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A NOTE ON THE REDUCIBILITY OF LINEAR DIFFERENTIAL EQUATIONS WITH QUASIPERIODIC COEFFICIENTS

XIAOPING YUAN and ANA NUNES

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The system $\dot{x} = (A+\epsilon Q(t))x$, where A is a constant matrix whose eigenvalues are not necessarily simple and Q is a quasiperiodic analytic matrix, is considered. It is proved that, for most values of the frequencies, the system is reducible.

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1. Introduction and results. Consider the quasiperiodic linear differential equation

$$\dot{x} = (A+\epsilon Q(t))x \tag{1.1}$$

with x an n-dimensional vector, A a constant square matrix of order n, and Q a square matrix of order n, quasiperiodic in time t. We say that a change of variables $x = P(t)y$ is a Lyapunov-Perron (LP) transformation if $P(t)$ is nonsingular and $P(t)$, $P^{-1}(t)$, and $\dot{P}(t)$ are bounded for all $t \in \mathbb{R}$. Moreover, if P, P^{-1}, and $\dot{P}$ are quasiperiodic in time t, we refer to $x = P(t)y$ as a quasiperiodic LP transformation. If there is a quasiperiodic LP transformation $x = P(t)y$ such that y satisfies the equation

$$\dot{y} = By \tag{1.2}$$

with B a constant matrix, then we say that (1.1) is reducible.

The concept of the reducibility was first considered by Lyapunov (see [5]). There are several authors who investigated the reducibility of (1.1) (see, e.g., [1, 2, 6]). The present paper complements the results obtained by Jorba and Simó [2], which we will briefly recall. To this end, we will introduce some notation and definitions that will be used throughout the paper.

We say that a function F is a quasiperiodic function in time t, with the basic frequencies $\omega = (\omega_1,\dots,\omega_r)$, if there exists a function $\mathcal{F}(\theta_1,\dots,\theta_r)$ which is 2π-periodic in all its arguments θ_j, $j = 1,\dots,r$, and such that $F(t) = \mathcal{F}(\omega_1 t,\dots,\omega_r t)$. We call $\mathcal{F}$ the hull of $F(t)$. The function F will be called analytic quasiperiodic in a strip of width δ if, furthermore, $\mathcal{F}$ is analytic in the complex strip $|\mathrm{Im}\theta| < \delta$.

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be the eigenvalues of A and $\lambda^0(\epsilon) = (\lambda_1^0(\epsilon), \ldots, \lambda_n^0(\epsilon))$ the eigenvalues of $\bar{A} := A + \epsilon\bar{Q}$, where $\bar{Q}$ is the average of $Q(t)$,

$$\bar{Q} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} Q(t)\,dt. \tag{1.3}$$

Assume that $Q(t)$ is analytic on the strip of width $\delta_0 > 0$ and that the vector $(\lambda, \sqrt{-1}\omega)$ satisfies the nonresonance conditions

$$\left|\sqrt{-1}k\cdot\omega + l\cdot\lambda\right| \geq \frac{c}{|k|^\gamma}, \tag{1.4}$$

where $l \in \mathbb{Z}^n$ with $|l| = 0, 2$ and $0 \neq k \in \mathbb{Z}^r$. It was shown by Jorba and Simó [2] that (1.1) is reducible for ϵ in some Cantorian set $\mathscr{E} \subset (0, \epsilon_0)$, with ϵ_0 sufficiently small, provided that

(1) the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A are different;
(2) the eigenvalues $\lambda_1^0(\epsilon), \ldots, \lambda_n^0(\epsilon)$ of $\bar{A}$ satisfy

$$\left|\frac{d}{d\epsilon}\left(\lambda_i^0(\epsilon) - \lambda_j^0(\epsilon)\right)\right|_{\epsilon=0} > 2\rho > 0 \tag{1.5}$$

for some constant ρ and any $1 \leq i < j \leq n$.

In [2], the basic idea is to kill the small perturbation $\epsilon Q(t)$ by KAM iteration. Condition (2) is used to overcome the problem arising from the frequency shift which comes up in this procedure. By a well-known theorem [3, pages 113–115], condition (1) guarantees that the eigenvalues $\lambda_j^0(\epsilon)$ of $\bar{A} = A + \epsilon\bar{Q}$ are differentiable in ϵ, and that therefore condition (2) can be imposed.

A natural question is: what happens when condition (1) or (2) is not satisfied? The main result of the present paper is the following theorem which gives an answer to this question.

