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FLAT COVERS OF REPRESENTATIONS OF THE QUIVER A_∞

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Rooted quivers are quivers that do not contain $A_\infty \equiv \cdots \to \bullet \to \bullet$ as a subquiver. The existence of flat covers and cotorsion envelopes for representations of these quivers have been studied by Enochs et al. The main goal of this paper is to prove that flat covers and cotorsion envelopes exist for representations of A_∞. We first characterize finitely generated projective representations of A_∞. We also see that there are no projective covers for representations of A_∞, which adds more interest to the problem of the existence of flat covers.

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1. Introduction. It is known that, for an arbitrary quiver Q, the category of representations by modules of Q is a Grothendieck category with a generating system of projective representations. An explicit construction of such a system was given in [6], where, furthermore, flat representations were characterized, not for all quivers, but for a large class of them called rooted quivers. This characterization of flat representations of rooted quivers was very useful in proving that every representation of a rooted quiver admits a flat cover and a cotorsion envelope, see [6, Theorem 4.3]. Rooted quivers were also characterized in [6] as those quivers with no path of the form $\cdots \to \bullet \to \bullet$. In this paper, we study flat representations and flat covers of representations of the quiver $A_\infty \equiv \cdots \to \bullet \to \bullet$ as a first step in the treatment of nonrooted quivers.

In Section 2, we characterize finitely generated and projective representations of A_∞, and this will allow us to define flat representations as direct limits of them. To do this, we give necessary and sufficient conditions for a representation P of A_∞ to be projective. Finally, we give an example which shows that projective covers of representations of A_∞ do not exist in general. This makes Theorem 5.3, the main result of Section 5, more interesting: we prove that any representation by modules of A_∞ has a flat cover. This will be done using the techniques developed by Eklof and Trlifaj in [4] concerning cotorsion theories cogenerated by sets in the categories of modules (see [7] for a more detailed explanation about cotorsion theories) and the generalizations of these techniques to Grothendieck categories with projective generators given in [1].

2. Preliminaries. All rings considered in this paper will be associative with identity and, unless otherwise specified, they are not necessarily commutative. The letter R will usually denote a ring.

A quiver Q is a directed graph whose edges are called arrows. An arrow of a quiver from a vertex v_1 to a vertex v_2 is denoted by $a : v_1 \to v_2$ or $v_1 \xrightarrow{a} v_2$. A quiver Q may be thought of as a category in which the objects are the vertices of Q and the morphisms are the paths (a path is a sequence of arrows) of Q.

A representation by modules X of a given quiver Q is a functor $X : Q \to R$-Mod. Such a representation is determined by giving a module $X(v)$ (or X_v) for each vertex v of Q and a homomorphism $X(a) : X(v_1) \to X(v_2)$ for each arrow $a : v_1 \to v_2$ of Q. A morphism η between two representations X and Y is a natural transformation, so it is a family η_v such that $Y(a) \circ \eta_{v_1} = \eta_{v_2} \circ X(a)$ for any arrow $a : v_1 \to v_2$ of Q. Thus, the representation of a quiver Q by modules over a ring R is a category denoted by $(Q, R\text{-Mod})$, which is a Grothendieck category with enough projectives.

As usual, A_∞ will be used to denote the quiver

$$\cdots \longrightarrow v_n \longrightarrow v_{n-1} \longrightarrow \cdots \longrightarrow v_1 \longrightarrow v_0, \tag{2.1}$$

where v_i, $i = 1, \ldots, n$, are the vertices of the quiver; representation will mean representation by modules of A_∞. For a representation M of A_∞, and for an arrow $a : v_n \to v_{n-1}$, we will often use the notation $f_n : M_n \to M_{n-1}$ or $M_n \xrightarrow{f_n} M_{n-1}$ to refer to $M(a) : M(v_n) \to M(v_{n-1})$, $n \geq 1$. A morphism ψ between two representations will be a family $\{\psi_k : k \geq 1\}$ satisfying the conditions above.

Since we will prove the existence of flat covers and cotorsion envelopes making use of the techniques developed by Eklof and Trlifaj (see [4]) over cotorsion theories, we introduce some general definitions on covers and envelopes and recall what is understood by a pair of classes cogenerated by a set.

