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HILBERT SERIES AND APPLICATIONS TO GRADED RINGS

SELMA ALTINOK

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This paper contains a number of practical remarks on Hilbert series that we expect to be useful in various contexts. We use the fractional Riemann-Roch formula of Fletcher and Reid to write out explicit formulas for the Hilbert series $P(t)$ in a number of cases of interest for singular surfaces (see Lemma 2.1) and 3-folds. If X is a $\mathbb{Q}$-Fano 3-fold and $S \in |-K_X|$ a $K3$ surface in its anticanonical system (or the general elephant of X), polarised with $D = \mathbb{O}_S(-K_X)$, we determine the relation between $P_X(t)$ and $P_{S,D}(t)$. We discuss the denominator $\prod(1-t^{a_i})$ of $P(t)$ and, in particular, the question of how to choose a reasonably small denominator. This idea has applications to finding $K3$ surfaces and Fano 3-folds whose corresponding graded rings have small codimension. Most of the information about the anticanonical ring of a Fano 3-fold or $K3$ surface is contained in its Hilbert series. We believe that, by using information on Hilbert series, the classification of $\mathbb{Q}$-Fano 3-folds is too close. Finding $K3$ surfaces are important because they occur as the general elephant of a $\mathbb{Q}$-Fano 3-fold.

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1. Introduction. We work with graded rings $R = \bigoplus_{n\geq 0} R_n$ that are finitely generated over an algebraically closed field k of characteristic 0 and satisfy $R_0 = k$. The *Hilbert function* of R is the numerical function $P_n = \dim R_n$ for $n \geq 0$; the *Hilbert series* $P(t)$ or $P_R(t)$ of R is the formal power series defined by $P(t) = \sum P_n t^n$. It is elementary and well known that $P(t)$ is a rational function of t. In fact, if $x_1,\dots,x_d$ are homogeneous elements of weight $\operatorname{wt} x_i = a_i$ generating R (or more generally, generating a subring over which R is finite), then $\prod(1-t^{a_i})P(t) = Q(t)$ is a polynomial.

2. Fractional Riemann-Roch formula

2.1. Surfaces with Du Val singularities. We use the definitions and notation of Reid [9] for singularities. If S is a projective surface with Du Val singularities and D a Weil divisor on S, then some multiple rD is Cartier, and there is a formula [9, Theorem 9.1]

$$\chi(S,\mathbb{O}_S(D)) = \chi(\mathbb{O}_S) + \frac{1}{2}(D^2 - DK_S) + \sum_P c_P(D), \tag{2.1}$$

where $c_P(D) \in \mathbb{Q}$ is a fractional contribution due to the singularity of S and

$\mathcal{O}_S(D)$ at P. Here, $DK_S \in \mathbb{Z}$ is the intersection number with the canonical class, and $D^2 \in \mathbb{Q}$ the self-intersection of the $\mathbb{Q}$-Cartier divisor D. Moreover, $c_P(D)$ can be written as a sum

$$c_P(D) = -\sum_{(r,a)\in\mathcal{B}} \frac{a(r-a)}{2r} \tag{2.2}$$

over a *basket* $\mathcal{B} = \{(r,a)\}$ with $0 < a < r$ and each a is coprime to r. Here, the basket appears simply as a list of combinatorial data for computing the right-hand side of (2.2); Reid [9, Section 9] interprets (2.1) and (2.2) as the singularity of S, and $\mathcal{O}_S(D)$ at P has the same effect on $\chi(S,\mathcal{O}_S(D))$ as a basket $\mathcal{B}$ of virtual cyclic quotient singularities of type $(1/r)(1,-1)$, at which $\mathcal{O}_S(D)$ is locally isomorphic to the eigensheaf of ε^a (see [9, Chapter III] for definition).

It follows from this interpretation and the proof of [9, Theorem 9.1] that for $n \in \mathbb{Z}$,

$$c_P(nD) = -\sum_{(r,a)\in\mathcal{B}} \frac{\overline{na}(r-\overline{na})}{2r}, \tag{2.3}$$

where, for each $(r,a) \in \mathcal{B}$, the bar stands for the smallest positive residue modulo r.

Lemma 2.1. *Let S be a surface with Du Val singularities and D a Weil divisor. We assume that $H^i(S,\mathcal{O}_S(nD)) = 0$ for all $i > 0$ and for all $n \geq 1$. Then, the graded ring $R(S,D) = \bigoplus_{n\geq 0} H^0(S,\mathcal{O}_S(nD))$ has Hilbert series*

$$\begin{aligned} P_S(t) = {} & \frac{1+(\chi-1)t}{1-t} + \frac{t+t^2}{2(1-t)^3}D^2 - \frac{t}{2(1-t)^2}DK_S \\ & - \sum_{(r,a)\in\mathcal{B}} \frac{\sum_{n=1}^{r-1} \overline{an}(r-\overline{an})t^n}{2r(1-t^r)}. \end{aligned} \tag{2.4}$$

Here, $\chi = \chi(\mathcal{O}_S)$.