Theorem 1.1. *Let $\Omega_0 \subset \mathbb{R}_+^r$ be a compact set with positive Lebesgue measure and assume that $Q(t)$ is quasiperiodic with frequency $\omega \in \Omega_0$ and analytic in the strip of width $\delta_0 > 0$. Then, for a sufficiently small positive constant γ, there exist a subset $\Omega \subset \Omega_0$ with $\mathrm{Meas}(\Omega_0 \setminus \Omega) = \mathrm{Meas}(\Omega_0)(1 - O(\gamma^{1/n^2}))$ and a sufficiently small constant $\epsilon^\star = \epsilon^\star(\delta_0, \gamma) > 0$ such that for any $\epsilon \in (0, \epsilon^\star)$, system (1.1) is reducible. More exactly, there is an analytic quasiperiodic transformation $x = P(t)y$ such that (1.1) is changed into*

$$\dot{y} = By, \tag{1.6}$$

where B is a constant matrix with $\|A - B\| = O(\epsilon)$.

The proof is based on the construction of an iterative lemma, Lemma 2.1. In this construction, a finite number of terms in the Fourier expansion of the perturbation are killed in each iteration, and the remainder is included in the higher-order perturbation. The averaged perturbation is included in the time-independent term. To solve the homological equation, avoiding the problem of small divisors, certain frequencies must be removed from the original frequency set Ω_0 at each iteration step. Showing that the remaining frequencies form a big subset of Ω_0 through the estimates of Section 3 concludes the proof.

Remark 1.2. When one of λ_j is not simple, the functions $\lambda_j^0(\epsilon)$ are not necessarily differentiable in ϵ. Therefore, in the hypothesis of Theorem 1.1, we have to regard the tangent frequencies ω, instead of ϵ, as the parameters used to overcome the frequency shift in KAM iterative steps. Thus, we cannot find explicitly a tangent frequency vector ω satisfying some Diophantine conditions such that Theorem 1.1 holds true. On the other hand, in Theorem 1.1 it is not necessary to excise a subset of small measure from $(0,\epsilon^*)$. In this sense, Theorem 1.1 complements the results of [2]. Yet another complementary approach is that of [1], where ω is fixed and reducibility is proved for "most" matrices A.

2. Proof of Theorem 1.1.

The proof of Theorem 1.1 is based on Newton iteration. Before we state the main iterative lemma, we need to introduce some notation.

In the following, we denote by C, C_1, $C_2,\ldots$ positive constants which arise in the estimates, by $\mathcal{Q}$ the hull of a quasiperiodic function $Q(t)$, and by $\tilde{\mathcal{Q}}$ the average of $\mathcal{Q}$ on the r-torus. For a matrix-valued function $Q(t)$, define

$$\|Q\|^D := \sup_{t\in D}\|Q(t)\|, \tag{2.1}$$

where $\|\cdot\|$ is the sup-norm of the matrix.

Denote by m the number of the iterative step, and let

(1) ϵ_m be the sequence that bounds the size of the perturbation before the mth iteration step with $\epsilon_m = \epsilon^{(1+\rho)^{m-1}}$ and $\rho = 1/3$, for example;

(2) δ_m be the sequence that measures the size of the analyticity domain in the angular variables after m iteration steps with

$$\delta_m = \delta_0 - \frac{\delta_0}{2}\left(1+\cdots+m^{-2}\right)\sum_{j=1}^{\infty} j^{-2} \quad \text{for } m \geq 1; \tag{2.2}$$

(3) $U_m = U(\delta_m) = \{\theta \in (\mathbb{C}/2\pi\mathbb{Z})^N : |\operatorname{Im}\theta| < \delta_m\}$;

(4) q_m be the sequence that measures the width of the analyticity domain in the frequency space after m iteration steps with $q_m = \epsilon_{m+1}^{1/4n^2}$, where n is the dimensional number of system (1.1);

(5) $C(m)$ be a constant of the form $C_1 m^{C_2}$.

Let $\Omega_0 = \Pi_0 \supset \Pi_1 \supset \cdots \supset \Pi_{m-1}$ be the closed sets in $\mathbb{R}_+^r$ and let $\Pi_m \subset \Pi_{m-1}$ be as defined inductively in Section 3. Let $\mathbb{O}_l$ be the complex q_l-neighborhood of Π_l for $l = 0,1,\ldots,m$. Assume that, after $m-1$ steps of Newton iteration, we get a quasiperiodic linear differential equation

$$\dot{x} = (A_{m-1} + \epsilon_m Q_m(t;\omega))x, \tag{2.3}$$

where the following conditions are satisfied:

(H1)$_m$ $A_{m-1} = A + \epsilon_1 \tilde{\mathfrak{Q}}_1(\omega) + \cdots + \epsilon_{m-1}\tilde{\mathfrak{Q}}_{m-1}(\omega)$, $m \geq 2$, $A_0 = A$ with $\tilde{\mathfrak{Q}}_l(\omega)$ analytic in $\mathbb{O}_l$, and $\|\tilde{\mathfrak{Q}}_l\|^{\mathbb{O}_l} \leq 1$ for $l = 1,\dots,m-1$;

(H2)$_m$ the hull $\mathfrak{Q}_m$ of $Q_m(t;\omega)$ is analytic in $U_{m-1}\times\mathbb{O}_{m-1}$ and

$$\|\mathfrak{Q}_m(\theta,\omega)\|^{U_{m-1}\times\mathbb{O}_{m-1}} \leq 1. \tag{2.4}$$