Recall from [5] that, given a class $\mathcal{F}$ of objects in an abelian category $\mathcal{A}$, an $\mathcal{F}$-precover (resp., an $\mathcal{F}$-pre-envelope) of an object $C \in \mathrm{Ob}(\mathcal{A})$ is a morphism $F \xrightarrow{\varphi} C$ ($C \xrightarrow{\varphi} F$) with $F \in \mathcal{F}$ such that $\mathrm{Hom}(F', F) \to \mathrm{Hom}(F', C) \to 0$ (resp., $\mathrm{Hom}(C, F') \to \mathrm{Hom}(F, F') \to 0$) is exact for every $F' \in \mathcal{F}$. If, moreover, every $f : F \to F$ such that $\varphi \circ f = \varphi$ (resp., $f \circ \varphi = \varphi$) is an automorphism, then φ is said to be an $\mathcal{F}$-cover (resp., an $\mathcal{F}$-envelope). For the same class $\mathcal{F}$, $\mathcal{F}^\perp$ will denote the class of all objects C of $\mathcal{A}$ such that $\mathrm{Ext}^1(F, C) = 0$ for every $F \in \mathcal{F}$. Then, the pair of classes $(\mathcal{F}, \mathcal{F}^\perp)$ is said to be cogenerated by a set if there exists a set of objects of $\mathcal{A}$, say Z, such that $C \in \mathcal{F}^\perp$ if and only if $\mathrm{Ext}^1(F, C) = 0$ for every $F \in Z$.

We also have to recall the definition of a finitely generated representation of a quiver.

Definition 2.1. Let Q be a quiver, D a representation of Q, and Z a set of elements of D. The subrepresentation of D generated by the set Z is defined as the intersection of all representations of Q containing Z. The representation D is said to be finitely generated provided that D is generated by a finite subset of elements, or equivalently, if it is finitely generated as an object of the category of representations of Q.

It follows immediately from Definition 2.1 that a representation S of the quiver A_∞ is finitely generated if and only if it is of the form

$$S \equiv \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow S_n \xrightarrow{f_n} S_{n-1} \longrightarrow \cdots \longrightarrow S_1 \xrightarrow{f_1} S_0 \tag{2.2}$$

for some natural number $n \geq 1$ and with S_i finitely generated as an R-module for all $i \geq 1$.

3. Projective representations. Projective representations of A_∞ by vector spaces over a field K were characterized in [2, Example, page 102] as those representations P such that the homomorphisms $P_n \to P_{n-1}$ are always injections and that $\varprojlim P_n = 0$. The condition is indeed necessary; however, we will now give an example of a representation of A_∞ by vector spaces satisfying these two conditions, but which is not a projective representation. Notice first that it is immediate to see that any representation of A_∞ of the form

$$T \equiv \cdots 0 \longrightarrow U \xrightarrow{\text{id}} U \longrightarrow \cdots \longrightarrow U \xrightarrow{\text{id}} U, \tag{3.1}$$

where U is a projective R-module, is projective since $\operatorname{Hom}_{(A_\infty, R\text{-Mod})}(T, M) \cong \operatorname{Hom}_R(U, M(v_n))$ (where n is the position where U appears for the first time in T) for every representation M of A_∞.

Example 3.1. Let K be a field and L the following representation of A_∞:

$$L \equiv \cdots \subseteq \prod_{i=2}^{\infty} K \subseteq \prod_{i=1}^{\infty} K \subseteq \prod_{i=0}^{\infty} K. \tag{3.2}$$

For every $n \in \mathbb{N}$, we consider the representation $P(n)$ of A_∞ given by

$$P(n) \equiv \cdots \longrightarrow 0 \longrightarrow K \xrightarrow{\text{id}} \cdots \xrightarrow{\text{id}} K, \tag{3.3}$$

where the first K appears in nth place. The direct sum $\oplus_{n=0}^{\infty} P(n)$ is a projective representation of A_∞ since each $P(n)$ is a projective representation. Furthermore, it is easy to see that $\oplus_{n=0}^{\infty} P(n)$ is a projective generator for the category of representations by K-modules of the quiver A_∞, so if L was a projective representation of A_∞, then L should be a direct summand of $(\oplus_{n=0}^{\infty} P(n))^{(X)}$ for some set X. We prove that L cannot be contained in $(\oplus_{n=0}^{\infty} P(n))^{(X)}$, so we will have a contradiction.

