IJMMS 2004:12, 637–646
PII. S0161171204203076
http://ijmms.hindawi.com

ON THE NUMBER OF REPRESENTATIONS OF POSITIVE INTEGERS BY QUADRATIC FORMS AS THE BASIS OF THE SPACE $S_4(\Gamma_0(47),1)$

AHMET TEKCAN and OSMAN BİZİM

Received 11 March 2002

The number of representations of positive integers by quadratic forms $F_1 = x_1^2 + x_1x_2 + 12x_2^2$ and $G_1 = 3x_1^2 + x_1x_2 + 4x_2^2$ of discriminant -47 are given. Moreover, a basis for the space $S_4(\Gamma_0(47),1)$ are constructed, and the formulas for $r(n;F_4)$, $r(n;G_4)$, $r(n;F_3 \oplus G_1)$, $r(n;F_2 \oplus G_2)$, and $r(n;F_1 \oplus G_3)$ are derived.

2000 Mathematics Subject Classification: 11E25, 11E76.

1. Introduction. A real binary quadratic form F is a polynomial in two variables x_1 and x_2 of the shape $F = F(x_1,x_2) = ax_1^2 + bx_1x_2 + cx_2^2$ with real coefficients a, b, c. The discriminant of F is defined by the formula $b^2 - 4ac$ and is denoted by $\Delta(F)$, where F is an *integral form* if and only if $a,b,c \in \mathbb{Z}$, and is *positive definite* if and only if $\Delta(F) < 0$ and $a,c > 0$. If $\gcd(a,b,c) = 1$, then F is called *primitive*.

Let $F_1 = ax_1^2 + bx_1x_2 + cx_2^2$ and $G_1 = dx_1^2 + ex_1x_2 + fx_2^2$ be two positive definite quadratic forms with discriminant $\Delta(F_1)$ and $\Delta(G_1)$, respectively. For each $k \geqslant 1$, let F_k and G_k denote the direct sum of k-copies of F_1 and G_1, respectively, where F_1 and G_1 have two variables, F_2 and G_2 have four variables, and therefore F_k and G_k have $2k$ variables.

Let

$$Q = Q(x_1,x_2,\dots,x_k) = \sum_{1\le r\le s\le k} b_{rs}x_rx_s \tag{1.1}$$

be a positive definite quadratic form of discriminant Δ in k (k is even) variables with integral coefficients b_{rs}. Consider the quadratic form

$$2Q = \sum_{r,s=1}^{k} a_{rs}x_rx_s, \quad (a_{rr} = 2b_{rr},\ a_{rs} = a_{sr} = b_{rs},\ r < s) \tag{1.2}$$

of discriminant $\check{D}$. Then $\Delta = (-1)^{k/2}\check{D}$. Let A_{rs} be the algebraic cofactors of elements a_{rs} in $\check{D}$, let $\delta = \gcd(A_{rr}/2, A_{rs})$, $(r,s = 1,2,\dots,k)$, let $N = \check{D}/\delta$ be the level of the form Q, and let $\chi(d)$ be the character of the form Q, that is, $\chi(d) = 1$ if Δ is a perfect square, but if Δ is not a perfect square and $2 \nmid \Delta$, then $\chi(d) = (d/|\Delta|)$ for $d > 0$ and $\chi(d) = (-1)^{k/2}\chi(-d)$ for $d < 0$, where $(d/|\Delta|)$ is the generalized Jacobi symbol.

A positive definite quadratic from in k variables of level N and character $\chi(d)$ is called a quadratic form of type $(-k/2,N,\chi)$. Let $P_v = P_v(x_1,x_2,\dots,x_k)$ be the spherical function of order v with respect to the quadratic form Q. Furthermore, let q denote an odd prime number.

Let $\Gamma(1)$ denote a full modular group and let Γ denote any subgroup of a finite index in $\Gamma(1)$. In particular,

$$\begin{aligned} \Gamma_0(N) &= \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) : c \equiv 0 \bmod (N)\right\}, \\ \Gamma_1(N) &= \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) : a \equiv d \equiv 1 \bmod (N)\right\}, \\ \Gamma(N) &= \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N) : b \equiv 0 \bmod (N)\right\}, \end{aligned} \tag{1.3}$$

for $N \in \mathbb{N}$.

