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CONSTANT MEAN CURVATURE HYPERSURFACES WITH CONSTANT δ-INVARIANT

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We completely classify constant mean curvature hypersurfaces (CMC) with constant δ-invariant in the unit 4-sphere S^4 and in the Euclidean 4-space $\mathbb{E}^4$.

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1. Introduction. A hypersurface in the unit round sphere S^{n+1} is called isoparametric if it has constant principal curvatures. It is known from [1] that an isoparametric hypersurface in S^4 is either an open portion of a 3-sphere or an open portion of the product of a circle and a 2-sphere, or an open portion of a tube of constant radius over the Veronese embedding. Because every isoparametric hypersurface in S^4 has constant mean curvature (CMC) and constant scalar curvature, it is interesting to determine all hypersurfaces with CMC and constant scalar curvature. In [2], it was proved that a closed hypersurface with CMC and constant scalar curvature in S^4 is isoparametric. Furthermore, complete hypersurfaces with CMC and constant scalar curvature in S^4 or in $\mathbb{E}^4$ have been completely classified in [9].

For each Riemannian n-manifold M^n with $n \geq 3$, the first author defined in [3, 4] the Riemannian invariant δ on M by

$$\delta(p) = \tau(p) - \inf K(p), \tag{1.1}$$

where $\tau = \sum_{i<j} K(e_i \wedge e_j)$ is the scalar curvature and $\inf K$ is the function assigning to each $p \in M^n$ the infimum of $K(\pi)$, π running over all planes in T_pM. Although the invariant δ and the scalar curvature are both Riemannian scalar invariants, they are very much different in nature.

It is known that the invariant δ plays some important roles in recent study of Riemannian manifolds and Riemannian submanifolds (see, e.g., [4, 5, 6, 7, 8, 10, 11, 12, 14, 15, 16]). In particular, it was proved in [3] that for any submanifold of a real space form $R^m(\epsilon)$ of constant curvature ϵ, one has the following general sharp inequality:

$$\delta \leq \frac{n^2(n-2)}{2(n-1)} H^2 + \frac{1}{2}(n+1)(n-2)\epsilon, \tag{1.2}$$

where H^2 is the squared mean curvature function and n is the dimension of the submanifold.

Clearly, every isoparametric hypersurface in S^4 or in $\mathbb{E}^4$ has constant mean curvature and constant δ-invariant. So, it is a natural problem to study hypersurfaces in S^4 and $\mathbb{E}^4$ with CMC and constant δ-invariant. The purpose of this paper is thus to classify such hypersurfaces.

Our main results are the following theorems.

Theorem 1.1. *A CMC hypersurface in the Euclidean* 4*-space* $\mathbb{E}^4$ *has constant* δ*-invariant if and only if it is one of the following:*

(1) *an isoparametric hypersurface;*
(2) *a minimal hypersurface with relative nullity greater than or equal to* 1*;*
(3) *an open portion of a hypercylinder* $N \times \mathbb{R}$ *over a surface* N *in* $\mathbb{E}^3$ *with CMC and nonpositive Gauss curvature.*

Theorem 1.2. *A CMC hypersurface* M *in the unit* 4*-sphere* S^4 *has constant* δ*-invariant if and only if one of the following two statements holds:*

(1) M *is an isoparametric hypersurface;*
(2) *there is an open dense subset* U *of* M *and a nontotally geodesic isometric minimal immersion* $\phi : B^2 \to S^4$ *from a surface* B^2 *into* S^4 *such that* U *is an open subset of* $NB^2 \subset S^4$*, where* NB^2 *is defined by*

$$N_pB^2 = \left\{\xi \in T_{\phi(p)}S^4 : \langle \xi,\xi\rangle = 1,\ \langle \xi,\phi_*(T_pB^2)\rangle = 0\right\}. \tag{1.3}$$

In contrast to [2, 9], we do not make any global assumption on the hypersurfaces in Theorems 1.1 and 1.2.