Let

$$A_m = A_{m-1} + \epsilon_m \tilde{\mathfrak{Q}}_m. \tag{2.5}$$

Then (2.3) can be rewritten as

$$\dot{x} = (A_m + \epsilon_m Q_m^*(t))x, \tag{2.6}$$

where $Q_m^*(t) = Q_m(t) - \tilde{\mathfrak{Q}}_m$. Following [2], we will find a change of variables

$$x = (E + \epsilon_m P_m(t))y, \tag{2.7}$$

where E is the unit matrix such that (2.3) is changed into

$$\dot{x} = (A_m + O(\epsilon_{m+1}))x \tag{2.8}$$

verifying conditions (H1)$_{m+1}$ and (H2)$_{m+1}$. This change of variables is given by the following lemma.

Lemma 2.1 (iterative lemma). *Assume that (H1)$_m$ and (H2)$_m$ are fulfilled. Then there is a quasiperiodic LP transformation*

$$x = (E + P_m(t))y, \tag{2.9}$$

where $P_m(t)$ is quasiperiodic with frequency ω and its hull $\mathcal{P}_m(\theta;\omega)$ is analytic in $U_m\times\mathbb{O}_m$ such that (2.3) is changed into

$$\dot{y} = (A_m + \epsilon_{m+1}Q_{m+1}(t))y, \tag{2.10}$$

where A_m and Q_{m+1} satisfy the conditions (H1)$_{m+1}$ and (H2)$_{m+1}$.

Proof. Rewrite (2.3) as

$$\dot{x} = (A_m + \epsilon_m Q_m^*(t))x, \tag{2.11}$$

where $Q_m^*(t) = Q_m(t) - \tilde{\mathfrak{Q}}_m$ and $A_m = A_{m-1} + \tilde{\mathfrak{Q}}_m$. Hence, we can write

$$\mathfrak{Q}_m^*(\theta,\omega) = \sum_{0\neq k\in\mathbb{Z}^r} \hat{\mathfrak{Q}}_m^*(k;\omega)e^{\sqrt{-1}k\cdot\theta}, \tag{2.12}$$

where $\hat{\mathcal{Q}}_m^*(k;\omega)$ is the k Fourier coefficient of $\mathcal{Q}_m^*(\theta,\omega)$ in θ. Let $M_m = (1/b_m)|\ln\epsilon_m|$, where $b_m = \delta_{m-1}-\delta_m$, and let

$$\begin{aligned}\mathcal{Q}_m^{*1}(\theta,\omega) &= \sum_{0\neq|k|\leq M_m}\hat{\mathcal{Q}}_m^*(k;\omega)e^{\sqrt{-1}k\cdot\theta},\\ \mathcal{Q}_m^{*2}(\theta,\omega) &= \sum_{|k|>M_m}\hat{\mathcal{Q}}_m^*(k;\omega)e^{\sqrt{-1}k\cdot\theta},\end{aligned}\tag{2.13}$$

so that

$$\mathcal{Q}_m^*(\theta,\omega) = \mathcal{Q}_m^{*1}(\theta,\omega)+\mathcal{Q}_m^{*2}(\theta,\omega).\tag{2.14}$$

We claim that

$$\left|\left|\mathcal{Q}_m^{*2}(\theta,\omega)\right|\right|^{U_m\times\mathbb{O}_m}\leq C(m)\epsilon_m,\tag{2.15}$$

where $C(m)$ is a constant of the form $C_1m^{C_2}$. In fact,

$$\begin{aligned}\left|\left|\mathcal{Q}_m^{*2}(\theta,\omega)\right|\right|^{U_m\times\mathbb{O}_m} &\leq \sum_{|k|>M_m}\left|\left|\hat{\mathcal{Q}}_m^*(k;\omega)\right|\right|^{\mathbb{O}_m}\left|e^{\sqrt{-1}k\cdot\theta}\right|^{U_m}\\ &\leq \sum_{|k|>M_m}\left|\left|\hat{\mathcal{Q}}_m^*(\theta;\omega)\right|\right|^{U_{m-1}\times\mathbb{O}_m}e^{-|k|\delta_{m-1}}e^{|k|\delta_m}\\ &\leq \sum_{|k|>M_m}e^{-|k|(\delta_{m-1}-\delta_m)} = \epsilon_m\sum_{|k|>0}e^{-|k|(\delta_{m-1}-\delta_m)}\\ &\leq C(m)\epsilon_m.\end{aligned}\tag{2.16}$$

Next, we perform the change of variables as in (2.7), where E is the unit matrix in $\mathbb{R}^n$, to transform (2.10) into

$$\dot{y} = \Big((E+\epsilon_mP_m)^{-1}(A_m+\epsilon_m(A_mP_m-\dot{P}_m+Q_m^{*1})+\epsilon_{m+1}Q_{m+1})\Big)y,\tag{2.17}$$

where

$$\epsilon_{m+1}Q_{m+1} = \epsilon_mQ_m^{*2}+\epsilon_m^2(E+\epsilon_mP_m)^{-1}Q_m^*P_m.\tag{2.18}$$

