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CHARACTERIZATION OF THE AUTOMORPHISMS HAVING THE LIFTING PROPERTY IN THE CATEGORY OF ABELIAN p-GROUPS

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Let p be a prime. It is shown that an automorphism α of an abelian p-group A lifts to any abelian p-group of which A is a homomorphic image if and only if $\alpha = \pi\,\mathrm{id}_A$, with π an invertible p-adic integer. It is also shown that if A is a torsion group or torsion-free p-divisible group, then id_A and $-\mathrm{id}_A$ are the only automorphisms of A which possess the lifting property in the category of abelian groups.

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1. Introduction. Every inner automorphism of a group G has the property that it extends to an automorphism of any group containing G as subgroup. Schupp [4] showed that this extension property characterizes inner automorphisms in the category of groups. Pettet [3] gave an easier proof of Schupp's result and proved at the same time that the inner automorphisms of a group G are also characterized by the lifting property in the category of groups. In [1], we characterized the automorphisms of abelian p-groups having the extension property in the category of abelian p-groups, as well as those having the extension property in the category of all abelian groups.

Let $\mathscr{C}$ be a full subcategory of the category of abelian groups. An automorphism α of $A \in \mathscr{C}$ has the lifting property in $\mathscr{C}$ if, for all $B \in \mathscr{C}$ and any epimorphism $s : B \to A$, there exists $\tilde{\alpha} \in \mathrm{Aut}(B)$ such that $s \circ \tilde{\alpha} = \alpha \circ s$, in other words, the diagram

$$\begin{array}{ccc} B & \xrightarrow{\;s\;} & A \\ {\scriptstyle\tilde{\alpha}}\big\downarrow & & \big\downarrow{\scriptstyle\alpha} \\ B & \xrightarrow{\;s\;} & A \end{array} \tag{1.1}$$

commutes. In this note, we show that an automorphism α of a p-group A (with p being a prime number) has the lifting property in the category of abelian p-groups if and only if $\alpha = \pi\,\mathrm{id}_A$, with π an invertible p-adic number. We also determine the automorphisms of an abelian group A having the lifting property in the category of all abelian groups, when A is either torsion or p-divisible torsion-free. In both cases they are id_A and $-\mathrm{id}_A$.

We will use the notation introduced in [2].

2. The lifting property in the category of the p-groups. Let p be a prime number.

Lemma 2.1. *Let α be an automorphism of a p-group A having the lifting property in the category of abelian p-groups. If C is subgroup of A with $\alpha(C) = C$, then the restriction of α to C also has the lifting property in the category of abelian p-groups.*

Proof. Let $\mu : B \to C \to 0$ be an exact sequence. It follows from [2, page 108] that we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Ker}\mu & \xrightarrow{i} & B & \xrightarrow{\mu} & C & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle j} & & \\ 0 & \longrightarrow & \operatorname{Ker}\mu & \xrightarrow{\lambda} & F & \xrightarrow{\gamma} & A & \longrightarrow & 0, \end{array} \tag{2.1}$$

where i and j are the canonical injections. It is easy to show that F is again a p-group, then there exists $\tilde{\alpha} \in \operatorname{Aut}(F)$ such that $\gamma\tilde{\alpha} = \alpha\gamma$. If we put, for any $b \in B$, $\tilde{\alpha}(\sigma(b)) = \sigma(\gamma(b))$, then $\gamma \in \operatorname{Aut}(B)$ and $\mu\gamma = \alpha_0\mu$, with α_0 the restriction of α to C. □

Lemma 2.2. *Let A be a torsion group and $n \in \mathbb{N}^*$. Then there exists an abelian group B and an epimorphism $\mu : B \to A$ such that $B[n] \subseteq \operatorname{Ker}\mu$, where $B[n] = \{b \in B \mid nb = 0\}$.*

Proof. For $a \in A$, we put $B_a = \langle x_a \rangle$, where $o(x_a) = o(a)$ and $\mu_a : B_a \to A$ is defined by $\mu_a(x_a) = a$. If we put $B = \bigoplus_{a \in A} B_a$ and $\mu : B \to A$, where $\mu(x_a) = \mu_a(x_a)$, for all $a \in A$, then μ is an epimorphism and $B[n] \subseteq \operatorname{Ker}\mu$. □

Theorem 2.3. *Let A be an abelian p-group and an automorphism α of A has the lifting property in the category of abelian p-groups if and only if $\alpha = \pi \operatorname{id}_A$, where π is an invertible p-adic number.*

Proof. One implication is clear. Assume that α has the lifting property in the category of abelian p-groups. The proof of the fact that $\alpha = \pi \operatorname{id}_A$ goes in three steps.