It is immediate to observe that the kernel of any morphism $L \to P(n)$ contains the subrepresentation

$$T \equiv \cdots \subseteq \prod_{n+3}^{\infty} K \subseteq \prod_{n+2}^{\infty} K \subseteq \prod_{n+1}^{\infty} K \subseteq \cdots \subseteq \prod_{n+1}^{\infty} K, \tag{3.4}$$

where the K-module $\prod_{n+2}^{\infty} K$ is the corresponding module to the vertex v_{n+2}. Then, any morphism $L \to P(n)$ factors through the quotient L/T (which is clearly a finitely generated representation by the comments made in Section 2), and then we have

$$\operatorname{Hom}(L, P(n)^{(X)}) \cong \operatorname{Hom}\left(\frac{L}{T}, P(n)^{(X)}\right). \tag{3.5}$$

Suppose we have

$$L \subseteq \left(\oplus_{n=0}^{\infty} P(n)\right)^{(X)} \cong \oplus_{n=0}^{\infty} P(n)^{(X)} \tag{3.6}$$

for some set X. Then, for any natural number n, we have a morphism

$$L \hookrightarrow \oplus_{n=0}^{\infty} P(n)^{(X)} \longrightarrow P(n)^{(X)}, \tag{3.7}$$

and by (3.5), we see that there exists a finite subset $X_n \subseteq X$ such that (3.7) factors through

$$L \longrightarrow P(n)^{(X_n)} \hookrightarrow P(n)^{(X)}. \tag{3.8}$$

Let $X' = \cup_{n=0}^{\infty} X_n$. Then, we have that in fact $L \subseteq \oplus_{n=0}^{\infty} P(n)^{(X')}$, and this is impossible in general since $(\oplus_{n=0}^{\infty} P(n)^{(X')})(v_0)$ has a countable base and $L(v_0)$ does not in general (take, e.g., $\mathbb{Q}$).

In Propositions 3.2 and 3.3, we give, respectively, necessary and sufficient conditions for a representation of A_∞ to be projective in the general case where R is an arbitrary ring. These will lead to a characterization of finitely generated projective representations of A_∞ (Proposition 3.4).

Proposition 3.2. *Let $P \equiv \cdots \to P_n \xrightarrow{f_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{f_1} P_0$ be a projective representation of A_∞. Then the following statements hold:*

(1) *P_n is a projective R-module for every $n \in \mathbb{N}$;*

(2) *f_n is a splitting monomorphism for every $n \in \mathbb{N}$;*

(3) *$\varprojlim P_n = 0$.*

Proof. The first statement is easy: take $n \in \mathbb{N}$ and let $M \xrightarrow{h} N$ be an epimorphism of R-modules. Now, if $P_n \xrightarrow{g_n} N$ is a morphism of R-modules, we may extend h and g_n to morphisms of representations by putting M and id_M (resp., N and id_N) for $k \geq n$ and 0 on the rest. We finally apply that P is a projective representation to get the desired extension $P_n \to M$ of g_n.

Consider now the family of representations

$$K^n \equiv \cdots 0 \longrightarrow \cdots \longrightarrow P_n \xrightarrow{\mathrm{id}} P_n \longrightarrow \cdots \longrightarrow P_n \tag{3.9}$$

of A_∞ (first P_n is in nth position) for all $n \in \mathbb{N}$. Then, the direct sum $\oplus_{n\geq 0}K^n$ can be considered as the representation

$$\cdots \oplus_{k\geq n} P_k \xrightarrow{\lambda_n} \oplus_{k\geq n-1} P_k \longrightarrow \cdots \longrightarrow \oplus_{k\geq 1} P_k \xrightarrow{\lambda_1} \oplus_{k\geq 0} P_k, \tag{3.10}$$

where each λ_j is the canonical injection.