As an immediate application of Theorem 1.1, we have the following corollary.

Corollary 1.3. *Let* M *be a complete hypersurface of Euclidean* 4*-space* $\mathbb{E}^4$*. Then* M *has constant* δ*-invariant and nonzero CMC if and only if* M *is one of the following hypersurfaces:*

(1) *an ordinary hypersphere;*
(2) *a spherical hypercylinder:* $\mathbb{R} \times S^2$*;*
(3) *a hypercylinder over a circle:* $\mathbb{E}^2 \times S^1$*.*

2. Preliminaries. Let $R^m(4)$ denote the complete simply connected space form $R^4(\epsilon)$ of constant curvature ϵ. Let M be a hypersurface of an $R^4(\epsilon)$. Denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections of M^n and $R^4(\epsilon)$, respectively. Then the Gauss and Weingarten formulas of M^n in $R^4(\epsilon)$ are given, respectively, by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \qquad \tilde{\nabla}_X \xi = -AX \tag{2.1}$$

for tangent vector fields X, Y, and unit normal vector field ξ, where h denotes the second fundamental form and A the shape operator. The second fundamental form and the shape operator are related by

$$\langle AX,Y\rangle = \langle h(X,Y),\xi\rangle. \tag{2.2}$$

The mean curvature H of M in $R^4(\epsilon)$ is defined by $H = (1/3)\operatorname{trace} A$. A hypersurface is called a CMC hypersurface if it has CMC.

Denote by R the Riemann curvature tensor of M. Then the *equation of Gauss* is given by

$$\begin{aligned} R(X,Y;Z,W) = (\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle)\epsilon + \langle h(X,W),h(Y,Z)\rangle \\ - \langle h(X,Z),h(Y,W)\rangle \end{aligned} \tag{2.3}$$

for vectors X, Y, Z, and W tangent to M. The Codazzi equation is given by

$$(\nabla_X A)Y = (\nabla_Y A)(X). \tag{2.4}$$

Since A is a symmetric endomorphism of T_pM, $p \in M$, we have three eigenvalues a, b, and c with three independent unit eigenvectors e_1, e_2, and e_3 so that

$$Ae_1 = ae_1, \qquad Ae_2 = be_2, \qquad Ae_3 = ce_3, \tag{2.5}$$

where $A = A_{e_4}$. The functions a, b, and c are called the principal curvatures and e_1, e_2, and e_3 the principal directions.

With respect to the frame fields e_1, e_2, and e_3 of M chosen above, let ω^1, ω^2, and ω^3 be the field of dual frames and let $\omega_B^A, A, b = 1,2,3,4$, be the connection forms associated with e_1, e_2, e_3, and e_4. Then the structure equations of M in $R^4(\epsilon)$ are given by

$$d\omega^i = -\sum_{j=1}^{3} \omega_j^i \wedge \omega^j, \qquad \omega_j^i + \omega_i^j = 0, \tag{2.6}$$

$$d\omega_j^i = \sum_{k=1}^{3} \omega_k^i \wedge \omega_k^j + \omega_i^4 \wedge \omega_j^4 + \epsilon\omega^i \wedge \omega^j, \tag{2.7}$$

$$d\omega_i^4 = \sum_{j=1}^{3} \omega_j^4 \wedge \omega_j^i, \quad i,j = 1,2,3. \tag{2.8}$$

Moreover, from (2.5), we have

$$\omega_1^4 = a\omega^1, \qquad \omega_2^4 = b\omega^2, \qquad \omega_3^4 = c\omega^3. \tag{2.9}$$

Without loss of generality, we may choose e_1, e_2, and e_3 such that $a \ge b \ge c$. It is well known that a, b, and c are continuous on M and differentiable on the open subset $U = \{p \in M : a(p) > b(p) > c(p)\}$. The principal directions e_1, e_2, and e_3 can be chosen to be differentiable on U.