Let $G_k(\Gamma,\chi)$ and $S_k(\Gamma,\chi)$ denote the space of entire modular and cusp forms, respectively, of type (k,Γ,χ). If $F(\tau) \in G_k(\Gamma,\chi)$, then in the neighbourhood of the cusps $\zeta = i\infty$;

$$F(\tau) = \sum_{m=m_0\geqslant 0}^{\infty} a_m z^m, \quad a_{m_0} \neq 0. \tag{1.4}$$

The order of an entire modular form $F(\tau) \neq 0$ of type (k,Γ,χ) at the cusps $\zeta = i\infty$ with respect to Γ is

$$\operatorname{ord}(F(\tau), i\infty, \Gamma) = m_0. \tag{1.5}$$

Let

$$\begin{aligned} \wp(\tau; Q(X), P_v(X), h) &= \sum_{n_i \equiv h_i (\bmod N)} P_v(n_1,\dots,n_k) z_N^{(1/N)Q(n_1,\dots,n_k)}, \\ \wp(\tau; Q(X), P_v(X)) &= \sum_{n=1}^{\infty} \left(\sum_{Q(X)=n} P_v(X) \right) z^n, \end{aligned} \tag{1.6}$$

where $Q(X) = 1/2,\ \sum_{r,s=1}^{k} a_{rs} x_r x_s$ is a quadratic form of type $(k/2,N,\chi)$, $P_v(X)$ is a spherical function of order v with respect to the $Q(X)$, $n_1,\dots,n_k$ are integers, and $h = (h_1,\dots,h_k)$, where h_i are integers such that

$$\sum_{s=1}^{k} a_{rs} h_s \equiv 0 (\bmod N), \quad (r = 1,\dots,k). \tag{1.7}$$

As well known, to each positive definite quadratic form Q, there corresponds the theta series

$$\wp(\tau; Q) = 1 + \sum_{n=1}^{\infty} r(n; Q) z^n, \tag{1.8}$$

where $r(n;Q)$ is the number of representations of a positive integer n by the quadratic form Q.

Lemma 1.1. *Any positive definite quadratic form Q of type $(-k,q,1), k>2$, corresponds to one and the same Eisenstein series*

$$E(\tau;Q) = 1+\sum_{n=1}^{\infty}\left(\alpha\sigma_{k-1}(n)z^n+\beta\sigma_{k-1}(n)z^{qn}\right), \tag{1.9}$$

where

$$\alpha = \frac{i^k}{\rho_k}\frac{q^{k/2}-i^k}{q^k-1}, \quad \beta = \frac{1}{\rho_k}\frac{q^k-i^kq^{k/2}}{q^k-1}, \quad \rho_k = (-1)^{k/2}\frac{(k-1)!}{(2\pi)^k}\zeta(k), \tag{1.10}$$

$\zeta(k)$ is the zeta function of Riemann, and $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ [1].

Lemma 1.2. *If Q is a primitive quadratic form of type $(-k,q,1)$, $2|k$, then the difference $\wp(\tau;Q)-E(\tau;Q)$ is a cusp form of type $(-k,q,1)$* [1].

Lemma 1.3. *The homogeneous quadratic polynomials in k variables*

$$\varphi_{rs} = x_rx_s-\frac{1}{k}\frac{A_{rs}}{\check{D}}2Q, \quad (r,s=1,2,\dots,k) \tag{1.11}$$

are spherical functions of second order with respect to the positive definite quadratic form Q in k variables [1].

Lemma 1.4. *If Q is a quadratic form of type $(-k/2,N,\chi)$ and P_v is the spherical function of order v with respect to Q, then the generalized theta series*

$$\wp(\tau;Q,P_v) = \sum_{n=1}^{\infty}\left(\sum_{Q=n}P_v\right)z^n \tag{1.12}$$

is a cusp form of type $(-(k/2+v),N,\chi)$ [1].