Step 1. We suppose that A is reduced. Let $x \in A$ be such that $\langle x \rangle$ is a direct summand of A. We prove that $\alpha(x) \in \langle x \rangle$.

Put $\langle x \rangle \bigoplus A' = A$ and let $E(A')$ be the injective envelope of A'. We put

$$A'' = \{y \in E(A') \mid p^n y \in A'\}, \tag{2.2}$$

where $o(x) = p^n$. We consider the group $B = \langle x \rangle \bigoplus A''$; the map $s : B \to A$ defined by

$$s(mx + y) = mx + p^n y, \tag{2.3}$$

for all $m \in \mathbb{Z}$ and $y \in A''$, is an epimorphism. Therefore, there exists $\tilde{\alpha} \in \operatorname{Aut}(B)$ such that $s\tilde{\alpha} = \alpha s$. We can write $\tilde{\alpha}(x) = kx + a''$, with $k \in \mathbb{Z}$ and $a'' \in A''$. Now

$$s\tilde{\alpha}(x) = kx + p^n a'' = kx = \alpha s(x) = \alpha(x) \tag{2.4}$$

because $p^n a'' = 0$, thus $\alpha(x) \in \langle x \rangle$. Let B be a basic subgroup of A, $B = \bigoplus_{n\geq 1} B_n$, and, for any $n \geq 1$, $B_n = 0$ or B_n is a direct sum of torsion cyclic groups of order p^n. We suppose $B_n \neq 0$ for $n \geq 1$, so $B_n = \bigoplus_{i\in I}\langle x_i \rangle$ such that $o(x_i) = p^n$, for all $i \in I$, since B_n is a direct summand of A (see [2, page 138]). With $m_i \in \mathbb{Z}$, $\alpha(x_i) = m_i x_i$. Let $(i,j) \in I^2$ with $i \neq j$. We can write $A = \langle x_i \rangle \bigoplus A_i$ with $x_j \in A_i$. It is easy to see that $\langle x_i + x_j \rangle \bigoplus A_i = A$, so $\alpha(x_i + x_j) = m(x_i + x_j)$, hence $p^n \mid (m_i - m_j)$. Then there is $k_n \in \mathbb{Z}$ such that $\alpha(b) = k_n b$, for all $b \in B_n$. For $(m,n) \in \mathbb{N}^2$ where $1 \leq m < n$, $B_m \bigoplus B_n$ is a direct summand of A [2, page 138] and it is easy to see that $p^m \mid (k_n - k_m)$.

Let π be the p-adic number defined by $(k_n)_{n\geq 0}$ (with $k_0 = 0$ and $k_n = k_{n-1}$ if $B_n = 0$). Then $\alpha(b) = \pi b$, for all $b \in B$. Since A is reduced, it follows that $\alpha = \pi\, \mathrm{id}_A$ (see [2, page 145]).

Step 2. We suppose that A is divisible. Therefore, $A = \bigoplus_{i\in I} A_i$ with $A_i \cong \mathbb{Z}(p^\infty)$, for all $i \in I$ (see [2, page 104]). We consider the direct product $E = \prod_{n\geq 1}\langle x_n \rangle$, where $o(x_n) = p^n$, for all $n \geq 1$. For all $n \geq 1$, let $e_n \in E$ be defined by

$$f_m(e_n) = \begin{cases} 0 & \text{if } m < n, \\ p^{m-n} x_m & \text{if } m \geq n, \end{cases} \tag{2.5}$$

where $f_m : E \to \langle x_m \rangle$ is the canonical projection. Let C be the following subgroup of E:

$$C = \left(\bigoplus_{n\geq 1}\langle x_n \rangle\right) + \langle \{e_n \mid n \geq 1\} \rangle. \tag{2.6}$$

It is easy to see that $C/(\bigoplus_{n\geq 1}\langle x_n \rangle) \cong \mathbb{Z}(p^\infty)$.