Let p be any given point in M. If $0 > b \geq c$ at p, then, after replacing ξ by $-\xi$ and interchanging a and c, we obtain $a \geq b > 0$ and $b \geq c$.

3. Lemmas. We follow the notations given in Section 2. Throughout this paper, we will choose e_1, e_2, e_3, and e_4 so that $a \geq b \geq 0$ and $b \geq c$.

Lemma 3.1. *For each point $p \in M$, either*
(a) $\inf K = bc + \epsilon$ *with* $c \geq 0$ *at* p*, or*
(b) $\inf K = ac + \epsilon$ *with* $c \leq 0$ *at* p.

Proof. Recall that we have assumed that $a \geq b \geq 0$ and $b \geq c$ at p. Let P be any 2-plane in T_pM. Then P must intersects the 2-plane $\text{Span}\{e_1, e_2\}$. Thus, there exists an orthonormal basis $\{X, Y\}$ of P such that $X \in P \cap \text{Span}\{e_1, e_2\}$ and

$$\begin{aligned} X &= \cos\theta e_1 + \sin\theta e_2, \\ Y &= \pm\sin\theta\cos\phi e_1 \mp \cos\theta\cos\phi e_2 + \sin\phi e_3 \end{aligned} \tag{3.1}$$

for some θ and ϕ with $\theta \in [0, \pi)$, $\phi \in [0, \pi]$. It is easy to see that the sectional curvature $K(P)$ of P is given by

$$K(P) = ab\cos^2\phi + c(a\cos^2\theta + b\sin^2\theta)\sin^2\phi + \epsilon. \tag{3.2}$$

We regard the sectional curvature at p as a function $K(\theta, \phi)$ of θ and ϕ.

If $c \geq 0$, (3.2) can be expressed as

$$K(\theta, \phi) = ac + a(b - c)\cos^2\phi - c(a - b)\sin^2\theta\sin^2\phi + \epsilon, \tag{3.3}$$

which implies that $K(\theta, \phi) \geq bc + \epsilon$ with the equality holding at $(\theta, \phi) = (\pi/2, \pi/2)$.

If $c \leq 0$, we can express (3.2) as

$$K(\theta, \phi) = bc + b(a - c)\cos^2\phi + c(a - b)\cos^2\theta\sin^2\phi + \epsilon, \tag{3.4}$$

which implies that $K(\theta, \phi) \geq ac + \epsilon$ with the equality holding at $(\theta, \phi) = (0, \pi/2)$. □

Lemma 3.2. *On the open subset U on which M has three distinct principal curvatures, the following equations hold:*

$$e_2 a = (a-b)\omega_1^2(e_1), \tag{3.5}$$

$$e_3 a = (a-c)\omega_1^3(e_1), \tag{3.6}$$

$$e_3 b = (b-c)\omega_2^3(e_2), \tag{3.7}$$

$$e_1 b = (b-a)\omega_2^1(e_2), \tag{3.8}$$

$$e_1 c = (c-a)\omega_3^1(e_3), \tag{3.9}$$

$$e_2 c = (c-b)\omega_3^2(e_3), \tag{3.10}$$

$$(c-b)\omega_3^2(e_1) = (c-a)\omega_3^1(e_2), \tag{3.11}$$

$$(b-c)\omega_2^3(e_1) = (b-a)\omega_2^1(e_3), \tag{3.12}$$

$$(a-b)\omega_1^2(e_3) = (a-c)\omega_1^3(e_2). \tag{3.13}$$

Proof. The proof follows from Codazzi's equation and is a straightforward computation. □

4. Proofs of Theorems 1.1 and 1.2. We use the same notations as before. Let M be a (connected) CMC hypersurface with constant δ-invariant in $R^4(\epsilon)$. Then the scalar curvature τ of M is given by

$$\tau = ab + bc + ac + 3\epsilon. \tag{4.1}$$

From the constancy of the mean curvature, we have

$$a + b + c = r_1 \tag{4.2}$$

for some constant r_1. By combining Lemma 3.1 with (1.1) and (4.1), we obtain

(i) $\delta = a(b+c) + 2\epsilon$ with $c \geq 0$, or

(ii) $\delta = b(a+c) + 2\epsilon$ with $c \leq 0$.