Lemma 1.5. *If Q_1 and Q_2 are quadratic forms of types (k_1,N,χ_1) and (k_2,N,χ_2), respectively, then the quadratic form $Q_1\oplus Q_2$, direct sum of Q_1 and Q_2, is a quadratic form of type $(k_1+k_2,N,\chi_1\chi_2)$* [1].

Lemma 1.6. *If Q is a quadratic form of type (k,N,χ), then*

$$\wp(\tau;Q(x),P_v(x)) \in \begin{cases} G_{v+k/2}(\Gamma_0(N),\chi); & \text{if } v<0,\\ S_{v+k/2}(\Gamma_0(N),\chi); & \text{if } v>0,\end{cases} \tag{1.13}$$

see [1].

2. The number of representations of positive integers by quadratic forms. In this note, we consider the quadratic forms $F_1 = x_1^2+x_1x_2+12x_2^2$ and $G_1 = 3x_1^2+x_1x_2+4x_2^2$ of discriminant -47. Firstly, we investigate which positive integers can be represented by F_1, G_1, F_2, G_2, or $F_1\oplus G_1$, and then we construct a basis for the cusp space

$S_4(\Gamma_0(47),\chi)$. Moreover, we derive the formulas for $r(n;F_4)$, $r(n;G_4)$, $r(n;F_3\oplus G_1)$, $r(n;F_2\oplus G_2)$, and $r(n;F_1\oplus G_3)$.

For the quadratic form $F_1 = x_1^2+x_1x_2+12x_2^2$, $b_{11}=1$, $b_{12}=b_{21}=1/2$, and $b_{22}=12$. Therefore, $a_{11}=2$, $a_{12}=a_{21}=1/2$, and $a_{22}=24$. Thus, $A_{11}=24$ and $A_{22}=2$. Here, $\check{D}=47$ since $\Delta=(-1)\check{D}$. Also $\delta=1$ and $N=\check{D}/\delta$. Therefore, F_1 is a quadratic form of type $(-1,\Gamma_0(47),\chi)$. Similarly, for the quadratic form $G_1=3x_1^2+x_1x_2+4x_2^2$, $b_{11}=3$, $b_{12}=b_{21}=1/2$, and $b_{22}=4$. Therefore $a_{11}=6$, $a_{12}=a_{21}=1/2$, and $a_{22}=8$. Thus $A_{11}=8$ and $A_{22}=6$. For $\check{D}=47$, since $\Delta=(-1)\check{D}$, $\delta=1$, and $N=47$, therefore, G_1 is a quadratic form of type $(-1,\Gamma_0(47),\chi)$.

Let n be a positive integer. Then the equation

$$F_1(x_1,x_2)=x_1^2+x_1x_2+12x_2^2=n \tag{2.1}$$

(1) has two integral solutions $(-1,0)$ and $(1,0)$ for $n=1$;
(2) has no integral solution for $n=2,3$, and 5;
(3) has two integral solutions $(-2,0)$ and $(2,0)$ for $n=4$.

Hence according to (1.8), we have

$$\wp(\tau;F_1)=1+2z+2z^4+\cdots. \tag{2.2}$$

From (2.2), we get

$$\wp(\tau;F_2)=\wp^2(\tau;F_1)=1+4z+4z^2+4z^4+8z^5+\cdots. \tag{2.3}$$

Similarly, the equation

$$G_1(x_1,x_2)=3x_1^2+x_1x_2+4x_2^2=n \tag{2.4}$$

(1) has no integral solution for $n=1,2$ and 5;
(2) has two integral solution $(-1,0)$ and $(1,0)$ for $n=3$;
(3) has two integral solutions $(0,1)$ and $(0,-1)$ for $n=4$.

