout of inclusion maps. When we pinch A to a point throughout (3.11), the mapping cylinders become mapping cones C_f and C_g and Z becomes the one-point union (coproduct) $C_f \vee C_g$. Thus (3.11) turns into the diagram

$$\begin{array}{ccc} * & \longrightarrow & C_g \\ \downarrow & & \downarrow {\scriptstyle \iota_g} \\ C_f & \xrightarrow{\iota_f} & C_f \vee C_g. \end{array} \tag{3.13}$$

Following Walker, we designate the homotopy pullback of ι_f and ι_g by $C_f \hat{*} C_g$. We thus obtain the diagram

$$\begin{array}{ccccc} & & C_f & & \\ & \nearrow & \downarrow & \nwarrow & \\ C_f \hat{*} C_g & & C_f \vee C_g & & * \\ & \searrow & \uparrow & \swarrow & \\ & & C_g & & \end{array} \tag{3.14}$$

and the pinching maps induce a map of diagram (3.12) to diagram (3.14). The component

$$P \longrightarrow C_f \hat{*} C_g \tag{3.15}$$

of this map of the diagrams is Walker's Hopf invariant. If we write ρ for the component

$$\rho : Z \longrightarrow C_f \vee C_g \tag{3.16}$$

of this map of diagrams, then this ρ is a two-counion map and plays the same role as in our definition of the Hopf invariant.

We conjecture that Walker's Hopf invariant is closely related to the map F_ρ of (3.8), with $X = Z$ and $Y = C_f \vee C_g$.

(ii) It is plain that the maps κ of the factorization (3.9) for various values of n may be related. We write κ_n and σ_n for the maps κ and σ of (3.8). Then $\sigma_1 = 1_1 - 1_2 : A \to A \vee A$ and there are maps

$$q = \sigma_1 \vee 1_3 \vee \cdots \vee 1_{n+2} : \bigvee_{n+1} A \longrightarrow \bigvee_{n+2} A, \qquad r : \bigvee_{n+2} A \longrightarrow \bigvee_{n+1} A, \tag{3.17}$$

where r projects *off* the second summand A. Then

$$rq = 1 : \bigvee_{n+1} A \longrightarrow \bigvee_{n+1} A, \qquad q\sigma_n = \sigma_{n+1} : A \longrightarrow \bigvee_{n+2} A. \tag{3.18}$$

Now let $\bar{q} : \prod_n A \to \prod_{n+1} A$ map $\underline{a}$ to $(*, \underline{a})$ and let $\bar{r} : \prod_{n+1} A \to \prod_n A$ project *off* the first component. Then the diagram

$$\begin{array}{ccc} \bigvee_{n+1} A & \underset{r}{\overset{q}{\rightleftarrows}} & \bigvee_{n+2} A \\ \Big\downarrow{\scriptstyle\langle \Delta, j\rangle} & & \Big\downarrow{\scriptstyle\langle \Delta, j\rangle} \\ \prod_n A & \underset{\bar{r}}{\overset{\bar{q}}{\rightleftarrows}} & \prod_{n+1} A \end{array} \tag{3.19}$$

commutes so that there are induced maps, which we again write as q, r, thus

$$F_{A,n} \underset{r}{\overset{q}{\rightleftarrows}} F_{A,n+1} \tag{3.20}$$

such that

$$rq = 1 : F_{A,n} \longrightarrow F_{A,n}, \qquad q\kappa_n = \kappa_{n+1} : A \longrightarrow F_{A,n+1}. \tag{3.21}$$

We may use the maps q to pass to the limit $F_{A\infty}$, obtaining a map $\kappa_\infty : A \to F_{A\infty}$ which is independent of n and thus truly universal.