When $U = \{p \in M : a(p) > b(p) > c(p)\}$ is empty, M is an isoparametric hypersurface since the mean curvature and the δ-invariant are both constant. Thus, from now on, we may assume that U is nonempty and work on U.

We will treat Cases (i) and (ii) on U separately.

Case (i) ($\delta = a(b+c) + 2\epsilon$, $c \geq 0$). Since δ is constant, we get $a(b+c) = r_2 - 2\epsilon$ for some constant r_2. Combining this with (4.2) yields

$$a = c_1, \qquad b + c = c_2 \tag{4.3}$$

for some constants c_1 and c_2. For simplicity, let

$$\omega_3^2(e_1) = \mu, \qquad \omega_1^2(e_2) = f, \qquad \omega_2^3(e_2) = g, \qquad \omega_2^3(e_3) = h. \tag{4.4}$$

If b and c are constant, then M is isoparametric. So, we assume that b and c are nonconstant on U. Using (4.3), we get $e_j b = -e_j c$, $j = 1,2,3$. Thus, Lemma 3.2 gives

$$\omega_1^3(e_1) = \omega_1^2(e_1) = 0, \tag{4.5}$$

$$e_1 b = (a-b)f = (c-a)\omega_1^3(e_3), \tag{4.6}$$

$$e_2 b = (b-c)h, \tag{4.7}$$

$$e_3 b = (b-c)g. \tag{4.8}$$

From (4.5), we know that the integral curves of e_1 are geodesics in U. Applying (3.12), (3.13), (4.6), (4.7), and (4.8), we find

$$\begin{gathered} e_1 b = (a-b)f, \qquad e_2 b = (c-b)h, \\ e_3 b = (b-c)g, \qquad e_j a = 0, \qquad e_j c = -e_j b, \end{gathered} \tag{4.9}$$

$$\omega_1^2 = f\omega^2 + \frac{b-c}{b-a}\mu\omega^3, \tag{4.10}$$

$$\omega_1^3 = \frac{b-c}{c-a}\mu\omega^2 + \frac{a-b}{c-a}f\omega^3, \tag{4.11}$$

$$\omega_2^3 = -\mu\omega^1 + g\omega^2 + h\omega^3, \tag{4.12}$$

for $j = 1,2,3$. By applying (2.6), (4.9), (4.10), (4.11), and (4.12), we find

$$\begin{aligned} d\omega^1 &= \left(\frac{b-c}{a-b} + \frac{b-c}{c-a}\right)\mu\omega^2\wedge\omega^3, \\ d\omega^2 &= f\omega^1\wedge\omega^2 + \frac{a-c}{b-a}\mu\omega^1\wedge\omega^3 + g\omega^2\wedge\omega^3, \\ d\omega^3 &= \frac{a-b}{a-c}\mu\omega^1\wedge\omega^2 - \frac{a-b}{a-c}f\omega^1\wedge\omega^3 + h\omega^2\wedge\omega^3. \end{aligned} \tag{4.13}$$

Using $(\nabla_{e_2}e_1 - \nabla_{e_1}e_2 - [e_2,e_1])b = 0$, we get

$$(a-b)e_2 f + (b-c)e_1 h = 2(c-a)fh + \frac{(b-a)(b-c)}{c-a}\mu g. \tag{4.14}$$

Similarly, from $(\nabla_{e_3}e_1 - \nabla_{e_1}e_3 - [e_3,e_1])b = (\nabla_{e_3}e_2 - \nabla_{e_2}e_3 - [e_3,e_2])b = 0$, we get

$$\begin{gathered} (a-b)e_3 f + (c-b)e_1 g = \frac{(a-c)(c-b)}{b-a}\mu h + \frac{2a(b+c) - b^2 - c^2 - 2a^2}{c-a} fg, \\ e_3 h + e_2 g = \frac{c-b}{a-c}\mu f. \end{gathered} \tag{4.15}$$