Proof. The first three terms of (2.4) expand out as

$$\begin{aligned} 1+\chi t+\chi t^2+\cdots, \qquad & \frac{(t+2^2t^2+3^2t^3+\cdots)D^2}{2}, \\ & -\frac{(t+2t^2+3t^2+\cdots)DK_S}{2}, \end{aligned} \tag{2.5}$$

respectively, corresponding to the sum over nD of first three terms of (2.1). For each element of the basket, the denominator $1-t^r$ has the effect of repeating the contribution $-\overline{na}(r-\overline{na})/2r$ from (2.3) periodically over the intervals $[0,r],[r,2r],\ldots$ (zero at the endpoints), giving the last term of (2.1). □

2.2. 3-**folds with canonical singularities.** Let X be a projective 3-fold with canonical singularities and A a Weil divisor. To use the formulas of [9, Section 10], assume that, at every singular point $P \in X$, we have $\mathcal{O}_X(A) \cong \mathcal{O}_X(lK_X)$ for some l (possibly depending on P). Then, [9, Theorem 10.2] states that

$$\chi(X,A) = \chi(\mathcal{O}_X) + \frac{1}{6}A^3 - \frac{1}{4}A^2K_X + \frac{1}{12}A(K_X^2 + c_2) + \sum c_Q(A), \tag{2.6}$$

where $c_Q(A)$ is a sum $\sum_{\mathcal{B}} c(r,a,l)$ taken over a basket $\mathcal{B} = \{(r,a,l)\}$, with a and l coprime to r, and the contributions are

$$c(r,a,l) = -\frac{r^2-1}{12r}l + \sum_{i=1}^{l-1}\frac{\overline{ai}(r-\overline{ai})}{2r} \tag{2.7}$$

$$= \frac{r^2-1}{12r}(r-l) - \sum_{i=l}^{r-1}\frac{\overline{ai}(r-\overline{ai})}{2r}. \tag{2.8}$$

The interpretation here is that $c(r,a,l)$ is the contribution from a singularity of type $(1/r)(a,1,-1)$ at which A is locally the ε^{al} eigensheaf.

We can write out the other terms in the Hilbert series by analogy with Lemma 2.1 (see the following section). The contribution made by each element $(r,a,l) \in \mathcal{B}$ to the Hilbert series is thus

$$\sum_{n=1}^{\infty} c(r,a,nl)t^n = \frac{1}{1-t^r}\sum_{n=1}^{r-1} c(r,a,nl)t^n. \tag{2.9}$$

Remark 2.2. The two formulas (2.7) and (2.8) are equal because

$$\sum_{i=1}^{r-1}\overline{ai}(r-\overline{ai}) = \sum_{i=1}^{r-1} i(r-i) = \frac{1}{6}r(r^2-1); \tag{2.10}$$

moreover, they also hold for l not in $[0,r]$ (the expression only depends on l modulo r).

In dealing with a single divisor A, we may assume that l is coprime to r; but the proof of [9, Section 10] is valid for any l. Thus, we can use formulas (2.7) and (2.8) for the contribution $c(r,a,nl)$ for any n in calculating $\chi(nA)$.

3. Fano 3-folds and $K3$ surfaces

3.1. Fano 3-folds. Let X be a Fano 3-fold, that is, a 3-fold with canonical singularities and ample anticanonical class $A = -K_X$. Standard use of vanishing gives

$$P_n = h^0(X,nA) = \chi(\mathcal{O}(nA)), \quad \text{for } n \geq 0. \tag{3.1}$$

Now, $\chi(\mathcal{O}_X) = 1$ and the basket of X is $\mathcal{B} = \{(r,a,-1)\}$, and by [9, Corollary 10.3], we have

$$\frac{1}{12}Ac_2 = -\frac{1}{12}K_Xc_2 = 2 - \sum_{\mathcal{B}}\frac{r^2-1}{12r}. \tag{3.2}$$

Using all this, formulas (2.6) and (2.7) specialise to give

$$P_n = 2n+1+\frac{1}{12}n(n+1)(2n+1)A^3+\sum_{\mathcal{B}}\left(-\frac{r^2-1}{12r}n+c(r,a,-n)\right). \tag{3.3}$$