(iii) There is a relation between the duals of the constructions in this paper and some aspects of Selick's paper [10]. In [10, Section 3, page 408], Selick's D is a dual for the present Hopf invariant based on two summands for the particular case of a multiplication. Selick's D_n is a sum of duals of the Hopf invariant associated to $e_{Y'}$, $|Y'| \geq 2$, in the case of a multiplication. Selick's f_i is the dual of our $e_{Y'} f$ for the case $|Y'| = i$ and multiplication. In [10, Lemma 1, page 408], he proves that the n-fold multiplication of f is a sum of $2^n - 1$ summands. In the proof of [10, Lemma 5, page 410], he starts with the summands of [10, Lemma 1] and then moves the n summands that correspond to $e_{Y'}$, $|Y'| = 1$, to the side of f, getting on the right-hand side a sum of $2^n - n - 1$ summands that correspond to $e_{Y'}$, $|Y'| \geq 2$, mirroring our result in Theorem 2.8.

4. The Hilton quadratic matrix calculus. The quadratic matrix calculus defined in [7] and used in [2, 7, 9] can be obtained by using the present paper idempotents of span 2. We here use [2] as a source and deduce some of the results of [2, Section 4].

Let $\alpha : S^{4m-1} = A = X \to S^{2m} \vee S^{2m} \vee \cdots \vee S^{2m} = \bigvee_K S^{2m} = Y$ be given as on [2, page 373]. Then α is a K-counion map as defined in the introduction and thus induces idempotents $\{e_{Y'}, |Y'| \geq 2\}$ on the group $[S^{4m-1}, \flat Y]$.

It can be observed that for $|Y'| = 2$, $\flat Y'$ is the fiber of the map $S^{2m} \vee S^{2m} \to S^{2m} \times S^{2m}$, which, as in [4], is equivalent to $\Omega(S^{2m}) * \Omega(S^{2m})$, which, by James' celebrated formula [8], equals $S^{4m-1} \cup e^{6m-2} \cup \cdots$. Thus $[S^{4m-1}, \flat Y'] = [S^{4m-1}, S^{4m-1}] = \mathcal{Z}$.

Also, for $|Y'| = 3$, say $Y' = \{a, b, c\}$, $\flat Y'$ can be embedded in the following commutative diagram:

$$
\begin{array}{ccccc}
\flat Y'_{a,b,c} & \longrightarrow & \overline{\flat Y'_{a,b,c}} & \longrightarrow & \flat Y_{a,b} \times \flat Y_{a,c} \times \flat Y_{b,c} \\
 & & \downarrow & & \downarrow \\
 & & S_a^{2m} \vee S_b^{2m} \vee S_c^{2m} & \longrightarrow & (S_a^{2m} \vee S_b^{2m}) \times (S_a^{2m} \vee S_c^{2m}) \times (S_b^{2m} \vee S_c^{2m}) \\
 & & \downarrow & & \downarrow \\
 & & S_a^{2m} \times S_b^{2m} \times S_c^{2m} & \longrightarrow & (S_a^{2m} \times S_b^{2m}) \times (S_a^{2m} \times S_c^{2m}) \times (S_b^{2m} \times S_c^{2m})
\end{array}
\tag{4.1}
$$

and by the well-known Hilton-Milnor Theorem, the first cell in $\flat Y_{a,b,c}$ is $(S_a^{2m-1} * S_b^{2m-1}) \wedge S_c^{2m-1}$ which is $(6m-3)$-connected so that $[A, \flat Y'_{a,b,c}] = 0$. Thus the only idempotents in Theorem 2.5 correspond to span less than three. In particular, on $[A, Y]$, we have

$$1 = \sum_{i=1}^{K} e_i + \sum_{1 \leq i < j \leq K} e_{i,j}, \tag{4.2}$$

and in particular

$$f = \sum_{i=1}^{K} f_i + \sum_{1 \leq i < j \leq K} f_{i,j}, \tag{4.3}$$

where $f_{i,j}$ factors as

$$S^{4m-1} \longrightarrow \flat Y_{i,j} \xrightarrow{\simeq} S^{4m-1} \xrightarrow{w_2} S_i^{2m} \vee S_j^{2m} \xrightarrow{\text{inc}} Y. \tag{4.4}$$

Thus $f_{i,j}$ is determined completely by the degree of the map $S^{4m-1} \to S^{4m-1}$, denoted by a_{ij} on [2, page 373].