We would like to have

$$(E+\epsilon_mP_m)^{-1}(A_m+\epsilon_m(A_mP_m-\dot{P}_m+Q_m^{*1})) = A_m\tag{2.19}$$

and this implies that

$$\dot{P}_m = A_mP_m-P_mA_m+Q_m^{*1}.\tag{2.20}$$

In order to solve this equation, we consider

$$\omega\cdot\frac{\partial\mathcal{P}_m}{\partial\theta} = A_m\mathcal{P}_m-\mathcal{P}_mA_m+\mathcal{Q}_m^{*1},\tag{2.21}$$

where $\mathcal{P}$ is the hull of $P(t)$. Write

$$\mathcal{P}_m(\theta,\omega) = \sum_{0\neq|k|\leq M_m} \hat{\mathcal{P}}_m(k;\omega)e^{\sqrt{-1}k\cdot\theta}. \tag{2.22}$$

Then we get

$$\sqrt{-1}(k\cdot\omega)\hat{\mathcal{P}}_m(k) = A_m\hat{\mathcal{P}}_m(k) - \hat{\mathcal{P}}_m(k)A_m + \hat{\mathcal{Q}}_m^{*1}(k), \quad 0<|k|\leq M_m, \tag{2.23}$$

where we omit the dependence on ω to simplify the notation. That is,

$$(\sqrt{-1}(k\cdot\omega)E - A_m)\hat{\mathcal{P}}_m(k) + \hat{\mathcal{P}}_m(k)A_m = \hat{\mathcal{Q}}_m^{*1}(k), \quad 0<|k|\leq M_m. \tag{2.24}$$

By Lemmas A.2 and 3.1, (2.24) is solvable for $\omega\in\mathcal{O}_m$ and

$$\begin{aligned}
||\hat{\mathcal{P}}_m(k;\omega)||^{\mathcal{O}_m} &\leq \left|\left|((\sqrt{-1}k\cdot\omega)E_{n^2} - E_n\otimes A_m + A_m^T\otimes E_n)^{-1}\right|\right|^{\mathcal{O}_m} ||\hat{\mathcal{Q}}_m^*(k)||^{\mathcal{O}_m}\\
&\leq \frac{C|k|^\tau}{\gamma_m}||\mathcal{Q}_m^*||^{U_{m-1}\times\mathcal{O}_{m-1}}e^{-|k|\delta_{m-1}}\\
&\leq \frac{C|k|^\tau}{\gamma_m}||\mathcal{Q}_m||^{U_{m-1}\times\mathcal{O}_{m-1}}e^{-|k|\delta_{m-1}}\\
&\leq \frac{C|k|^\tau}{\gamma_m}e^{-|k|\delta_{m-1}},
\end{aligned} \tag{2.25}$$

where in the last inequality we have used $(\text{H2})_m$. Therefore,

$$\begin{aligned}
||\mathcal{P}_m(\theta,\omega)||^{U_m\times\mathcal{O}_m} &\leq \sum_{0<|k|\leq M_m} ||\hat{\mathcal{P}}_m(k;\omega)||^{\mathcal{O}_m}\,|e^{\sqrt{-1}k\cdot\theta}|^{U_m}\\
&\leq \sum_{k\in\mathbb{Z}^r}\frac{C|k|^\tau}{\gamma_m}e^{-|k|(\delta_{m-1}-\delta_m)}\\
&\leq \frac{C}{\gamma_m}\sum_{k\in\mathbb{Z}^r}|k|^\tau e^{-C_3|k|(\delta_0/m^2)} \leq C(m),
\end{aligned} \tag{2.26}$$

where the last inequality follows from Lemma A.1. Then, the function

$$P_m(t) = \mathcal{P}_m(\omega_1 t,\ldots,\omega_r t;\omega) \tag{2.27}$$

solves (2.20). By (2.26) and (2.18), it is easy to show that $\|\mathcal{Q}_{m+1}\|^{U_m\times\mathcal{O}_m}\leq 1$. We omit the details. □

Proof of Theorem 1.1. Obviously, (1.1) satisfies the conditions $(\text{H})_m$ with $m=1$. In fact, condition $(\text{H2})_1$ may always be fulfilled by a suitable rescaling of ϵ. Thus, by Lemma 2.1, there exists a sequence of transformations $x=(E+\epsilon_m P_m(t))y$, $m=1,2,\ldots$, such that the hulls $\mathcal{P}_m$ of the $P_m(t)$ are analytic in the domains $U_m\times\mathcal{O}_m$, and

$$||\mathcal{P}_m||^{U_m\times\mathcal{O}_m} \leq C(m). \tag{2.28}$$

Let

$$U_\infty \times \mathbb{O}_\infty = \bigcap_{m=1}^{\infty} U_m \times \mathbb{O}_m. \tag{2.29}$$

