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ON HOPF GALOIS HIRATA EXTENSIONS

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Let H be a finite-dimensional Hopf algebra over a field k, H^* the dual Hopf algebra of H, and B a right H^*-Galois and Hirata separable extension of B^H. Then B is characterized in terms of the commutator subring $V_B(B^H)$ of B^H in B and the smash product $V_B(B^H)\#H$. A sufficient condition is also given for B to be an H^*-Galois Azumaya extension of B^H.

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1. Introduction. Let H be a finite-dimensional Hopf algebra over a field k, H^* the dual Hopf algebra of H, and B a right H^*-Galois extension of B^H. In [3], the class of H^*-Galois Azumaya extensions was investigated and in [8], it was shown that B is a Hirata separable extension of B^H if and only if the commutator subring $V_B(B^H)$ of B^H in B is a left H-Galois extension of C, where C is the center of B (see [8, Lemma 2.1, Theorem 2.6]). The purpose of the present paper is to characterize a right H^*-Galois and Hirata separable extension B of B^H in terms of the commutator subring $V_B(B^H)$ and the smash product $V_B(B^H)\#H$. Let B be a right H^*-Galois extension of B^H such that $B^H = B^{H^*}$. Then the following statements are equivalent:

(1) B is a Hirata separable extension of B^H,
(2) $V_B(B^H)$ is an Azumaya C-algebra and $V_B(V_B(B^H)) = B^H$,
(3) $V_B(B^H)$ is a right H^*-Galois extension of C and a direct summand of $V_B(B^H)\#H$ as a $V_B(B^H)$-bimodule,
(4) $V_B(B^H)$ is a right H^*-Galois extension of C and $V_B(B^H)\#H$ is a direct summand of a finite direct sum of $V_B(B^H)$ as a bimodule over $V_B(B^H)$.

Moreover, an equivalent condition is given for a right H^*-Galois and Hirata separable extension B of B^H to be an H^*-Galois Azumaya extension which was studied in [3, 7]. Also, let B be a right H^*-Galois and Hirata separable extension of B^H and A a subalgebra of B^H over C such that B^H is a projective Hirata separable extension of A containing A as a direct summand as an A-bimodule. Then $V_{B^H}(A)$ is a separable subalgebra of B^H over C, and there exists an H-submodule algebra D in B which is separable over C such that $D^H = V_{B^H}(A)$ and $D \cong V_{B^H}(A) \otimes_Z F$ as Azumaya Z-algebras, where Z is the center of D and F is an Azumaya Z-algebra in D.

2. Basic definitions and notations. Throughout, H denotes a finite-dimensional Hopf algebra over a field k with comultiplication Δ and counit ε, H^* the dual Hopf algebra of H, B a left H-module algebra, C the center of B, $B^H = \{b \in B \mid hb = \varepsilon(h)b$ for all $h \in H\}$ which is called the H-invariants of B, and $B\#H$ the smash product of B with H, where $B\#H = B \otimes_k H$ such that for all $b\#h$ and $b'\#h'$ in $B\#H$, $(b\#h)(b'\#h') = \sum b(h_1 b')\#h_2 h'$, where $\Delta(h) = \sum h_1 \otimes h_2$. The ring B is called a right H^*-Galois extension of B^H if B is a right H^*-comodule algebra with structure map $\rho : B \to B \otimes_k H^*$ such that $\beta : B \otimes_{B^H} B \to B \otimes_k H^*$ is a bijection, where $\beta(a \otimes b) = (a \otimes 1)\rho(b)$.

For a subring A of B with the same identity 1, we denote the commutator subring of A in B by $V_B(A)$. We call B a separable extension of A if there exist $\{a_i, b_i$ in B, $i = 1,2,\ldots,m$, for some integer $m\}$ such that $\sum a_i b_i = 1$ and $\sum b a_i \otimes b_i = \sum a_i \otimes b_i b$ for all b in B, where $\otimes$ is over A. An Azumaya algebra is a separable extension of its center. A ring B is called a Hirata separable extension of A if $B \otimes_A B$ is isomorphic to a direct summand of a finite direct sum of B as a B-bimodule. A right H^*-Galois extension B is called an H^*-Galois Azumaya extension if B is separable over B^H which is an Azumaya algebra over C^H. A right H^*-Galois extension B of B^H is called an H^*-Galois Hirata extension if B is also a Hirata separable extension of B^H. Throughout, an H^*-Galois extension means a right H^*-Galois extension unless it is stated otherwise.