For $i = j$, $f_{i,j} = f_j$ factors as

$$A = S^{4m-1} \xrightarrow{f} Y = \bigvee_K S_i^{2m} \xrightarrow{\pi_j} S_j^{2m} \xrightarrow{\text{inc}_j} Y. \tag{4.5}$$

Now $\pi_j f : S^{4m-1} \to S^{2m}$ has a Hilton-Milnor invariant obtained by considering $\rho f : S^{4m-1} \to S^{2m} \vee S^{2m}$, where ρ is the comultiplication map. The invariant of f is a map $a_{i,i} : S^{4m-1} \to S^{4m-1}$. Thus the following matrix is obtained:

$$\underline{H}(f) = \begin{cases} a_{ij} = a_{ji}, & i \neq j, \\ a_{ii}, & i = j, \end{cases} \tag{4.6}$$

defined on [2, page 373] and called the Hilton-Hopf quadratic form of f.

Given any map $f : \bigvee_J S^{4m-1} \to \bigvee_K S^{2m}$, it is determined by the inclusions $f_j : S^{4m-1} \xrightarrow{\text{in}_j} \bigvee_J S^{4m-1} \xrightarrow{f} \bigvee_K S^{2m}$, each of which is determined by a $K \times K$ integer matrix as above. Then f is determined by J integer matrices which can be written using juxtaposition as a single $K \times JK$ integer matrix composed of J blocks:

$$\underline{H}(f) = (\underline{H}(f_1), \underline{H}(f_2), \ldots, \underline{H}(f_J)) \tag{4.7}$$

as described on [2, page 375].

Finally, for a cogroup A and $Y = \bigvee_K S^m$, any map $\phi : A \to Y$ has a decomposition as above: $\phi = \sum_{|Y'| \geq 1} e_{Y'} \phi$. If $A = Y = \bigvee_K S^m$, then every $e_{Y'}, |Y'| > 1$, factors through a multiconnected space and is thus null. Thus we have $\phi = \sum_{j=1}^K \phi_j$, where ϕ_j is a map $Y \to Y \xrightarrow{\pi_j} S_j^m \hookrightarrow Y$. Each map out of a wedge is determined by the restrictions $S_j^m \hookrightarrow Y \xrightarrow{\phi_j} S_j^m$ so that $\phi : Y \to Y$ is determined by $K \times K$ maps $S_i^m \overset{\text{in}_i}{\hookrightarrow} Y \xrightarrow{\phi} Y \xrightarrow{\pi_j} S_j^m$, which are determined by degrees. Thus $\phi = \{\deg(\phi_{j,i})\} = \underline{A}(\phi)$ is a $K \times K$ matrix, defined exactly as in [2, Lemma 4.5(iii)]. (The reader may find, in what follows, some superficial changes of notation from that of [2].) We now describe *the matrix calculus.*

Suppose the following given composition:

$$\bigvee_J S^{4m-1} \xrightarrow{\psi} \bigvee_J S^{4m-1} \xrightarrow{f} \bigvee_K S^{2m} \xrightarrow{\phi} \bigvee_K S^{2m}. \tag{4.8}$$

Then ψ is determined by a $J \times J$ integer degree matrix $\underline{A}(\psi)$, ϕ by a $K \times K$ integer degree matrix $\underline{A}(\phi)$, and f leads to an integer $K \times JK$ matrix of the Hilton quadratic form $\underline{H}(f)$. Also $\phi f \psi$ creates an integer $K \times JK$ matrix of the quadratic form $\underline{H}(\phi f \psi)$. Then, for $i < j$, the (i,j)th term of the matrix $\underline{H}(\phi f \psi)$ is given by using the projection $\pi_{i,j} \phi f \psi : \bigvee_K S^{4m-1} \to S_i^{2m} \vee S_j^{2m}$ and a lifting $\bigvee_J S^{4m-1} \to S^{4m-1} \xrightarrow{w_{i,j}} S_i^{2m} \vee S_j^{2m}$.

Map (4.8) can be presented using inclusions in the following way:

$$\mathrm{com}_{i,j}^k : S_k^{4m-1} \xrightarrow{\mathrm{inc}_k} \bigvee_J S^{4m-1} \xrightarrow{\phi f \psi} \bigvee_K S^{4m-1} \xrightarrow{\pi_{i,j}} S_i^{2m} \vee S_j^{2m} \tag{4.9}$$

for $1 \le i < j \le K$ and $1 \le k \le J$.