By computing $d\omega_1^2$ and applying (4.10), (4.11), (4.12), (4.13), and Cartan's structure equations, we obtain

$$e_1 f = \frac{2(b-c)}{a-c}\mu^2 - f^2 - ab - \epsilon, \tag{4.16}$$

$$e_1\left(\frac{b-c}{b-a}\mu\right) = \left\{\frac{b-c}{a-b} + \frac{2(a-b)}{a-c}\right\}\mu f, \tag{4.17}$$

$$e_3 f + e_2\left(\frac{b-c}{a-b}\mu\right) = \frac{b+c-2a}{c-a}\left\{fg - \frac{b-c}{a-b}\mu h\right\}. \tag{4.18}$$

Similarly, by computing $d\omega_1^3$ and $d\omega_2^3$, and by applying (4.10), (4.11), (4.12), (4.13), and Cartan's structure equations, we obtain

$$e_1\left(\frac{b-c}{c-a}\mu\right) = \left\{\frac{2a^2+2c^2+b^2-ab-3ac-bc}{(a-c)^2}\right\}\mu f, \tag{4.19}$$

$$e_1\left(\frac{a-b}{c-a}f\right) = -ac - \epsilon - \left(\frac{a-b}{c-a}\right)^2 f^2 + \frac{2(b-c)}{b-a}\mu^2, \tag{4.20}$$

$$e_3\left(\frac{c-b}{c-a}\mu\right) + e_2\left(\frac{a-b}{c-a}f\right) = \frac{2a-b-c}{a-c}\left\{fh + \frac{b-c}{a-b}\mu g\right\}, \tag{4.21}$$

$$e_2\mu + e_1 g = \frac{a-b}{c-a}\mu h - fg, \tag{4.22}$$

$$e_1 h + e_3\mu = \frac{a-b}{a-c}fh + \frac{a-c}{a-b}\mu g, \tag{4.23}$$

$$e_2 h - e_3 g = \frac{2(b-c)^2\mu^2}{(a-b)(a-c)} + \frac{a-b}{a-c}f^2 - g^2 - h^2 - bc - \epsilon. \tag{4.24}$$

Combining (4.9), (4.16), and (4.20) yields

$$\begin{aligned}&2(2a-b-c)(a-b)^2 + 2(2a-b-c)(b-c)^2\mu^2\\ &\quad + (a-b)(a-c)\{ab(a-b) + ac(a-c) + (2a-b-c)\epsilon\} = 0,\end{aligned} \tag{4.25}$$

which is impossible unless $\epsilon < 0$, since we assume that $a > b > c \geq 0$ in Case (i).

Case (ii) ($\delta = b(a+c) + 2\epsilon$, $c \leq 0$). Since δ is constant, we get $b(a+c) = r_2 - 2\epsilon$ for some constant r_2. Combining this with (4.2) yields

$$b = c_3, \qquad a + c = c_4, \tag{4.26}$$

for some constants c_3 and c_4. For simplicity, let

$$\omega_3^1(e_2) = \tilde{\mu}, \qquad \omega_2^1(e_1) = \tilde{f}, \qquad \omega_1^3(e_1) = \tilde{g}, \qquad \omega_1^3(e_3) = \tilde{h}. \tag{4.27}$$

If a and c are constant, then M is isoparametric. So, from now on, we may assume that a and c are nonconstant on U. Using (4.26), we get

$$e_j a = -e_j c, \quad j = 1, 2, 3. \tag{4.28}$$

Thus, Lemma 3.2 yields

$$\omega_2^3(e_2) = \omega_2^1(e_2) = 0, \tag{4.29}$$

$$e_1 a = (c-a)\tilde{h}, \qquad e_2 a = (b-a)\tilde{f}, \qquad e_3 a = (a-c)\tilde{g}. \tag{4.30}$$