3. The H^*-Galois Hirata extensions. In this section, we will characterize an H^*-Galois Hirata extension B of B^H in terms of the commutator subring $V_B(B^H)$ of B^H in B and the smash product $V_B(B^H)\#H$. A relationship between an H^*-Galois Hirata extension and an H^*-Galois Azumaya extension is also given. We begin with some properties of an H^*-Galois Hirata extension B of B^H. Throughout, we assume $B^H = B^{H^*}$.

Lemma 3.1. *If A_1 and A_2 are H^*-Galois extensions such that $A_1^H = A_2^H$ and $A_1 \subset A_2$, then $A_1 = A_2$.*

Proof. By [3, Theorem 5.1], there exist $\{x_i, y_i \in A_1 \mid i = 1,2,\ldots,n\}$ for some integer n such that, for all $h \in H$, $\sum x_i(hy_i) = T(h)1_{A_1}$, where $T \in \int_{H^*}^r$, the set of right integrals in H^*. Let $t \in \int_H^l$, the set of left integrals in H, such that $T(t) = 1$, then $\{x_i, f_i = t(y_i -) \mid i = 1,2,\ldots,n\}$ is a dual basis of the finitely generated and projective right module A_1 over A_1^H. Since $A_1 \subset A_2$ such that $A_1^H = A_2^H$, $\{x_i, f_i \mid i = 1,2,\ldots,n\}$ is also a dual basis of the finitely generated and projective right module A_2 over A_1^H. This implies that $A_1 = A_2$. □

Lemma 3.2. *If B is an H^*-Galois Hirata extension of B^H, then B^H is a direct summand of B as a B^H-bimodule.*

Proof. We use the argument as given in [2]. Since B is an H^*-Galois and a Hirata separable extension of B^H, $V_B(B^H)$ is a left H-Galois extension of C (see [8, Lemma 2.1, Theorem 2.6]). Hence, $V_B(B^H)$ is a finitely generated and projective module over C (see [3, Theorem 2.2]). Let $\Omega = \mathrm{Hom}_C(V_B(B^H), V_B(B^H))$. Since C is commutative, $V_B(B^H)$ is a progenerator of C. Thus, B is a right Ω-module such that $B \cong V_B(B^H) \otimes_C \mathrm{Hom}_\Omega(V_B(B^H), B) \cong V_B(B^H) \otimes_C B^{H^*}$ as C-algebras, where $f(1) \in B^{H^*}$ for each $f \in \mathrm{Hom}_\Omega(V_B(B^H), B)$ by the proof of [2, Lemma 2.8]. But $V_B(V_B(B^H)) = B^H$ (see [2, Lemma 2.5]), so $B \cong V_B(B^H) \otimes_C B^H$. This implies that $V_B(B^H)$ is an H^*-Galois extension of C (see [2, Lemma 2.8]); and so C is a direct summand of $V_B(B^H)$ as a C-bimodule (see [2, Corollaries 1.9 and 1.10]). Therefore, B^H is a direct summand of B as a B^H-bimodule. □

By the proof of Lemma 3.2, $V_B(B^H)$ is an H^*-Galois extension of C.

Corollary 3.3. *If B is an H^*-Galois Hirata extension of B^H, then $V_B(B^H)$ is an H^*-Galois extension of C.*

Corollary 3.4. *If B is an H^*-Galois Hirata extension of B^H, then $B = B^H \cdot V_B(B^H)$ and the centers of B, B^H, and $V_B(B^H)$ are the same C.*

Proof. By Corollary 3.3, $V_B(B^H)$ is an H^*-Galois extension of C, so $B^H \cdot V_B(B^H)$ is also an H^*-Galois extension of B^H $(= (B^H \cdot V_B(B^H))^H)$ with the same Galois system as $V_B(B^H)$ (see [3, Theorem 5.1]). Noting that $B^H \cdot V_B(B^H) \subset B$, we conclude that $B = B^H \cdot V_B(B^H)$ by Lemma 3.1. Moreover, $V_B(V_B(B^H)) = B^H$ (see [8, Lemma 2.5]), so the centers of B^H, $V_B(B^H)$, and B are the same C. □