Now, consider $P_n - P_{n-1}$. To handle the second term, we use

$$n(n+1)(2n+1)-(n-1)(n)(2n-1) = 6n^2. \tag{3.4}$$

By (2.8), the bracketed expression inside the sum over $\mathcal{B}$ equals

$$-\sum_{i=-n}^{r-1}\frac{\overline{ai}(r-\overline{ai})}{2r}, \tag{3.5}$$

and the difference from n to $n-1$ is just one term of the sum. We obtain

$$P_n - P_{n-1} = 2+\frac{1}{2}n^2A^3-\sum_{(r,a,-1)\in\mathcal{B}}\frac{\overline{an}(r-\overline{an})}{2r}. \tag{3.6}$$

Since $P_n - P_{n-1}$ is the coefficient of t^n in $(1-t)P_X(t)$, arguing as in Lemma 2.1 gives the following result.

Corollary 3.1. *The Hilbert series of a Fano* 3*-fold* X *is*

$$P_X(t) = \frac{1+t}{(1-t)^2}+\frac{t(1+t)}{2(1-t)^4}A^3-\sum_{(r,a,-1)\in\mathcal{B}}\frac{\sum_{n=1}^{r-1}\overline{an}(r-\overline{an})t^n}{2r(1-t)(1-t^r)}. \tag{3.7}$$

3.2. $K3$ **surfaces.** For a $K3$ surface S with Du Val singularities and a Weil divisor D, Lemma 2.1 specialises to give the Hilbert series of $P_n = h^0(S,nD)$ in the form

$$P_S(t) = \frac{1+t}{1-t}+\frac{t(1+t)}{2(1-t)^3}D^2-\sum_{(r,a)\in\mathcal{B}}\frac{\sum_{n=1}^{r-1}\overline{an}(r-\overline{an})t^n}{2r(1-t^r)}. \tag{3.8}$$

Corollary 3.2. *If* X *is a Fano* 3*-fold polarised by* $A = -K_X$ *and* $S \in |-K_X|$ *a* $K3$ *surface polarised by* $D = A_{|S}$*, then*

$$P_S(t) = (1-t)P_X(t). \tag{3.9}$$

This result follows, of course, from the restriction exact sequence

$$0 \longrightarrow \mathcal{O}_X((n-1)A) \longrightarrow \mathcal{O}_X(nA) \longrightarrow \mathcal{O}_S(nD) \longrightarrow 0. \tag{3.10}$$

The point, however, is that the corollary gives a formula for the Hilbert series $P_X(t)$ of the Fano 3-fold in terms of simpler data for a $K3$ surface. This formula is valid even if there is no $K3$ surface $S \in |-K_X|$, for example, if $P_1(X) = 0$ so that $|-K_X| = \varnothing$. Compare Corti, Pukhlikov, and Reid [5, Remark 7.2.3].

4. Applications. In [2], we studied polarised $K3$ surfaces S in terms of their graded rings $R(S,D)$. We also studied Fano 3-folds X in a similar way. The first step in our strategy consisted of finding suitable weights a_0, a_1, and a_2 so that

$$(1-t^{a_0})(1-t^{a_1})(1-t^{a_2})P(t) \tag{4.1}$$

is a polynomial with positive coefficients and a_0, a_1, and a_2 are "fairly small." This is a combinatorial analogue of finding a polynomial subring $k[x_0,x_1,x_2] \subset R(S,D)$ over which $R(S,D)$ is a finite-free module of "fairly small rank." If $\mathcal{B} = \{(r,a)\}$ is the basket of S, then each $(1-t^r)$ appears in the denominator so that a first necessary condition for this is that each r divides some a_i.

We have the obvious bound $\sum n \le 19$ for the number and types of Du Val singularities A_n, D_n, and E_n on a $K3$ surface S. This implies the bound $\sum(r-1) \le 19$ for the basket $\mathcal{B} = \{(r,a)\}$ on a $K3$ surface (see [6, Theorem III.9.20] and [7, Theorem II.8.21]). For a Fano 3-fold X, a similar bound on the basket $\{(r,a,-1)\}$ is provided by an argument of Kawamata [8] who first proves that $Ac_2 > 0$; then (3.2) implies

$$\sum\left(r-\frac{1}{r}\right) < 24. \tag{4.2}$$

Thus, there are only finitely many possibilities for the basket $\mathcal{B}$. An easy calculation shows that in either case, at most, 5 distinct values of r occur.

In [2], we develop a procedure based on these ideas to find all possible Hilbert series for $K3$ surfaces and Fano 3-folds with graded ring of given codimension. We give explicit lists [1] of codimension 3 and 4 cases (comparable to the lists of hypersurfaces and codimension 2 in Fletcher [7]) and, in most cases, settle the question of the existence of the varieties.