It is clear that the map $\varphi : \oplus_{n\geq 0}K^n \to P$ given by

$$\varphi_n\big((x_j)_{j\geq n}\big) = x_n + \sum_{k\geq 1} f_{n+1} \circ \cdots \circ f_{n+k}(x_{n+k}) \tag{3.11}$$

is a morphism of representations and that it is in fact an epimorphism of representations. But P is projective by hypothesis, so there exists $\phi : P \to \oplus_{k\geq 0}K^n$ with $\varphi \circ \phi = \mathrm{id}_P$, which means that $\varphi_n \circ \phi_n = \mathrm{id}_{P_n}$, for all $n \in \mathbb{N}$. If we now look at the canonical projections $\pi_n : \oplus_{k\geq n-1}P_k \to \oplus_{k\geq n}P_k$, we see that $\mathrm{id}_{P_n} = \varphi_n \circ \phi_n = \varphi_n \circ \pi_n \circ \lambda_n \circ \phi_n = \varphi_n \circ \pi_n \circ \phi_{n-1} \circ f_n$, where the last equality holds since ϕ is a morphism of representations (so, $\lambda_n \circ \phi_n = \phi_{n-1} \circ f_n$ for all $n \geq 1$). Therefore, we immediately obtain that each f_n is a splitting monomorphism.

It only remains to prove (3). We have already seen that P is a direct summand of $\oplus_{n\geq 0}K^n$, that is, $P \oplus T = \oplus_{n\geq 0}K^n$ for some representation T of A_∞, and it is clear that $\oplus_{n\geq 0}K^n$ satisfies (3), so we are done since inverse limits commute with finite direct sums. □

Proposition 3.3. *Let $P \equiv \cdots \to P_n \xrightarrow{f_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{f_1} P_0$ be a representation of A_∞ and suppose that*

(1) *P_n is a projective R-module for every n;*
(2) *f_n is a splitting monomorphism for every n;*
(3) *there exists a set of epimorphisms $\{\alpha_n : P_{n-1} \to P_n \mid n \geq 1\}$ such that $\alpha_n \circ f_n = \mathrm{id}_{P_n}$ for every $n \in \mathbb{N}$ and that for any $x \in P_n$ there is a natural integer $k \geq 1$ with $\alpha_{n+k} \circ \cdots \circ \alpha_{n+1}(x) = 0$.*

Then P is a projective representation.

Proof. Let K^n and $\varphi : \oplus_{n\geq 0}K^n \to P$ be the representations of A_∞ and the epimorphism of representations given in Proposition 3.2. By the previous comments, each K^n is projective and so is $\oplus_{n\geq 0}K^n$. Therefore, if we prove that P is a direct summand of $\oplus_{n\geq 0}K^n$, we will have that P is also projective.

We want to define a morphism $w : P \to \oplus_{n\geq 0}K^n$ so that $\varphi \circ w = \mathrm{id}_P$. For any $x_n \in P_n$, we define

$$\begin{aligned} w_n(x_n) = \Big(&\ldots, \alpha_{n+j}\alpha_{n+j-1}\cdots\alpha_{n+1}(x_n) \\ &- f_{n+j+1}\alpha_{n+j+1}\alpha_{n+j}\alpha_{n+j-1}\cdots\alpha_{n+1}(x_n), \ldots, \alpha_{n+1}(x_n) \\ &- f_{n+2}\alpha_{n+2}\alpha_{n+1}(x_n), x_n - f_{n+1}\alpha_{n+1}(x_n)\Big) \end{aligned} \tag{3.12}$$

for each $j \geq 0$. Notice that w_n is well defined by condition (3).

With this definition, it is an easy computation to check that $\lambda_n \circ w_n = w_{n-1} \circ f_n$ and that $\varphi_n \circ w_n(x_n) = x_n$ for all $x_n \in P_n$, for all $n \in \mathbb{N}$, so we are done. □

Using Propositions 3.2 and 3.3 and the comments given in Section 2, we immediately obtain the following result.