Theorem 3.5. *Let B be an H^*-Galois extension of B^H. The following statements are equivalent:*

(1) *B is a Hirata separable extension of B^H,*
(2) *$V_B(B^H)$ is an H^*-Galois extension of C and a direct summand of $V_B(B^H)\#H$ as a $V_B(B^H)$-bimodule,*
(3) *$V_B(B^H)$ is an Azumaya C-algebra and $V_B(V_B(B^H)) = B^H$,*
(4) *$V_B(B^H)$ is an H^*-Galois extension of C and $V_B(B^H)\#H$ is a direct summand of a finite direct sum of $V_B(B^H)$ as a bimodule over $V_B(B^H)$.*

Proof. (1)⇒(3). Since B is an H^*-Galois and a Hirata separable extension of B^H, by Lemma 3.2, B^H is a direct summand of B as a B^H-bimodule. Thus, $V_B(V_B(B^H)) = B^H$ and $V_B(B^H)$ is a separable C-algebra (see [4, Propositions 1.3 and 1.4]). But the center of $V_B(B^H)$ is C by Corollary 3.4, so $V_B(B^H)$ is an Azumaya C-algebra.

(3)⇒(1). Since $V_B(B^H)$ is an Azumaya C-algebra and B is a bimodule over $V_B(B^H)$, $B \cong V_B(B^H) \otimes_C V_B(V_B(B^H)) = V_B(B^H) \otimes_C B^H$ as a bimodule over $V_B(B^H)$ (see [1, Corollary 3.6, page 54]). Noting that $B \cong V_B(B^H) \otimes_C B^H$ is also an isomorphism as C-algebras and that $V_B(B^H)$ is an Azumaya C-algebra, we conclude that $V_B(B^H) \otimes_C B^H$ is a Hirata separable extension of B^H; and so B is a Hirata separable extension of B^H.

(3)⇒(2). By the proof of (3)⇒(1), $B \cong V_B(B^H) \otimes_C B^H$ such that $V_B(B^H)$ is a finitely generated and projective module over C, so $V_B(B^H)$ is an H^*-Galois extension of C (see [2, Lemma 2.8]). Moreover, since $V_B(B^H)$ is an Azumaya

C-algebra, $V_B(B^H)$ is a direct summand of $V_B(B^H)\otimes_C(V_B(B^H))^{\circ}$ as a $V_B(B^H)$-bimodule, where $(V_B(B^H))^{\circ}$ is the opposite algebra of $V_B(B^H)$. But $V_B(B^H)\otimes_C(V_B(B^H))^{\circ}\cong \mathrm{Hom}_C(V_B(B^H),V_B(B^H))\cong V_B(B^H)\#H$ (see [3, Theorem 2.2]), so $V_B(B^H)$ is a direct summand of $V_B(B^H)\#H$ as a $V_B(B^H)$-bimodule.

(2)⇒(3). Since $V_B(B^H)$ is an H^*-Galois extension of C, $B^H\cdot V_B(B^H)$ is an H^*-Galois extension of $(B^H\cdot V_B(B^H))^H$. But $(B^H\cdot V_B(B^H))^H=B^H$, so $B^H\cdot V_B(B^H)$ and B are H^*-Galois extensions of B^H such that $B^H\cdot V_B(B^H)\subset B$. Hence, $B^H\cdot V_B(B^H)=B$ by Lemma 3.1. Thus, the centers of B and $V_B(B^H)$ are the same C. Moreover, $V_B(B^H)$ is a direct summand of $V_B(B^H)\#H$ as a $V_B(B^H)$-bimodule by hypothesis, so it is a separable C-algebra (see [3, Theorem 2.3]). Thus, $V_B(B^H)$ is an Azumaya C-algebra. But then $B\cong V_B(B^H)\otimes_C V_B(V_B(B^H))$. On the other hand, by hypothesis, $V_B(B^H)$ is an H^*-Galois extension of C, so $B\cong V_B(B^H)\otimes_C B^H$ (see [2, Lemma 2.8]). Therefore, $V_B(V_B(B^H))=B^H$.