The *numerical data* of a $K3$ surface S with D is $P_1 = h^1(S,D)$, where $0 \le P_1 \le 3+\operatorname{codim} S$ and the basket $\mathcal{B} = \{(r,a)\}$. We can rewrite D^2 from the formula of P_1 in terms of P_1 and the basket $\mathcal{B}$,

$$D^2 = 2(P_1-2)+\sum\frac{a(r-a)}{r}. \tag{4.3}$$

We produced the lists of codimension 3 and 4 [1] by searching all possible $K3$ surfaces of given numerical data. We give an example below to show how this search is carried out. More details of applications are given in the Singapore paper [3].

Example 4.1. A typical example is

$$D^2 = -2+\frac{1}{2}+\frac{1}{2}+\frac{3}{4}+\frac{4}{5}. \tag{4.4}$$

That is, the numerical data are the basket $\mathcal{B} = \{(2,1),(2,1),(4,1),(5,1)\}$ and $P_1 = 1$ (see formula (4.3)). This is #39 in the codimension 3 list [1]. Since $P_1 = 1$, there is a generator of degree 1, say x. From formula (3.8),

$$\begin{aligned} P_S(t) = \frac{1+t}{1-t} + \frac{t(1+t)}{2(1-t)^3}\frac{11}{20} - 2\times\frac{t}{2\cdot 2(1-t^2)} \\ - \frac{3t+4t^2+3t^3}{2\cdot 4(1-t^4)} - \frac{4t+6t^2+6t^3+4t^4}{2\cdot 5(1-t^5)}. \end{aligned} \tag{4.5}$$

To kill its denominators, we must have at least two more generators, say t_1 and v, whose degrees are divisible by 4 and 5, respectively. Therefore, the smallest choice of (a_1,a_2,a_3) is $(1,4,5)$. After simplifying, we obtain

$$(1-t)(1-t^4)(1-t^5)P_S(t) = t^{10}+t^8+t^7+2t^6+t^5+2t^4+t^3+t^2+1. \tag{4.6}$$

This looks like the Hilbert series of an Artinian ring with further generators y, z, t_2 of degrees 2, 3, 4 so that the possible candidates S is in the weighted projective space $\mathbb{P}(1,2,3,4,4,5)$ (see [2, 6, 7] for definition). To find the structure of its resolution, multiply (3.6) again by $(1-t^2)(1-t^3)(1-t^4)$,

$$\begin{aligned} &(1-t)(1-t^2)(1-t^3)(1-t^4)^2(1-t^5)P_S(t) \\ &\quad = 1-t^6-t^7-2t^8-t^9+t^{10}+2t^{11}+t^{12}+t^{13}-t^{19}. \end{aligned} \tag{4.7}$$

From here, we read off the shape of the resolution of $R(S,D)$ over $A = k[x,y,z,t_1,t_2,v]$, namely,

$$\begin{aligned} &0 \longrightarrow A(-19) \xrightarrow{P^t} A(-13)\oplus A(-12)\oplus A(-11)\oplus A(-11)\oplus A(-10) \\ &\xrightarrow{M} A(-9)\oplus A(-8)\oplus A(-8)\oplus A(-7)\oplus A(-6) \xrightarrow{P} A \longrightarrow R \longrightarrow 0, \end{aligned} \tag{4.8}$$

where P is a 5×1 vector and M is a 5×5 skew-symmetric matrix. In other words, we expect 5 relations in degrees 6, 7, 8, 8, and 9, and 5 syzygies in degrees 10, 11, 11, 12, and 13. Note that $n = 1+2+3+4+4+5$ corresponds to the canonical class of $\mathbb{P}(1,2,3,4,4,5)$ and hence the canonical divisor K_S of S is $\mathbb{O}(19-n)$, which is trivial. The shape of the polynomial, together with the Buchsbaum-Eisenbud theorem on Gorenstein rings in codimension 3 (see [4]), gives us the equations of the relations as the Paffian of a 5×5 skew-symmetric matrix M with degrees

$$\begin{pmatrix} 0 & 2 & 2 & 3 & 4 \\ 2 & 0 & 3 & 4 & 5 \\ 2 & 3 & 0 & 4 & 5 \\ 3 & 4 & 4 & 0 & 6 \\ 4 & 5 & 5 & 6 & 0 \end{pmatrix}. \tag{4.9}$$

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Selma Altinok: Department of Mathematics, Arts and Science Faculty, Adnan Menderes University, Aydın 09010, Turkey

E-mail address: saltinok43@hotmail.com