Thus there are $K \times JK$ terms as above. Then we have the composition

$$\mathrm{com}_{i,j}^k : S_k^{4m-1} \xrightarrow{\mathrm{inc}_k} \bigvee_J S^{4m-1} \xrightarrow{\psi} \bigvee_J S^{4m-1} \xrightarrow{f} \bigvee_K S^{2m} \xrightarrow{\phi} \bigvee_K S^{2m} \overset{\pi_{i,j}}{\twoheadrightarrow} S_i^{2m} \vee S_j^{2m}. \tag{4.10}$$

This last map equals

$$\pi_{i,j}\phi f \psi \,\mathrm{inc}_k = \pi_{i,j}\phi\Big(\sum_{1\le p<q\le K} e_{p,q} + \sum_{s=1}^{K} e_s\Big) f \Big(\sum_{r=1}^{J} er\Big)\psi \,\mathrm{inc}_k. \tag{4.11}$$

As e_s factors through S^{2m} and is null into S^{4m-1}, then $\mathrm{com}_{i,j}^k$ equals

$$\sum_{1\le p<q\le K,\ 1\le r\le J} \pi_{i,j}\phi e_{p,q} f e_r \psi \,\mathrm{inc}_k. \tag{4.12}$$

Any of the summands is a composition of three maps:

$$\pi_r \psi \,\mathrm{inc}_k : S_k^{4m-1} \xrightarrow{\mathrm{inc}_k} \bigvee_J S^{4m-1} \xrightarrow{\psi} \bigvee_J S^{4m-1} \xrightarrow{\pi_r} S_r^{4m-1}, \tag{4.13}$$

$$\pi_{p,q} f \,\mathrm{inc}_r : S_r^{4m-1} \xrightarrow{\mathrm{inc}_r} \bigvee_J S^{4m-1} \xrightarrow{f} \bigvee_K S^{2m} \xrightarrow{\pi_{p,q}} S_p^{2m} \vee S_q^{2m}, \tag{4.14}$$

$$\pi_{i,j}\phi \,\mathrm{inc}_{p,q} : S_p^{2m} \vee S_q^{2m} \xrightarrow{\mathrm{inc}_{p,q}} \bigvee_K S^{2m} \xrightarrow{\phi} \bigvee_K S^{2m} \xrightarrow{\pi_{i,j}} S_i^{2m} \vee S_j^{2m}. \tag{4.15}$$

The map in (4.13) is by definition $\underline{A}(\psi)_r^k$. The second part (4.14) defines a lifting $S_r^{4m-1} \to S_{p,q}^{4m-1}$ which is, by definition, $\underline{H}(f)_{p,q}^r$; and the map of the fibers $S_{p,q}^{4m-1} \to S_{i,j}^{4m-1}$ obtained from (4.14) is induced by applying the matrix $\overline{\phi}_{i,j}^{p,q}$, which is a 2×2 submatrix of $\underline{A}(\phi)$.

Thus we get the formula

$$\underline{H}(\phi f \psi)_{i,j}^k = \underline{A}(\phi)_{i,j}^{p,q}\underline{H}(f)_{p,q}^r \underline{A}(\psi)_r^k. \tag{4.16}$$

Recall that $\underline{A}(\phi)_{i,j}^{p,q} = \underline{A}(\phi)_i^p \otimes \underline{A}(\phi)_j^q$. Now we consider the matrices as written in [2].

Thus $\underline{H}(f)_{p,q}^r$ is written in the $(p,(r-1)K+q)$th spot. While $\underline{H}(f)_{i,j}^k$ is written in the $(i,(k-1)K+j)$th spot. The way to express the indices of matrices so as to accord with the multiplication is

$$(i,p)(p,(r-1)K+q)((r-1)K+q,(k-1)K+j). \tag{4.17}$$

The correct matrix setup for this is $\underline{A}(\phi)\underline{H}(f)(\underline{A}(\psi)\otimes\underline{A}(\alpha)^t)$, which is the first line of [2, Lemma 4.8].

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