Hence according to (1.8), we have

$$\wp(\tau;G_1)=1+2z^3+2z^4+\cdots. \tag{2.5}$$

From (2.5), we get

$$\wp(\tau;G_2)=\wp^2(\tau;G_1)=1+4z^3+4z^4+\cdots. \tag{2.6}$$

From (2.2) and (2.5), we have

$$\wp(\tau;F_1\oplus G_1)=\wp(\tau;F_1)\wp(\tau;G_1)=1+2z+2z^3+8z^4+4z^5+\cdots. \tag{2.7}$$

Moreover, we get

$$\begin{aligned}\wp(\tau;F_3) &= 1+6z+12z^2+8z^3+6z^4+24z^5+\cdots;\\ \wp(\tau;F_4) &= 1+8z+24z^2+32z^3+24z^4+48z^5+\ldots;\\ \wp(\tau;G_3) &= 1+6z^3+6z^4+\cdots;\\ \wp(\tau;G_4) &= 1+8z^3+8z^4+\cdots;\\ \wp(\tau;F_1\oplus G_3) &= 1+2z+6z^3+20z^4+12z^5+\cdots;\\ \wp(\tau;F_2\oplus G_2) &= 1+4z+4z^2+4z^3+24z^4+40z^5+\cdots;\\ \wp(\tau;F_3\oplus G_1) &= 1+6z+12z^2+10z^3+20z^4+60z^5+\cdots.\end{aligned} \tag{2.8}$$

Now consider the quadratic forms F_2, G_2, and $F_1\oplus G_1$. From Lemma 1.5, they are of type $(-2,\Gamma_0(47),1)$.

Theorem 2.1. *For the quadratic form F_2,*

(1) $\varphi_{11}=x_1^2-(12/47)F_2$ *is a spherical function of second order with respect to F_2;*
(2) $\wp(\tau;F_2,\varphi_{11})=(1/47)(46z+116z^2+184z^4+460z^5+\cdots)\in S_4(\Gamma_0(47),1)$*;*
(3) $\operatorname{ord}(\wp(\tau;F_2,\varphi_{11}),i\infty,\Gamma_0(47))=1$.

Proof. If we take $k=4$, $Q=F_2$, and $r=s=1$, then from Lemma 1.3 we have $\varphi_{11}=x_1^2-(12/47)F_2$, which is a spherical function of second order with respect to F_2. The equation

$$F_2(x_1,x_2,x_3,x_4)=n \tag{2.9}$$

(1) has four integral solutions$(\pm1,0,0,0)$ and $(0,0,\pm1,0)$ for $n=1$;
(2) has four integral solutions $(1,0,\pm1,0)$ and $(-1,0,\pm1,0)$ for $n=2$;
(3) has no integral solutions for $n=3$;
(4) has four integral solutions $(\pm2,0,0,0)$ and $(0,0,\pm2,0)$ for $n=4$;
(5) has eight integral solutions $(-2,0,\pm1,0)$, $(-1,0,\pm2,0)$, $(1,0,\pm2,0)$, and $(2,0,\pm1,0)$ for $n=5$.

So we have from Lemma 1.4,

$$\begin{aligned}\wp(\tau;F_2,\varphi_{11}) &= \frac{1}{47}\{(47.1.2-12.1.4)z+(47.1.4-12.2.4)z^2\\ &\quad+(47.4.2-12.4.4)z^4+(47.4.4+47.1.4-12.5.8)z^5+\cdots\}\\ &= \frac{46}{47}z+\frac{116}{47}z^2+\frac{184}{47}z^4+\frac{460}{47}z^5+\cdots\in S_4(\Gamma_0(47),1).\end{aligned} \tag{2.10}$$

From (1.5) we have $\operatorname{ord}(\wp(\tau;F_2,\varphi_{11}),i\infty,\Gamma_0(47))=1$. □

Theorem 2.2. *For the quadratic form G_2,*

(1) $\varphi_{11}=x_1^2-(4/47)G_2$ *and* $\varphi_{22}=x_2^2-(3/47)G_2$ *are spherical functions of second order with respect to G_2;*
(2) $\wp(\tau;G_2,\varphi_{11})=(1/47)(46z^3-64z^4+\cdots)\in S_4(\Gamma_0(47),1)$*;*

(3) $\wp(\tau;G_2,\varphi_{22}) = (1/47)(-36z^3+46z^4+\cdots) \in S_4(\Gamma_0(47),1)$;
(4) $\text{ord}(\wp(\tau;G_2,\varphi_{11}),i\infty,\Gamma_0(47)) = \text{ord}(\wp(\tau;G_2,\varphi_{22}),i\infty,\Gamma_0(47)) = 3$.