Then, all the $\mathcal{P}_m$, $m = 1,2,\dots$, are well defined in the domain $U_\infty \times \mathbb{O}_\infty$. Set

$$\begin{aligned} \mathit{\Phi}(\theta,\omega) &= \cdots \circ (E+\epsilon_m \mathcal{P}_m(\theta,\omega)) \circ \cdots \circ (E+\epsilon_2\mathcal{P}_2(\theta,\omega)) \circ (E+\epsilon_1\mathcal{P}_1(\theta,\omega)), \\ \Phi(t;\omega) &= \cdots \circ (E+\epsilon_m P_m(t;\omega)) \circ \cdots \circ (E+\epsilon_2 P_2(t;\omega)) \circ (E+\epsilon_1 P_1(t;\omega)). \end{aligned} \tag{2.30}$$

Note that $\|E+\epsilon_m\mathcal{P}_m(\theta;\omega)\|^{U_\infty\times\mathbb{O}_\infty} \le 1+\epsilon_m C(m) \le 1+2^{-m}$. We see that $\mathit{\Phi}$, and thus Φ, are well defined. Let

$$x = \Phi(t)y. \tag{2.31}$$

Since $\epsilon_m\|\mathfrak{Q}_m\|^{U_\infty\times\mathbb{O}_\infty} \le \epsilon_m \to 0$ as $m \to \infty$, the transformation $x = \Phi(t)y$ changes (1.1) into

$$\dot{y} = By, \tag{2.32}$$

where $B = A+\sum_{j=1}^{\infty}\epsilon_m\tilde{\mathfrak{Q}}_m$. This, together with Lemma 3.3, completes the proof of Theorem 1.1. □

3. Estimates on the allowed frequencies set. Let Π_l $(0 \le l \le m-1)$ be a sequence of compact subsets of $\mathbb{R}^r_+$ with $\Omega = \Pi_0 \supset \Pi_1 \supset \cdots \supset \Pi_{m-1}$ and denote by $\mathbb{O}_l$ the complex q_l-neighborhood of Π_l, $l = 0,\dots,m-1$. Recall that

$$A_m(\omega) = A+\epsilon_1\tilde{\mathfrak{Q}}_1(\omega)+\cdots+\epsilon_m\tilde{\mathfrak{Q}}_m(\omega), \tag{3.1}$$

where, for $l = 1,\dots,m-1$, the $\tilde{\mathfrak{Q}}_l(\omega)$ are analytic, and real for real arguments in the domain $\mathbb{O}_l$, and $\|\tilde{\mathfrak{Q}}_l(\omega)\|^{\mathbb{O}_l} \le 1$; and $\tilde{\mathfrak{Q}}_m(\omega)$ is analytic, and real for real arguments in the domain $\mathbb{O}_{m-1}$, and $\|\tilde{\mathfrak{Q}}_m(\omega)\|^{\mathbb{O}_{m-1}} \le 1$. Denote by $|\cdot|_d$ the determinant of a matrix. Let

$$\mathcal{R}_k(m) := \left\{\omega \in \Pi_{m-1} : \left| \, |(\sqrt{-1}k\cdot\omega)E_{n^2} - E_n \otimes A_m + A_m^T \otimes E_n|_d \right| < \frac{\gamma_m}{|k|^{\tau_1}}\right\}, \tag{3.2}$$

where $\gamma_m = \gamma/m^{2n^2}$ and $\tau_1 = (r+1)n^2$,

$$\Pi_m = \Pi_{m-1} \setminus \bigcup_{0<|k|\le M_m} \mathcal{R}_k(m), \tag{3.3}$$

where $M_m = |\ln\epsilon_m|/(\delta_{m-1}-\delta_m)$ is the number of Fourier coefficients we must consider at the mth step of the iteration, and denote by $\mathbb{O}_m$ the complex q_m-neighborhood of Π_m.

Lemma 3.1. *Let $\tau = \tau_1 + n^2 - 1$ and*

$$G(\omega) = (\sqrt{-1}k\cdot\omega)E_{n^2} - E_n \otimes A_m(\omega) + A_m^T(\omega)\otimes E_n. \tag{3.4}$$