Equation (4.28) shows that the integral curves of e_2 are geodesics in U. Applying (3.12), (3.13), (4.29), and (4.30), we find

$$\omega_1^2 = -\tilde{f}\omega^1 - \frac{a-c}{a-b}\tilde{\mu}\omega^3, \tag{4.31}$$

$$\omega_1^3 = \tilde{g}\omega^1 - \tilde{\mu}\omega^2 + \tilde{h}\omega^3, \tag{4.32}$$

$$\omega_2^3 = \frac{a-c}{c-b}\tilde{\mu}\omega^1 + \frac{a-b}{b-c}\tilde{f}\omega^3. \tag{4.33}$$

By applying (2.6), (4.31), (4.32), and (4.33), we find

$$\begin{aligned} d\omega^1 &= -\tilde{f}\omega^1\wedge\omega^2 + \tilde{g}\omega^1\wedge\omega^3 + \frac{b-c}{a-b}\tilde{\mu}\omega^2\wedge\omega^3, \\ d\omega^2 &= -\left(\frac{a-c}{a-b} + \frac{a-c}{b-c}\right)\tilde{\mu}\omega^1\wedge\omega^3, \\ d\omega^3 &= \frac{a-b}{b-c}\tilde{\mu}\omega^1\wedge\omega^2 + \tilde{h}\omega^1\wedge\omega^3 + \frac{a-b}{b-c}\tilde{f}\omega^2\wedge\omega^3. \end{aligned} \tag{4.34}$$

Using $(\nabla_{e_2}e_1 - \nabla_{e_1}e_2 - [e_2,e_1])a = 0$, we find

$$(a-b)e_1\tilde{f} - (a-c)e_2\tilde{h} = 2(b-a)\tilde{f}\tilde{h} + \frac{(a-b)(a-c)}{b-c}\tilde{\mu}\tilde{g}. \tag{4.35}$$

Similarly, from $(\nabla_{e_3}e_1 - \nabla_{e_1}e_3 - [e_3,e_1])a = (\nabla_{e_3}e_2 - \nabla_{e_2}e_3 - [e_3,e_2])a = 0$, we get

$$\begin{aligned} (a-b)e_3\tilde{f} + (a-c)e_2\tilde{g} &= \frac{(a-c)(b-c)}{a-b}\tilde{\mu}\tilde{h} + \frac{2ab+2bc-2b^2-a^2-c^2}{b-c}\tilde{f}\tilde{g}, \\ e_3\tilde{h} + e_1\tilde{g} &= \frac{c-a}{b-c}\tilde{\mu}\tilde{f}. \end{aligned} \tag{4.36}$$

By computing $d\omega_1^2$ and applying (4.31), (4.32), and (4.33) and Cartan's structure equations, we obtain

$$e_2\tilde{f} = \frac{2(a-c)}{b-c}\tilde{\mu}^2 - \tilde{f}^2 - ab - \epsilon, \tag{4.37}$$

$$e_2\left(\frac{a-c}{a-b}\tilde{\mu}\right) = \left\{\frac{a-c}{b-a} - \frac{2(a-b)}{b-c}\right\}\tilde{\mu}\tilde{f}, \tag{4.38}$$

$$e_3\tilde{f} + e_1\left(\frac{a-c}{b-a}\tilde{\mu}\right) = \frac{a+c-2b}{c-b}\left\{\tilde{f}\tilde{g} + \frac{a-c}{a-b}\tilde{\mu}\tilde{h}\right\}. \tag{4.39}$$