Proof. Similarly, if we take $k = 4$, $Q = G_2$, $r = s = 1$, and $r = s = 2$, then from Lemma 1.3 we have $\varphi_{11} = x_1^2-(4/47)G_2$ and $\varphi_{22} = x_2^2-(3/47)G_2$, which are spherical functions of second order with respect to G_2. The equation

$$G_2(x_1,x_2,x_3,x_4) = n \tag{2.11}$$

(1) has no integral solutions for $n = 1,2$ and 5;
(2) has four integral solutions $(\pm 1,0,0,0)$ and $(0,0,\pm 1,0)$ for $n = 3$;
(3) has four integral solutions $(0,\pm 1,0,0)$ and $(0,0,0,\pm 1)$ for $n = 4$.

So we have from Lemma 1.4,

$$\begin{aligned}
\wp(\tau;G_2,\varphi_{11}) &= \frac{1}{47}\{(47.1.2-4.3.4)z^3+(47.0.4-4.4.4)z^4+\cdots\} \\
&= \frac{46}{47}z^3-\frac{64}{47}z^4+\cdots \in S_4(\Gamma_0(47),1), \\
\wp(\tau;G_2,\varphi_{22}) &= \frac{1}{47}\{(47.0.4-3.3.4)z^3+(47.1.2-3.4.4)z^4+\cdots\} \\
&= -\frac{36}{47}z^3+\frac{46}{47}z^4+\cdots \in S_4(\Gamma_0(47),1).
\end{aligned} \tag{2.12}$$

By definition $\text{ord}(\wp(\tau;G_2,\varphi_{11}),i\infty,\Gamma_0(47)) = \text{ord}(\wp(\tau;G_2,\varphi_{22}),i\infty,\Gamma_0(47)) = 3$. □

Theorem 2.3. *For the quadratic form $F_1\oplus G_1$,*
(1) $\varphi_{11} = x_1^2-(12/47)(F_1\oplus G_1)$ *and* $\varphi_{22} = x_2^2-(1/47)(F_1\oplus G_1)$ *are spherical functions of second order with respect to* $F_1\oplus G_1$;
(2) $\wp(\tau;F_1\oplus G_1,\varphi_{11}) = (1/47)(70z-72z^3+274z^4-52z^5+\cdots) \in S_4(\Gamma_0(47),1)$;
(3) $\wp(\tau;F_1\oplus G_1,\varphi_{22}) = (1/47)(-2z-6z^3-32z^4-20z^5+\cdots) \in S_4(\Gamma_0(47),1)$;
(4) $\text{ord}(\wp(\tau;F_1\oplus G_1,\varphi_{11}),i\infty,\Gamma_0(47)) = \text{ord}(\wp(\tau;F_1\oplus G_1,\varphi_{22}),i\infty,\Gamma_0(47)) = 1$.

Proof. If we take $k = 4$, $Q = F_1\oplus G_1$, $r = s = 1$, and $r = s = 2$, then from Lemma 1.3 we have $\varphi_{11} = x_1^2-(12/47)F_1\oplus G_1$ and $\varphi_{22} = x_2^2-(1/47)F_1\oplus G_1$, which are spherical functions of second order with respect to $F_1\oplus G_1$. The equation

$$F_1\oplus G_1(x_1,x_2,x_3,x_4) = n \tag{2.13}$$

(1) has two integral solutions $(\pm 1,0,0,0)$ for $n = 1$;
(2) has no integral solutions for $n = 2$;
(3) has two integral solutions $(0,0,\pm 1,0)$ for $n = 3$;
(4) has eight integral solutions $(\pm 2,0,0,0)$, $(1,0,\pm 1,0)$, $(0,0,0,\pm 1)$, and $(-1,0,\pm 1,0)$ for $n = 4$;
(5) has four integral solutions $(1,0,0,\pm 1)$ and $(-1,0,0,\pm 1)$ for $n = 5$.