Then, for $\omega \in \mathbb{O}_m$ and $0 < |k| \le M_m$, the inverse of $G(\omega)$ exists and it is analytic in the domain $\mathbb{O}_m$ with

$$||G^{-1}(\omega)||^{\mathbb{O}_m} \le C\gamma_m^{-1}|k|^{\tau}. \tag{3.5}$$

Proof. By the definition of Π_m, we get that for $\omega \in \Pi_m$ and $0 < |k| \le M_m$,

$$|\mathcal{M}_k(\omega)| \ge \frac{\gamma_m}{|k|^{\tau_1}}. \tag{3.6}$$

It is easy to see that

$$||G(\omega)||^{\mathbb{O}_m} \le C_5|k|, \quad k \ne 0, \tag{3.7}$$

where $C_5 = 2(\max\{|\omega| : \omega \in \Pi\} + \|A\| + 1)$. Since $\det G(\omega) = \mathcal{M}_k(\omega)$, $G^{-1}(\omega)$ exists for $\omega \in \Pi_m$ and

$$G^{-1}(\omega) = \frac{\mathrm{adj}G(\omega)}{\mathcal{M}_k(\omega)}, \tag{3.8}$$

where adj is the adjoint of a matrix. Thus, for $0 < |k| \le M_m$,

$$||G^{-1}(\omega)||^{\Pi_m} \le C_6\frac{|k|^{n^2-1}}{\gamma_m/|k|^{\tau_1}} = C_6\gamma_m^{-1}|k|^{\tau}. \tag{3.9}$$

Now, we assume that $\omega \in \mathbb{O}_m$. Then there is an $\omega_0 \in \Pi_m$ such that $|\omega - \omega_0| < q_m$. Thus,

$$\begin{aligned}
&||G^{-1}(\omega_0)||\,||G(\omega) - G(\omega_0)|| \\
&\quad \le ||G^{-1}(\omega)||^{\Pi_m}||\nabla_\omega G(\omega)||^{\mathbb{O}_m}\cdot|\omega-\omega_0| \\
&\quad \le C_6\gamma_m^{-1}|k|^{\tau}||G(\omega)||^{\mathbb{O}_{m-1}}\frac{q_m}{q_{m-1}-q_m} \\
&\quad \le C_6\gamma_m^{-1}|k|^{\tau+1}\frac{q_m}{q_{m-1}-q_m} \\
&\quad \le C_6M_m^{\tau+1}\gamma_m^{-1}\frac{q_m}{q_{m-1}-q_m} \\
&\quad \le \frac{C_6m^{(6+2r)n^2}|\ln\epsilon_m|^{\tau+1}\epsilon_{m+1}^{1/4n^2}(\gamma\delta_0)^{-1}}{\epsilon_m^{1/4n^2}-\epsilon_{m+1}^{1/4n^2}} \\
&\quad < \frac{1}{2}.
\end{aligned} \tag{3.10}$$

Therefore, $E+G^{-1}(\omega_0)(G(\omega)-G(\omega_0))$ has its inverse which is analytic in $\mathbb{O}_m$ since

$$\begin{aligned}&(E+G^{-1}(\omega_0)(G(\omega)-G(\omega_0)))^{-1}\\&\qquad=\sum_{j=0}^{\infty}(-G^{-1}(\omega_0)(G(\omega)-G(\omega_0)))^j.\end{aligned}\tag{3.11}$$

So, $G(\omega)$ has its inverse for $\omega\in\mathbb{O}_m$ and

$$\begin{aligned}||G^{-1}(\omega)||&=\left|\left|(E+G^{-1}(\omega_0)(G(\omega)-G(\omega_0)))^{-1}\cdot G^{-1}(\omega_0)\right|\right|\\&\le\left|\left|(E+G^{-1}(\omega_0)(G(\omega)-G(\omega_0)))^{-1}\right|\right|\cdot||G^{-1}(\omega_0)||\\&\le C\gamma_m^{-1}|k|^{\tau}.\end{aligned}\tag{3.12}$$

□

Lemma 3.2. *Let $K=n^2(n+1)^2(\operatorname{diam}\Pi_0)^{r-1}$. Then the Lebesgue measure of $\mathcal{R}_k(m)$ verifies*

$$\operatorname{Meas}\mathcal{R}_k(m)\le\frac{K\gamma^{1/n^2}}{|k|^{r+1}}\frac{1}{m^2}.\tag{3.13}$$

Proof. Recall that $q_l=\epsilon_{l+1}^{1/4n^2}$, let $q_l^1=(5/6)q_l+(1/6)q_{l+1}$, and denote by $\mathbb{O}_l^1$ the complex q_l^1-neighborhood of Π_l. Obviously, $\mathbb{O}_{l+1}\subset\mathbb{O}_l^1\subset\mathbb{O}_l$ and $\operatorname{dist}(\partial\mathbb{O}_l^1,\partial\mathbb{O}_l)=(1/6)(q_l-q_{l+1})>(1/12)q_l$. Noting that $\|\tilde{\mathfrak{Q}}_l\|^{\mathbb{O}_l}\le 1$ and using Cauchy's theorem, we get for $1\le s\le n^2$ and $0\le l\le m-1$,

$$\epsilon_l||\partial_\omega^s\tilde{\mathfrak{Q}}_l||^{\mathbb{O}_l^1}\le\epsilon_l(12q_l^{-1})^s||\tilde{\mathfrak{Q}}_l||^{\mathbb{O}_l}\le\epsilon_l^{1/2}.\tag{3.14}$$