Proposition 3.4. *Let P be a representation of A_∞. Then P is finitely generated projective if and only if there exists an $n \in \mathbb{N}$ such that*

(1) $P_k = 0$ *for every* $k > n$;
(2) P_k *is a finitely generated projective module for every* k *with* $0 \le k \le n$;
(3) $f_k : P_k \to P_{k-1}$ *is a split monomorphism for every* $k \in \mathbb{N}$.

We finish this section with an example showing that, in general, not every representation of A_∞ has a projective cover. This will raise the interest in the problem of the existence of flat covers for all representations of A_∞ (which will be treated in Section 5) as well as the characterization of rings for which every representation by modules of A_∞ has a projective cover.

Example 3.5. Let R be a ring and consider the representation

$$T \equiv \cdots \longrightarrow R \xrightarrow{\text{id}} R \xrightarrow{\text{id}} R \tag{3.13}$$

of A_∞, and for $n \in \mathbb{N}$, the subrepresentation

$$T^n \equiv \cdots \longrightarrow 0 \longrightarrow R \xrightarrow{\text{id}} \cdots \xrightarrow{\text{id}} R \tag{3.14}$$

with first R in nth position. It is clear that, for any $n_0 \in \mathbb{N}$, there exists an epimorphism $\oplus_{n \ge n_0} T^n \to T$ which of course is a projective precover of T (each T^n is a projective representation of A_∞). Suppose T has a projective cover S ($S \equiv \cdots \to S_{n+1} \xrightarrow{g_{n+1}} S_n \cdots \to S_1 \xrightarrow{g_1} S_0$). Then S is a direct summand of $\oplus_{n \ge n_0} T^n$, and so g_n is an isomorphism for all n, $0 \le n \le n_0$, and all $n_0 \in \mathbb{N}$, which contradicts the hypotheses of Proposition 3.2. Therefore, T does not have a projective cover.

4. Flat representations. Since the category of representations by modules of a quiver has enough projectives, we can define flat representations of A_∞ as direct limits of projective representations. Furthermore, it is easy to see that we can assume that a flat representation is a direct limit of finitely generated and projective representations which have been characterized in Proposition 3.4. This section is therefore devoted to characterizing flat representations of A_∞. This turns out to be very useful in proving that the pair $(\mathcal{F}, \mathcal{F}^\perp)$ (where $\mathcal{F}$ is the class of all flat representations of A_∞) is cogenerated by a set, as was noticed in [6], where the same result was proved for rooted quivers.

In [6, Proposition 3.4] it is stated that, for any quiver Q, if a representation F is flat, then $F(v)$ is a flat module for every vertex v of Q and that the homomorphism $\oplus_{t(a)=v}F(i(a)) \to F(v)$ is a pure monomorphism for all vertices v of Q, where $t(a)$ and $i(a)$ denote the terminal and initial vertices of the arrow a. Furthermore, it was also proved in [6, Theorem 3.7] that these conditions are sufficient for a representation to be flat provided that Q is rooted. Now we prove that this characterization also holds for a nonrooted quiver A_∞.

Proposition 4.1. *A representation F of the quiver A_∞ is flat if and only if the following statements hold:*

(1) *F_v is a flat module for every vertex v of A_∞;*

(2) *the homomorphism $f_{j+1}: F_{j+1} \to F_j$ is a pure injection for every $j \in \mathbb{N}$.*

Proof. As we have seen above, the conditions are necessary. We prove that they are also sufficient. For every $n \in \mathbb{N}$, we define the subrepresentation F^n of F given by

$$F^n \equiv \cdots 0 \longrightarrow 0 \longrightarrow F_n \xrightarrow{f_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0. \tag{4.1}$$

It is clear that $F = \varinjlim_{n\in\mathbb{N}} F^n$, so if F^n is a flat representation for any $n \in \mathbb{N}$, then F is also a flat representation of A_∞. But it is easy to see that F^n is a flat representation of A_∞ if and only if

$$F_n \xrightarrow{f_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \tag{4.2}$$

is a flat representation of the rooted quiver $v_n \to v_{n-1} \to \cdots \to v_1 \to v_0$. By hypothesis, F_j is a flat module for every $j \le n$ and the homomorphisms f_j are pure injections for every $j \in \{1,\dots,n\}$, so [6, Theorem 3.7] gives us that (4.2) is a flat representation for all $n \in \mathbb{N}$, and we are done. □

Throughout the rest of this paper, the class of all flat representations of A_∞ will be denoted by the symbol $\mathcal{F}$.