We choose $i \in I$ and $a_i \in A_i$. We want to show that $\alpha(a_i) \in A_i$. Let $j \in I$ with $j \neq i$. We put $A' = \bigoplus_{k\in I-\{j\}} A_k$ and we have $A = A_j \bigoplus A'$. Let $\gamma : C \to A_j$ be an epimorphism. If we suppose that $B = C \bigoplus A'$ and consider $s : B \to A$ which is defined by $s(c + a') = \gamma(c) + a'$ $(c \in C,\ a' \in A')$, then s is an epimorphism. Therefore, there exists $\tilde{\alpha} \in \operatorname{Aut}(B)$ such that $s\tilde{\alpha} = \alpha s$. Since A' is a maximal divisible subgroup of B, $\tilde{\alpha}(a') = a'$. Since $a_i \in A'$, then $\tilde{\alpha}(a_i) = \alpha(a_i) \in A'$. Thus for all $j \neq i$, $\alpha(a_i) \in \bigoplus_{k\neq j} A_k$, and therefore, $\alpha(a_i) \in A_i$. Then there is a p-adic number π_i such that $\alpha(a_i) = \pi_i a_i$, for all $a_i \in A_i$ (see [2, page 181]). For each $i \in I$, we put $A_i = \langle \{y_{i,n} \mid n \geq 1\} \rangle$ with $p y_{i,1} = 0$ and $p y_{i,n+1} = y_{i,n}$, for all $n \geq 1$. Let $(i,j) \in I^2$ with $i \neq j$. If we suppose that $z_n = y_{i,n} + y_{j,n}$ and $H = \langle \{z_n \mid n \geq 1\} \rangle$, then $H \cong \mathbb{Z}(p^\infty)$ and $A_i \bigoplus A_j = A_i \bigoplus H$. By the preceding arguments, there exists a p-adic number π such that $\alpha(h) = \pi h$, $\alpha h \in H$. Then we deduce that $\pi_i = \pi_j = \pi$.

Step 3. We suppose that A is an arbitrary abelian p-group. We can write $A = C \bigoplus D$ with C reduced and D divisible. We can also suppose that $C \neq 0$ and $D \neq 0$. We have $\alpha(D) = D$, and the restriction α_1 of α to D has the lifting property in the category of p-groups, by Lemma 2.1. Then there is a p-adic number π such that $\alpha(d) = \pi d$, for all $d \in D$.

Let $c_0 \in C$ with $o(c_0) = p^{n_0}$. we define the map $s : A \to A$ by

$$s(c+d) = c + p^{n_0} d, \tag{2.7}$$

for $(c,d) \in C \times D$. Then s is an epimorphism, and therefore, there exists $\tilde{\alpha} \in \operatorname{Aut}(A)$ such that $s\tilde{\alpha} = \alpha s$. Put $\tilde{\alpha}(c_0) = c_1 + d_1$. Then

$$s\tilde{\alpha}(c_0) = c_1 + p^{n_0} d_1 = c_1 = \alpha s(c_0) = \alpha(c_0), \tag{2.8}$$

and it follows that $\alpha(c_0) \in C$ and $\alpha(C) = C$. We show that $\alpha(c) = \pi c$, for all $c \in C$. To this end, take $\bigoplus_{i \in I} \langle c_i \rangle$ as a basic subgroup of C. We choose $i \in I$; $\langle c_i \rangle$ is a direct summand of C. Put $p^{n_i} = o(c_i)$ and $\bigoplus C_i = C$. Let $d_i \in D$ such that $o(d_i) = p^{n_i}$. We have

$$A = \langle c_i + d_i \rangle \bigoplus C_i \bigoplus D. \tag{2.9}$$

Then there exist a group G and an epimorphism $\eta : G \to C_i \bigoplus D$ such that $G[p^{n_i}] \subseteq \ker \eta$, by Lemma 2.2. We suppose that $B = \langle c_i + d_i \rangle \bigoplus G$, and we define $\mu : B \to G$ by $\mu(m(c_i + d_i) + g) = m(c_i + d_i) + \eta(g)$. Then μ is an epimorphism. Let $\tilde{\alpha} \in \operatorname{Aut}(B)$ be such that $\alpha\mu = \mu\tilde{\alpha}$. We have

$$\alpha\mu(c_i + d_i) = \alpha(c_i + d_i) = \alpha(c_i) + \pi d_i. \tag{2.10}$$

We put $\tilde{\alpha}(c_i + d_i) = k(c_i + d_i) + g_0$, then $\mu\tilde{\alpha}(c_i + d_i) = k(c_i + d_i)$ (because $\eta(g_0) = 0$). Thus $\alpha(c_i) + \pi d_i = k c_i + k d_i$, so $\alpha(c_i) = \pi c_i$, and therefore, $\alpha(c) = \pi c$, for all $c \in C$, by [2, page 145]. □

3. The lifting property in the category of abelian groups.

In this section, we show that, for a torsion or p-divisible torsion-free group A (p is a prime number), id_A and $-\mathrm{id}_A$ are the only automorphisms of A having the lifting property in the category of abelian groups.