So we have

$$
\begin{aligned}
&\wp(\tau;F_1\oplus G_1,\varphi_{11})\\
&\quad=\frac{1}{47}\{(47.1.2-12.1.2)z+(47.0.2-12.3.2)z^3+(47.4.2+47.1.6-12.4.8)z^4\\
&\qquad+(47.1.4-12.5.4)z^5+\cdots\}\\
&\quad=\frac{70}{47}z-\frac{72}{47}z^3+\frac{274}{47}z^4-\frac{52}{47}z^5+\cdots\in S_4(\Gamma_0(47),1),
\end{aligned}
\tag{2.14}
$$

$$
\begin{aligned}
&\wp(\tau;F_1\oplus G_1,\varphi_{22})\\
&\quad=\frac{1}{47}\{(47.0.2-1.1.2)z+(47.0.2-1.3.2)z^3+(47.0.8-1.4.8)z^4\\
&\qquad+(47.0.4-1.5.4)z^5+\cdots\}\\
&\quad=-\frac{2}{47}z-\frac{6}{47}z^3-\frac{32}{47}z^4-\frac{20}{47}z^5+\cdots\in S_4(\Gamma_0(47),1).
\end{aligned}
\tag{2.15}
$$

From (1.5), $\operatorname{ord}(\wp(\tau;F_1\oplus G_1,\varphi_{11}),i\infty,\Gamma_0(47)) = \operatorname{ord}(\wp(\tau;F_1\oplus G_1,\varphi_{22}),i\infty,\Gamma_0(47)) = 1$. □

The system of theta series in (2.10), (2.12), (2.14), and (2.15) are linearly independent since the fifth order determinant of the coefficients in the expansions of these theta series is different from zero. Since $|S_4(\Gamma_0(47),1)| = 5$, we proved Theorem 2.4.

Theorem 2.4. *The system of generalized fourfold theta series*

$$
\begin{aligned}
\wp(\tau;F_2,\varphi_{11}) &= \frac{1}{47}\sum_{n=1}^{\infty}\left(\sum_{F_2=n}47x_1^2-12n\right)z^n,\\
\wp(\tau;G_2,\varphi_{11}) &= \frac{1}{47}\sum_{n=1}^{\infty}\left(\sum_{G_2=n}47x_1^2-4n\right)z^n,\\
\wp(\tau;G_2,\varphi_{22}) &= \frac{1}{47}\sum_{n=1}^{\infty}\left(\sum_{G_2=n}47x_2^2-3n\right)z^n,\\
\wp(\tau;F_1\oplus G_1,\varphi_{11}) &= \frac{1}{47}\sum_{n=1}^{\infty}\left(\sum_{F_1\oplus G_1=n}47x_1^2-12n\right)z^n,\\
\wp(\tau;F_1\oplus G_1,\varphi_{22}) &= \frac{1}{47}\sum_{n=1}^{\infty}\left(\sum_{F_1\oplus G_1=n}47x_2^2-n\right)z^n
\end{aligned}
\tag{2.16}
$$

is a basis of the space $S_4(\Gamma_0(47),1)$*, of cusp forms of type* $(-4,\Gamma_0(47),1)$*.*

From Theorems 2.1, 2.2, and 2.3 we have following corollaries.

Corollary 2.5. *Let* F_k (G_k) *be the direct sum of* k*-copies of* F_1 (G_1) *of type* $(-k,\Gamma_0(47),1)$ *and let* φ_{rs} *be the spherical function of second order with respect to* F_k (G_k)*, then*

(1) $\operatorname{ord}(\wp(\tau;F_k,\varphi_{rs}),i\infty,\Gamma_0(47)) = 1$;
(2) $\operatorname{ord}(\wp(\tau;G_k,\varphi_{rs}),i\infty,\Gamma_0(47)) = 3$.

Corollary 2.6. *Let $F_i \oplus G_j$, $i,j \geqslant 1$, $i+j=k$ be the direct sum of i-copies of F_1 and j-copies of G_1 of type $(-k,\Gamma_0(47),1)$ and let φ_{rs} be the spherical function of second order with respect to $F_i \oplus G_j$, then*

$$\operatorname{ord}\left(\wp(\tau;F_i \oplus G_j,\varphi_{rs}),i\infty,\Gamma_0(47)\right)=1. \tag{2.17}$$

Now we give the formulas for $r(n;F_4)$, $r(n;G_4)$, $r(n;F_3\oplus G_1)$, $r(n;F_2\oplus G_2)$, and $r(n;F_1\oplus G_3)$ by the following theorem.