As an immediate consequence of the previous results, we have the following proposition.

Proposition 4.2. *Let F be a flat representation of A_∞ and G a subrepresentation of F in such a way that F/G is flat. Then G is also a flat representation of A_∞.*

5. Flat covers and cotorsion envelopes. We will now prove the existence of a flat cover and a cotorsion envelope for every representation of A_∞. This will be deduced from the fact that the pair of classes of flat representations and cotorsion representations of A_∞, $(\mathcal{F}, \mathcal{F}^\perp)$ is cogenerated by a set.

Definition 5.1. The cardinality $|X|$ of an arbitrary representation X of A_∞ is defined as

$$|X| = \left|\coprod_{v\in V} X(v)\right|, \tag{5.1}$$

where V denotes the set of all vertices of A_∞.

Theorem 5.2. *The pair of classes* $(\mathcal{F},\mathcal{F}^\perp)$ *in the category of representations of* A_∞ *is cogenerated by a set.*

Proof. Let

$$F \equiv \cdots \longrightarrow F_{n+1} \xrightarrow{f_{n+1}} F_n \xrightarrow{f_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{f_1} F_0 \tag{5.2}$$

be a flat representation of A_∞. Let x be any element of F (suppose that $x \in F_n$) and suppose $|R| = \aleph$. We know by [3, Lemma 1] that there exists a pure submodule $S_n^{(1)}$ of F_n with $x \in S_n^{(1)}$ such that $|S_n^{(1)}| \le \aleph$. We define $S_m^{(1)} = f_{m+1}(S_{m+1}^{(1)})$ for all $m \le n-1$ and $S_k^{(1)}$ as the inverse image of $S_{k-1}^{(1)}$ by means of f_k for all $k \ge n+1$. It is then clear that $|S_l^{(1)}| \le \aleph$ for all $l \in \mathbb{N}$ since f_l is a monomorphism for all $l \in \mathbb{N}$.

We consider the subrepresentation

$$S^{(1)} \equiv \cdots \longrightarrow S_{n+1}^{(1)} \xrightarrow{f_{n+1}^{(1)}} S_n^{(1)} \xrightarrow{f_n^{(1)}} S_{n-1}^{(1)} \longrightarrow \cdots \longrightarrow S_1^{(1)} \xrightarrow{f_1^{(1)}} S_0^{(1)} \tag{5.3}$$

of F, where the homomorphism $f_k^{(1)}$ is the restriction of f_k for every $k \in \mathbb{N}$. We observe that the quotient $F/S^{(1)}$ is such that the homomorphism

$$\overline{f_{n+1}^{(1)}} : \frac{F_{n+1}}{S_{n+1}^{(1)}} \longrightarrow \frac{F_n}{S_n^{(1)}} \tag{5.4}$$

is indeed a monomorphism.

Now $|S_n^{(1)}| \le \aleph$, so we also have that

$$\left|\frac{S_n^{(1)}}{\mathrm{Im}\,(f_{n+1}) \cap S_n^{(1)}}\right| \le \aleph. \tag{5.5}$$

Then, again using [3, Lemma 1], we find a pure submodule $T/\mathrm{Im}(f_{n+1})$ of $F_n/\mathrm{Im}(f_{n+1})$ such that

$$\frac{\mathrm{Im}\,(f_{n+1}) + S_n^{(1)}}{\mathrm{Im}\,(f_{n+1})} \subseteq \frac{T}{\mathrm{Im}\,(f_{n+1})} \tag{5.6}$$

and $|T/\operatorname{Im}(f_{n+1})| \le \aleph$. Now we may choose $S_n^{(2)}$ such that $S_n^{(1)} \subseteq S_n^{(2)} \subseteq F_n$ with

$$\frac{\operatorname{Im}(f_{n+1}) + S_n^{(2)}}{\operatorname{Im}(f_{n+1})} = \frac{T}{\operatorname{Im}(f_{n+1})} \tag{5.7}$$

and also $|S_n^{(2)}| \le \aleph$.