The combination of (3.1) and (3.14) leads to

$$||\partial_\omega^sA_m(\omega)||^{\mathbb{O}_{m-1}^1}\le\epsilon^{1/2}.\tag{3.15}$$

Let

$$B(\omega):=-E_n\otimes A_m(\omega)+A_m^T\otimes E_n.\tag{3.16}$$

Then

$$||\partial_\omega^sB(\omega)||^{\mathbb{O}_{m-1}^1}\le\epsilon^{1/2}.\tag{3.17}$$

Set

$$\mathcal{M}_k(\omega)=\left|(\sqrt{-1}k\cdot\omega)E_{n^2}+B(\omega)\right|_d.\tag{3.18}$$

We are now in a position to estimate $\partial_\omega^s\mathcal{M}_k$. To this end, write $B(\omega)=(b_{ij})$. Then

$$\mathcal{M}_k(\omega)=(\sqrt{-1})^{n^2}(k\cdot\omega)^{n^2}+\sum_{1\le l\le n^2-1}\phi_l(\omega)(k\cdot\omega)^{n^2-l},\tag{3.19}$$

where

$$\phi_l(\omega) = \sum_{1 \le j_i \le n^2} \sigma_{j_1 \cdots j_l} b_{1j_1} \cdots b_{lj_l} \tag{3.20}$$

and $\sigma_{j_1 \cdots j_l} \in \{-1, +1, -\sqrt{-1}, +\sqrt{-1}\}$.

Observe that for $1 \le l \le n^2$ and $\omega \in \mathbb{O}^1_{m-1}$,

$$\left|\partial^s_{\omega_1} b_{ij}\right| \le \left\|\partial^s_{\omega} B(\omega)\right\|^{\mathbb{O}^1_{m-1}} \le \epsilon^{1/2}. \tag{3.21}$$

Thus, for $\omega \in \mathbb{O}^1_{m-1}$ and $1 \le s \le n^2$,

$$\begin{aligned} \left|\frac{d^s}{d\omega_1^s}\left(b_{1j_1} \cdots b_{lj_l}\right)\right| &\le \left|\sum_{s_1+\cdots+s_l=s}\left(\frac{d^{s_1}}{d\omega_1^{s_1}} b_{1j_1}\right) \cdots \left(\frac{d^{s_l}}{d\omega_1^{s_l}} b_{1j_l}\right)\right| \\ &\le \epsilon^{(1/2)(s_1+\cdots+s_l)} \sum_{s_1+\cdots+s_l=s} 1 \\ &\le \left(2\epsilon^{1/2}\right)^s, \end{aligned} \tag{3.22}$$

and therefore,

$$\left|\frac{d^s}{d\omega_1^s}\phi_l(\omega)\right| \le \binom{n^2}{l}\left(2\epsilon^{1/2}\right)^s. \tag{3.23}$$

Without loss of generality, assume that $|k| = |k_1| + \cdots + |k_r| \le r|k_1|$. Hence, for every $\omega \in \mathbb{O}^1_{m-1}$,

$$\begin{aligned} &\left|\frac{d^{n^2}}{d\omega_1^{n^2}} \sum_{1 \le l \le n^2-1} \phi_l(\omega)(k\cdot\omega)^{n^2-l}\right| \\ &\quad \le \sum_{1 \le l \le n^2-1}\left|\frac{d^{n^2}}{d\omega_1^{n^2}}\left(\phi_l(\omega)(k\cdot\omega)^{n^2-l}\right)\right| \\ &\quad \le \sum_{1 \le l \le n^2-1}\ \sum_{l \le s \le n^2}\binom{n^2}{s}\left|\frac{d^{s^2}}{d\omega_1^{s^2}}\phi_l(\omega)\right|\left|\frac{d^{n^2-s}}{d\omega_1^{n^2-s}}(k\cdot\omega)^{n^2-l}\right| \\ &\quad \le \sum_{1 \le l \le n^2-1}\ \sum_{l \le s \le n^2}\binom{n^2}{l}\binom{n^2}{s}\left(2\epsilon^{1/2}\right)^s\left|k_1\right|^{n^2-s}|k\cdot\omega|^{s-l}n^2! \\ &\quad \le C_4\epsilon^{1/2}\left|k_1\right|^{n^2-1}n^2!, \end{aligned} \tag{3.24}$$

where C_4 is some constant which depends only on n, r, and on the maximum of $|\omega|$ in Π_0. Obviously,

$$\frac{d^{n^2}}{d\omega_1^{n^2}}(k\cdot\omega)^{n^2} = n^2!\left|k_1\right|^{n^2}. \tag{3.25}$$