(3)⇔(4). Since $V_B(B^H)$ is an H^*-Galois extension of C, it is a finitely generated and projective module over C and $\mathrm{Hom}_C(V_B(B^H),V_B(B^H))\cong V_B(B^H)\#H$ (see [3, Theorem 2.2]). But then $V_B(B^H)$ is a Hirata separable extension of C if and only if $V_B(B^H)\#H$ is a direct summand of a finite direct sum of $V_B(B^H)$ as a bimodule over $V_B(B^H)$ (see [5, Corollary 3]). Thus, $V_B(B^H)$ is an Azumaya C-algebra if and only if $V_B(B^H)$ is an H^*-Galois extension of C and $V_B(B^H)\#H$ is a direct summand of a finite direct sum of $V_B(B^H)$ as a bimodule over $V_B(B^H)$. □

By Theorem 3.5, we can obtain a relationship between the class of H^*-Galois Hirata extensions and the class of H^*-Galois Azumaya extensions which were studied in [3, 7].

Corollary 3.6. *Let B be an H^*-Galois Azumaya extension of B^H. Then B is an H^*-Galois Hirata extension of B^H if and only if $C=C^H$.*

Proof. (⇒) Since B is an H^*-Galois Hirata extension of B^H, $V_B(B^H)$ is an Azumaya algebra over C and a left H-Galois extension of C (see [8, Theorem 2.6]). Hence, $V_B(V_B(B^H))=B^H$ (see [8, Lemma 2.5]). Thus, $C\subset B^H$; and so $C=C^H$.

(⇐) Since B is an H^*-Galois Azumaya extension of B^H, $V_B(B^H)$ is separable over C^H (see [3, Lemma 4.1]). Since B is an H^*-Galois Azumaya extension of B^H again, $V_B(B^H)$ is an H^*-Galois extension of $(V_B(B^H))^H$ (see [3, Lemma 4.1]), so both $B^H\cdot V_B(B^H)$ and B are H^*-Galois extensions of B^H such that $B^H\cdot V_B(B^H)\subset B$. Hence, $B^H\cdot V_B(B^H)=B$ by Lemma 3.1. This implies that the center of $V_B(B^H)$ is C. But by hypothesis, $C=C^H$, so $V_B(B^H)$ is an Azumaya C-algebra. Hence, $V_B(B^H)$ is a Hirata separable extension of C. But $B=B^H\cdot V_B(B^H)\cong B^H\otimes_C V_B(B^H)$ as Azumaya C-algebras, so B is a Hirata separable extension of B^H. Thus, B is an H^*-Galois Hirata extension of B^H. □

Corollary 3.7. *Let B be an H^*-Galois Hirata extension of B^H. Then B is an H^*-Galois Azumaya extension of B^H if and only if B is an Azumaya C^H-algebra.*

Proof. (⇒) Since B is an H^*-Galois Azumaya extension of B^H, B^H is an Azumaya C^H-algebra and B is separable over B^H (see [3, Theorem 3.4]). Hence, B is separable over C^H by the transitivity of separable extensions. But B is an H^*-Galois Azumaya extension of B^H and an H^*-Galois Hirata extension of B^H by hypothesis, so $C = C^H$ by Corollary 3.6. This implies that B is an Azumaya C^H-algebra.

(⇐) By hypothesis, B is an Azumaya C^H-algebra. Hence, $C = C^H$. But B is an H^*-Galois Hirata extension of B^H, so $V_B(B^H)$ is an Azumaya subalgebra of B over C by Theorem 3.5(3). Since B is an H^*-Galois Hirata extension of B^H again, B is a Hirata separable extension of B^H and a finitely generated and projective module over B^H. Thus, $V_B(V_B(B^H)) = B^H$ (see [8, Lemma 2.5]); and so B^H $(= V_B(V_B(B^H)))$ is an Azumaya subalgebra of B over C^H by the commutator theorem for Azumaya algebras (see [1, Theorem 4.3, page 57]). This proves that B is an H^*-Galois Azumaya extension of B^H. □

4. Invariant subalgebras. For an H^*-Galois Hirata extension B as given in Theorem 3.5, let A be a subalgebra of B^H over C such that B^H is a projective Hirata separable extension of A and contains A as a direct summand as an A-bimodule. In this section, we show that $V_{B^H}(A)$ is the H-invariant subalgebra of a separable subalgebra D in B over C, that is, $D^H = V_{B^H}(A)$. We denote by $\mathcal{S}$ the set $\{A \mid A$ is a subalgebra of B^H over C such that B^H is a projective Hirata separable extension of A and contains A as a direct summand as an A-bimodule$\}$.