Therefore, we have that $F_n/(\operatorname{Im}(f_{n+1}) + S_n^{(2)})$ is a flat module, and then, $(\operatorname{Im}(f_{n+1}) + S_n^{(2)})/S_n^{(2)}$ is a pure submodule of $F_n/S_n^{(2)}$.

Let $S_m^{(2)} = f_{m+1}(S_{m+1}^{(2)})$ for all $m \le n-1$ and $S_k^{(2)}$ the inverse image of $S_{k-1}^{(2)}$ for every $k \ge n+1$. Again we have that $|S_l^{(2)}| \le \aleph$ for all $l \in \mathbb{N}$. Consider the subrepresentation

$$S^{(2)} \equiv \cdots \longrightarrow S_{n+1}^{(2)} \xrightarrow{f_{n+1}^{(2)}} S_n^{(2)} \xrightarrow{f_n^{(2)}} S_{n-1}^{(2)} \longrightarrow \cdots \longrightarrow S_1^{(2)} \xrightarrow{f_1^{(2)}} S_0^{(2)} \tag{5.8}$$

of F (which contains $S^{(1)}$ as a subrepresentation), where $f_k^{(2)}$ is the restriction of f_k for every $k \in \mathbb{N}$. Then

$$\overline{f_{n+1}^{(2)}} : \frac{F_{n+1}}{S_{n+1}^{(2)}} \longrightarrow \frac{F_n}{S_n^{(2)}} \tag{5.9}$$

is a pure homomorphism.

We now embed $S_{n+1}^{(2)}$ into a pure submodule $S_{n+1}^{(3)}$ of F_{n+1} with $|S_{n+1}^{(3)}| \le \aleph$. We define $G_n = f_{n+1}(S_{n+1}^{(3)})$ and, for every $k < n$, we denote the module $f_{k+1}(S_{k+1}^{(3)} + G_{k+1})$ by G_k and the module $S_k^{(2)} + G_k$ by $S_k^{(3)}$. For every $k > n+1$, let $S_k^{(3)}$ be the inverse image of $S_{k-1}^{(3)}$. Consider the representation

$$S^{(3)} \equiv \cdots \longrightarrow S_{n+2}^{(3)} \xrightarrow{f_{n+2}^{(3)}} S_{n+1}^{(3)} \xrightarrow{f_{n+1}^{(3)}} S_n^{(3)} \longrightarrow \cdots \longrightarrow S_1^{(3)} \xrightarrow{f_1^{(3)}} S_0^{(3)}. \tag{5.10}$$

Then

$$\overline{f_{n+2}^{(3)}} : \frac{F_{n+2}}{S_{n+2}^{(3)}} \longrightarrow \frac{F_{n+1}}{S_{n+1}^{(3)}} \tag{5.11}$$

is an injection by the same argument as in the case $S^{(1)}$, $S_{n+1}^{(3)}$ is a pure submodule of F_{n+1}, and, by the same argument we used for constructing $S^{(2)}$ from $S^{(1)}$, we get a representation $S^{(4)}$ such that $|S^{(4)}| \le \aleph$ and

$$\overline{f_{n+2}^{(4)}} : \frac{F_{n+2}}{S_{n+2}^{(4)}} \longrightarrow \frac{F_{n+1}}{S_{n+1}^{(4)}} \tag{5.12}$$

is a pure homomorphism.

We now turn over and embed $S_n^{(4)}$ into a pure submodule $S_n^{(5)}$ of F_n, with $|S_n^{(5)}| \le \aleph$, and construct representations $S^{(5)}$ and $S^{(6)}$ as before, with $|S^{(5)}| \le \aleph$, $|S^{(6)}| \le \aleph$,

$$\overline{f_{n+1}^{(5)}} : \frac{F_{n+1}}{S_{n+1}^{(5)}} \longrightarrow \frac{F_n}{S_n^{(5)}} \tag{5.13}$$

a monomorphism, and

$$\overline{f_{n+1}^{(6)}} : \frac{F_{n+1}}{S_{n+1}^{(6)}} \longrightarrow \frac{F_n}{S_n^{(6)}} \tag{5.14}$$

a pure homomorphism.