Proposition 3.1. *Let A be an abelian torsion group. Then an automorphism α of A has the lifting property in the category of abelian groups if and only if $\alpha = \mathrm{id}_a$ or $\alpha = -\mathrm{id}_a$.*

Proof. One implication is obvious. Assume that α has the lifting property in the category of abelian groups and consider the exact sequence

$$E : 0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0, \tag{3.1}$$

then, by the Cartan-Eilenberg theorem (see [2, page 218]), the sequence

$$0 = \operatorname{Hom}(A, \mathbb{Q}) \longrightarrow \operatorname{Hom}(A, \mathbb{Q}/\mathbb{Z}) \xrightarrow{E_*} \operatorname{Ext}(A, \mathbb{Z}) \longrightarrow \operatorname{Ext}(A, \mathbb{Q}) = 0 \tag{3.2}$$

is exact, where E_* is the map associating to $\xi \in \operatorname{Hom}(A, \mathbb{Q}/\mathbb{Z})$ with the class extension $E\xi$.

Let $E_1 : 0 \to \mathbb{Z} \xrightarrow{\lambda} B \xrightarrow{\mu} A \to 0$ be an extension of $\mathbb{Z}$ by A. Then there exists $\sigma \in \operatorname{Aut}(\mathbb{Z})$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\lambda} & B & \xrightarrow{\mu} & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\tilde{\alpha}} & & \downarrow{\scriptstyle\alpha} & & \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\lambda} & B & \xrightarrow{\mu} & A & \longrightarrow & 0. \end{array} \tag{3.3}$$

If $\sigma = \mathrm{id}_{\mathbb{Z}}$, then $E_1 \equiv E_1\alpha$, and if $\sigma = -\mathrm{id}_{\mathbb{Z}}$, then $E_1 \equiv E_1(-\alpha)$. Therefore, for all $\xi \in \operatorname{Hom}(A, \mathbb{Q}/\mathbb{Z})$, $E_*(\xi\alpha - \xi) = 0$ or $E_*(\xi\alpha + \xi) = 0$. Thus $\xi(\alpha - \mathrm{id}) = 0$ or $\xi(\alpha + \mathrm{id}) = 0$, for all $\xi \in \operatorname{Hom}(A, \mathbb{Q}/\mathbb{Z})$.

From the fact that $\mathbb{Q}/\mathbb{Z}$ is divisible, it follows that $\alpha = \mathrm{id}$ or $\alpha = -\mathrm{id}$. □

Proposition 3.2. *Let p be a prime number and A a p-divisible torsion-free group. Then an automorphism α of A has the lifting property in the category of abelian groups if and only if $\alpha = \mathrm{id}_a$ or $\alpha = -\mathrm{id}_a$.*

Proof. One implication is obvious. Suppose that α has the required lifting property, and consider the pure exact sequence

$$E : 0 \longrightarrow \mathbb{Z} \longrightarrow J_p \longrightarrow J_p/\mathbb{Z} \longrightarrow 0, \tag{3.4}$$

where J_p is the additive group of p-adic integers. By the theorem of Harrisson (see [2, page 231]), the sequence

$$\operatorname{Hom}(A, J_p) \longrightarrow \operatorname{Hom}(A, J_p/\mathbb{Z}) \xrightarrow{E_*} \operatorname{Pext}(A, \mathbb{Z}) \longrightarrow \operatorname{Pext}(A, J_p) \tag{3.5}$$

is exact. $\operatorname{Hom}(A, j_p) = 0$ because J_p contains no nonzero p-divisible subgroup and $\operatorname{Pext}(A, j_p) = 0$ because J_p is algebraically compact. Thus E_* is an isomorphism, and, as in the proof of Proposition 3.1, we find that $\alpha = \mathrm{id}$ or $\alpha = -\mathrm{id}$. □

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