Lemma 4.1. *Let B be an H^*-Galois Hirata extension of B^H. For any $A \in \mathcal{S}$, $V_B(A)$ is an H-submodule algebra of B and separable over C, and $(V_B(A))^H = V_{B^H}(A)$ which is a separable C-algebra.*

Proof. Since $A \in \mathcal{S}$, B^H is a projective Hirata separable extension of A and contains A as a direct summand as an A-bimodule. But B is an H^*-Galois Hirata extension of B^H, so B is a projective Hirata separable extension of B^H. Hence, by the transitivity property of projective Hirata separable extensions, B is a projective Hirata separable extension of A. Also B^H is a direct summand of B as a B^H-bimodule by Lemma 3.2, so A is a direct summand of B as an A-bimodule. Thus, $V_B(A)$ is a separable algebra over C (see [6, Theorem 1]). Moreover, it is clear that $(V_B(A))^H = V_{B^H}(A)$, so $V_{B^H}(A)$ is a separable C-algebra (see Corollary 3.4 and [6, Theorem 1]). □

Next we want to show which separable subalgebra of B^H over C is an H-invariant subring of an H-submodule algebra in B. Let $\mathcal{T} = \{E \subset B \mid E$ is a separable C-subalgebra of B^H and satisfies the double centralizer property in B^H such that $V_{B^H}(E) \in \mathcal{S}\}$. Next we show that for any $E \in \mathcal{T}$, E is the H-invariant subring of an H-submodule algebra D in B which is separable over C.

Theorem 4.2. *Let E be in $\mathcal{T}$. Then there exists an H-submodule algebra D in B which is separable over C such that $D^H = E$.*

Proof. Since E is in $\mathcal{T}$, $V_{B^H}(E)$ is in $\mathcal{S}$ such that $V_{B^H}(V_{B^H}(E)) = E$. Now by Lemma 4.1, $V_B(V_{B^H}(E))$ is an H-submodule algebra of B and separable over C such that $(V_B(V_{B^H}(E)))^H = V_{B^H}(V_{B^H}(E))$. But $V_{B^H}(V_{B^H}(E)) = E$, so

$$(V_B(V_{B^H}(E)))^H = E. \tag{4.1}$$

Let $D = V_B(V_{B^H}(E))$. Then D satisfies the theorem. □

By Theorem 4.2, we obtain an expression for the separable H-submodule algebra D for a given E in $\mathcal{T}$.

Corollary 4.3. *By keeping the notations as given in Theorem 4.2, let Z be the center of E. Then $D \cong E \otimes_Z V_D(E)$ as Azumaya Z-algebras.*

Proof. Since E satisfies the double centralizer property in B^H, $V_{B^H}(V_{B^H}(E)) = E$. Hence, the centers of E and $V_{B^H}(E)$ are the same Z. Similarly as given in the proof of Lemma 4.1, since $V_{B^H}(E)$ is in $\mathcal{S}$, B $(= B^H \cdot V_B(B^H))$ is a projective Hirata separable extension of $V_{B^H}(E)$ and contains $V_{B^H}(E)$ as a direct summand as a $V_{B^H}(E)$-bimodule by the transitivity property of projective Hirata separable extensions and the direct summand conditions. Thus, $V_{B^H}(E)$ satisfies the double centralizer property in B, that is, $V_B(V_B(V_{B^H}(E))) = V_{B^H}(E)$. This implies that the centers of $V_{B^H}(E)$ and $V_B(V_{B^H}(E))$ are the same. Therefore, D and E have the same center Z. Noting that D and E are separable C-algebras by Theorem 4.2, we conclude that E $(= D^H)$ is an Azumaya subalgebra of D over Z; and so $D \cong E \otimes_Z V_D(E)$ as Azumaya Z-algebras (see [1, Theorem 4.3, page 57]). □

Remark 4.4. When B is an H^*-Galois Azumaya extension of B^H, the correspondence $A \to V_B(A)$ as given in Lemma 4.1 recovers the one-to-one correspondence between the set of separable subalgebras of B^H and the set of H^*-Galois extensions in B containing $V_B(B^H)$ as given in [3].

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