Then embed $S_{n-1}^{(6)}$ into a pure submodule $S_{n-1}^{(7)}$ of F_{n-1} and find the representations $S^{(7)}$ and $S^{(8)}$ as before.

Turn over again and embed $S_n^{(8)}$ into a pure submodule $S_n^{(9)}$ of F_n with $|S_n^{(9)}| \leq \aleph$ and find the corresponding representations $S^{(9)}$ and $S^{(10)}$ as above. Then embed $S_{n+1}^{(10)}$ into a pure $S_{n+1}^{(11)}$ of F_{n+1} with $|S_{n+1}^{(11)}| \leq \aleph$ and construct $S^{(11)}$ and $S^{(12)}$. Repeat this argument of $S_{n+2}^{(12)}$, finding $S^{(13)}$ and $S^{(14)}$. Then, turn over again and continue this zigzag procedure.

We have then found a chain of subrepresentations $\{S^{(n)} \mid n \in \mathbb{N}\}$ of F. So if $S(1)$ is the direct limit $S(1) = \varinjlim S^{(n)}$ (which is a well-ordered direct union), we have that $|S(1)| \leq \aleph$, $S(1)_n$ is pure in F_n for all $n \in \mathbb{N}$, and

$$\overline{f_n} : \frac{F_n}{S(1)_n} \longrightarrow \frac{F_{n-1}}{S(1)_{n-1}} \tag{5.15}$$

is a pure injection for every $n-1 \in \mathbb{N}$ because the system of representations satisfying these properties is cofinal for every $n \in \mathbb{N}$. Then $F/S(1)$ is a flat representation by Proposition 4.1.

Since $F/S(1)$ is flat, we can choose any element $y \in F/S(1)$ and repeat the previous argument, obtaining a subrepresentation $S(2)/S(1)$ of $F/S(1)$ such that $|S(2)/S(1)| \leq \aleph$, $y \in S(2)_k/S(1)_k$ (if we suppose that $y \in F_k/S(2)_k$), and $F/S(2)$ is a flat representation.

Proceeding by (transfinite) induction, we can find for every successor ordinal number α a subrepresentation $S(\alpha)$ of F such that $F/S(\alpha)$ is flat and $|S(\alpha)| \leq \aleph$, while if β is a limit ordinal, we define $S(\beta) = \varinjlim_{\alpha<\beta} S(\alpha)$ (note that if β is a limit ordinal, then $F/S(\beta) = F/(\varinjlim S(\alpha)) \cong \varinjlim F/S(\alpha)$, but $F/S(\alpha)$ is flat for every $\alpha < \beta$ by construction, so $F/S(\beta)$ is also flat). It is now immediate that there exists an ordinal number μ such that F is the direct union of the chain of subrepresentations $\{S(\alpha) \mid \alpha < \mu\}$ (which is a continuous chain by construction). But $F/S(1)$ is flat, so by Proposition 4.2, $S(1)$ is a flat representation, and by construction, for any ordinal $\alpha + 1 < \mu$, the representations $F/S(\alpha)$ and $(F/S(\alpha))/(S(\alpha+1)/S(\alpha))$ are flat, so $S(\alpha+1)/S(\alpha)$ is also flat. Then, by [4, Lemma 1], we get that if Z is a set of representatives of flat representations S such that $|S| \leq \aleph$, then a representation C is cotorsion if and only if $C \in Z^{\perp}$, that is, the cotorsion theory $(\mathcal{F}, \mathcal{F}^{\perp})$ is cogenerated by the set Z. □

Theorem 5.3. *Every representation of A_∞ has a flat cover and a cotorsion envelope.*

Proof. The class $\mathcal{F}$ is closed under extensions and arbitrary direct limits, so every representation of A_∞ has a cotorsion envelope by [1, Corollary 2.11]. Furthermore, $\mathcal{F}$ contains all projective representations of A_∞, so by [1, Corollary 2.12], every representation of A_∞ has a flat cover. □

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