Similarly, by computing $d\omega_1^3$, $d\omega_2^3$, and by applying (4.31), (4.32), and (4.33) and Cartan's structure equations, we obtain

$$e_2\left(\frac{a-c}{c-b}\tilde{\mu}\right) = \left\{\frac{2b^2+2c^2+a^2-ab-3bc-ac}{(b-c)^2}\right\}\tilde{\mu}\tilde{f}, \tag{4.40}$$

$$e_2\left(\frac{a-b}{b-c}\tilde{f}\right) = -bc-\epsilon-\left(\frac{a-b}{c-b}\right)^2\tilde{f}^2+\frac{2(a-c)}{a-b}\tilde{\mu}^2, \tag{4.41}$$

$$e_3\left(\frac{a-c}{b-c}\tilde{\mu}\right)+e_1\left(\frac{a-b}{b-c}\tilde{f}\right) = \frac{2b-a-c}{b-c}\left\{\tilde{f}\tilde{h}-\frac{a-c}{a-b}\tilde{\mu}\tilde{g}\right\}, \tag{4.42}$$

$$e_1\tilde{\mu}+e_2\tilde{g} = \frac{a-b}{b-c}\tilde{\mu}\tilde{h}-\tilde{f}\tilde{g}, \tag{4.43}$$

$$e_2\tilde{h}+e_3\tilde{\mu} = \frac{b-a}{b-c}\tilde{f}\tilde{h}-\frac{b-c}{a-b}\tilde{\mu}\tilde{g}, \tag{4.44}$$

$$e_1\tilde{h}-e_3\tilde{g} = \frac{2(a-c)^2\tilde{\mu}^2}{(b-a)(b-c)}-\frac{a-b}{b-c}\tilde{f}^2-\tilde{g}^2-\tilde{h}^2-ac-\epsilon. \tag{4.45}$$

Applying (4.30), (4.37), and (4.41) yields

$$\begin{aligned}&2(2b-a-c)(a-b)^2\tilde{f}^2+2(2b-a-c)(a-c)^2\tilde{\mu}^2\\&\quad+(b-a)(b-c)\{ab(b-a)+bc(b-c)+(2b-a-c)\epsilon\}=0.\end{aligned} \tag{4.46}$$

Using (4.26), (4.30), and (4.38), we find

$$e_2\tilde{\mu} = \frac{2[(a-b)^2+(b-c)^2]}{(a-c)(c-b)}\tilde{\mu}\tilde{f}. \tag{4.47}$$

On the other hand, by differentiating (4.46) with respect to e_2 and using (4.26), (4.30), and (4.37), we obtain

$$\begin{aligned}&4(a+c-2b)(a-c)^2\tilde{\mu}(e_2\tilde{\mu})\\&\quad=b(a-b)(3a^3-13a^2b+10ab^2+7a^2c-4abc-2b^2c-3ac^2+bc^2+c^3)\tilde{f}\\&\qquad-8\frac{(a+c-2b)^2(a-b)(a-c)}{b-c}\tilde{\mu}^2\tilde{f}+8(a+c-2b)(a-b)^2\tilde{f}^3\\&\qquad-(a-b)(a+c-2b)(4b-3a-c)\epsilon\tilde{f}.\end{aligned} \tag{4.48}$$

Replacing $\tilde{f}^2$ in (4.48) by using (4.46) yields

$$\begin{aligned}&4(a+c-2b)(a-c)^2\tilde{\mu}(e_2\tilde{\mu})\\&\quad=3(a-b)(a-c)(a+c-2b)\tilde{\epsilon}\tilde{f}\\&\qquad+3b(a-b)(a-c)(a^2-3ab+2b^2+2ac-3bc+c^2)\tilde{f}\\&\qquad-8\frac{(a+c-2b)(a-c)[(a-b)^2+(b-c)^2]}{b-c}\tilde{\mu}^2\tilde{f}.\end{aligned} \tag{4.49}$$

Substituting (4.47) into (4.49) yields

$$\tilde{f}(a+c-2b)\{b(a+c-b)+\epsilon\} = 0. \tag{4.50}$$

CASE (ii-a) ($\tilde{f}=0$). In this case, (4.37) and (4.41) imply that

$$2(a-c)\mu^2 = (ab+\epsilon)(b-c) = (bc+\epsilon)(a-b). \tag{4.51}$$

The equality in (4.51) yields

$$b(ab+bc-2ac) = (a+c-2b)\epsilon. \tag{4.52}$$

CASE (ii-a.1) ($\tilde{f}=0$, $b \neq 0$). In this case, (4.26) and (4.52) imply that ac is constant. Hence, by (4.26), we know that both a and c are constant. Thus, M is isoparametric.