Thus, in $\mathbb{O}^1_{m-1}$, we have

$$\left|\frac{d^{n^2}}{d\omega_1^{n^2}}\mathcal{M}_k(\omega)\right| \geq n^2!|k_1|^{n^2}\left(1-C_4\epsilon^{1/2}|k_1|^{-1}\right) > \frac{1}{2}n^2!|k_1|^{n^2} \tag{3.26}$$

provided that ϵ is small enough so that $C_4\epsilon^{1/2} < 1/2$. Using (3.26) and Lemma A.3, we get

$$\begin{aligned}\operatorname{Meas}\mathcal{R}_k(m) &\leq n^2(n^2+1)\left(\frac{\gamma_m}{|k|^{\tau_1}}\right)^{1/n^2}(\operatorname{diam}\Pi_0)^{r-1}\\ &\leq \frac{K\gamma^{1/n^2}}{|k|^{r+1}}\cdot\frac{1}{m^2}.\end{aligned} \tag{3.27}$$

This completes the proof. □

By Lemma 2.1, the nested sequence of closed sets

$$\Omega_0 = \Pi_0 \supset \Pi_1 \supset \cdots \supset \Pi_m \supset \cdots \tag{3.28}$$

is defined inductively. The following lemma is a corollary of Lemma 3.2.

Lemma 3.3. *Let*

$$\Pi_\infty = \bigcap_{m=0}^{\infty}\Pi_m. \tag{3.29}$$

Then $\operatorname{Meas}\Pi_\infty = (\operatorname{Meas}\Pi_0)(1-O(\gamma^{1/n^2}))$.

Appendix

Lemma A.1. *For $\delta > 0$ and $\nu > 0$, the following inequality holds true:*

$$\sum_{k\in\mathbb{Z}^N} e^{-2|k|\delta}|k|^\nu \leq \left(\frac{\nu}{e}\right)^\nu\frac{1}{\delta^{\nu+N}}(1+e)^N. \tag{A.1}$$

Proof. This lemma can be found in [1]. We will find the value of $z \geq 1$ yielding a maximum value for the expression $\nu\ln z - \delta z$. Differentiating it in z and equating the result to zero, we get that $\nu/z = \delta$ and $z = \nu/\delta > 1$. From this it follows that

$$\nu\ln z - \delta z \leq \nu\left(\ln\frac{\nu}{\delta}-1\right). \tag{A.2}$$

This expression yields

$$z^\nu \leq \exp(\delta z)\exp\left(\nu\left(\ln\frac{\nu}{\delta}-1\right)\right) = \left(\frac{\nu}{e}\right)^\nu\frac{\exp(\delta z)}{\delta^\nu}. \tag{A.3}$$

Thus,

$$\sum_{k\in\mathbb{Z}^N} e^{-2|k|}|k|^\nu \le \left(\frac{\nu}{e}\right)^\nu \frac{1}{\delta^\nu}\sum_k e^{-|k|\delta} = \left(\frac{\nu}{e}\right)^\nu \frac{1}{\delta^\nu}\left(\frac{1+\exp(-\delta)}{1-\exp(-\delta)}\right)^N \le \left(\frac{\nu}{e}\right)^\nu \frac{1}{\delta^\nu}\left(\frac{1+e}{\delta}\right)^N. \tag{A.4}$$

□

Lemma A.2. *Let A, B, and C be $r\times r$, $s\times s$, and $r\times s$ matrices, respectively; and let X be an $r\times s$ unknown matrix. Then the matrix equation*

$$AX+XB=C \tag{A.5}$$

is solvable if and only if the vector equation

$$(E_s\otimes A+B^T\otimes E_r)X' = C' \tag{A.6}$$

is solvable, where $X' = (X_1^T,\dots,X_s^T)^T$, $C' = (C_1^T,\dots,C_s^T)^T$ if we write $X = (X_1,\dots,X_s)$ and $C = (C_1,\dots,C_s)$. Moreover,

$$\|X\| \le \left\|(E_s\otimes A+B^T\otimes E_r)^{-1}\right\|\|C\| \tag{A.7}$$

if the inverse exists.

Proof. This lemma can be found in many textbooks on matrix theory; for example, [4, page 256]. □

The following lemma can be found in [7, page 23].

Lemma A.3. *Let $\mathcal{I}$ be an interval in $\mathbb{R}^1$ and $\bar{\mathcal{I}}$ its closure. Suppose that $g:\bar{\mathcal{I}}\to\mathbb{C}$ is k times continuously differentiable. Let $\mathcal{I}_h = \{x\in\bar{\mathcal{I}} : |g(x)|\le h\}$, $h>0$. If, for some constant $d>0$, $|d^k g(x)/dx^k|\ge d$ for any $x\in\mathcal{I}$, then $\operatorname{Meas} I_h \le ch^{1/k}$ with $c = 2(2+3+\cdots+k+d^{-1})$.*

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Xiaoping Yuan: Department of Mathematics, Fudan University, Shanghai 200433, China

Current address: Centro de Matemática e Aplicações Fundamentais, Universidade de Lisboa, 1649-003 Lisboa, Portugal

Ana Nunes: Centro de Matemática e Aplicações Fundamentais, Universidade de Lisboa, 1649-003 Lisboa, Portugal

E-mail address: anunes@lmc.fc.ul.pt