CASE (ii-a.2) ($b=\tilde{f}=0$, $\epsilon=1$). In this case, (4.52) reduces to $a+c=2b$. So, M satisfies the equality case of inequality (1.2). Therefore, by applying [7, Theorem 2], we know that M is given by Theorem 1.2 (2).

CASE (ii-a.3) ($b=\tilde{f}=\epsilon=0$). In this case, (4.37) implies that $\tilde{\mu}=0$. Thus, by (4.31) and (4.33), we obtain $\omega_1^2=\omega_2^3=0$. On the other hand, from (4.29), we have $\nabla_{e_2}e_2=0$. Therefore, $\mathcal{D}_1=\operatorname{Span}\{e_1,e_3\}$ and $\mathcal{D}_2=\operatorname{Span}\{e_2\}$ are integrable distributions in M with totally geodesic leaves. Hence, M is locally the Riemannian product of a line and a Riemannian 2-manifold N^2. Moreover, because the second fundamental form h of M in $\mathbb{E}^4$ satisfies $h(\mathcal{D}_1,\mathcal{D}_2)=\{0\}$, Moore's lemma [13] implies that M is an open portion of a hypercylindrical $\mathbb{R}\times N^2 \subset \mathbb{E}\times\mathbb{E}^3=\mathbb{E}^4$. Furthermore, from the assumption on the shape operator of M in $\mathbb{E}^4$, we know that the mean curvature of N in $\mathbb{E}^3$ is constant and the Gauss curvature of N is nonpositive. Thus, we obtain case (3) of Theorem 1.1.

CASE (ii-b) ($\tilde{f}\neq 0$, $b=0$). In this case, (4.50) yields $(a+c)\epsilon=0$.

If $\epsilon=1$, then $a+c=0$. Hence, M is a minimal hypersurface satisfying the equality case of inequality (1.2). Thus, by applying [7, Theorem 2], we obtain case (2) of Theorem 1.2.

If $\epsilon=0$, then (4.46) implies that $a+c-2b=0$ due to $b=0$ and $a\neq b$. Hence, M satisfies the equality case of inequality (1.2). Since M has CMC, [7, Theorem 1] implies that M is either an isoparametric hypersurface or a minimal hypersurface which satisfies the equality $\delta=0$. Hence, we obtain either case (1) or case (2) of Theorem 1.1.

CASE (ii-c) ($b\neq 0$, $\tilde{f}\neq 0$). In this case, (4.50) yields

$$(a+c-2b)\{b(a+c-b)+\epsilon\} = 0. \tag{4.53}$$

If $a+c-2b=0$ holds, then (4.46) implies that $a(a-b)-c(b-c)=0$ which is impossible, since $a\geq 0$, $c\leq 0$, and $a>b>0$ by assumption. Therefore, we must have

$$\epsilon = b(b-a-c). \tag{4.54}$$

From (4.54), $\epsilon \geq 0$, and $b > 0$, we get

$$b \geq a+c. \tag{4.55}$$

On the other hand, by substituting (4.54) into (4.46), we find

$$(a+c-2b)\left[(a-c)^2\tilde{\mu}^2+(a-b)^2\tilde{f}^2\right] = b(b-a)^2(b-c)^2. \tag{4.56}$$

In particular, we obtain $a+c > 2b$. Combining this with (4.55) gives $b < 0$ which is a contradiction. Thus, this case is impossible.

The converse follows from [7, Theorem 2] and from direct computation.

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