Theorem 2.7. *For the quadratic forms F_4, G_4, $F_3\oplus G_1$, $F_2\oplus G_2$ and $F_1\oplus G_3$ we have the following formulas:*

$$\begin{aligned}
r(n;F_4) &= \frac{24}{221}\sigma_3^*(n)+\frac{1272}{29.221}\left(\sum_{F_2=n}47x_1^2-12n\right)\\
&\quad-\frac{6498200}{29.47.221}\left(\sum_{G_2=n}47x_1^2-4n\right)-\frac{8716584}{29.47.221}\left(\sum_{G_2=n}47x_2^2-3n\right)\\
&\quad+\frac{17864}{29.47.221}\left(\sum_{F_1\oplus G_1=n}47x_1^2-12n\right)+\frac{811736}{29.47.221}\left(\sum_{F_1\oplus G_1=n}47x_2^2-n\right),\\
r(n;G_4) &= \frac{24}{221}\sigma_3^*(n)-\frac{54}{29.221}\left(\sum_{F_2=n}47x_1^2-12n\right)\\
&\quad-\frac{1305584}{29.47.221}\left(\sum_{G_2=n}47x_1^2-4n\right)-\frac{1742266}{29.47.221}\left(\sum_{G_2=n}47x_2^2-3n\right)\\
&\quad+\frac{5046}{29.47.221}\left(\sum_{F_1\oplus G_1=n}47x_1^2-12n\right)+\frac{134592}{29.47.221}\left(\sum_{F_1\oplus G_1=n}47x_2^2-n\right),\\
r(n;F_3\oplus G_1) &= \frac{24}{221}\sigma_3^*(n)+\frac{609}{29.221}\left(\sum_{F_2=n}47x_1^2-12n\right)\\
&\quad+\frac{372911}{29.47.221}\left(\sum_{G_2=n}47x_1^2-4n\right)+\frac{416904}{29.47.221}\left(\sum_{G_2=n}47x_2^2-3n\right)\\
&\quad+\frac{5046}{29.47.221}\left(\sum_{F_1\oplus G_1=n}47x_1^2-12n\right)-\frac{52374}{29.47.221}\left(\sum_{F_1\oplus G_1=n}47x_2^2-n\right),\\
r(n;F_2\oplus G_2) &= \frac{24}{221}\sigma_3^*(n)+\frac{167}{29.221}\left(\sum_{F_2=n}47x_1^2-12n\right)\\
&\quad+\frac{1193705}{29.47.221}\left(\sum_{G_2=n}47x_1^2-4n\right)+\frac{1545330}{29.47.221}\left(\sum_{G_2=n}47x_2^2-3n\right)\\
&\quad+\frac{5046}{29.47.221}\left(\sum_{F_1\oplus G_1=n}47x_1^2-12n\right)-\frac{228953}{29.47.221}\left(\sum_{F_1\oplus G_1=n}47x_2^2-n\right),
\end{aligned}$$

$$
\begin{aligned}
r(n;F_1\oplus G_3) &= \frac{24}{221}\sigma_3^*(n) - \frac{54}{29.221}\left(\sum_{F_2=n} 47x_1^2 - 12n\right) \\
&\quad - \frac{302460}{29.47.221}\left(\sum_{G_2=n} 47x_1^2 - 4n\right) - \frac{418656}{29.47.221}\left(\sum_{G_2=n} 47x_2^2 - 3n\right) \\
&\quad + \frac{8250}{29.47.221}\left(\sum_{F_1\oplus G_1=n} 47x_1^2 - 12n\right) - \frac{54491}{29.47.221}\left(\sum_{F_1\oplus G_1=n} 47x_2^2 - n\right),
\end{aligned}
\tag{2.18}
$$

where

$$
\sigma_3^*(n) = \begin{cases} \sigma_3(n) & \text{if } 47 \nmid n, \\ \sigma_3(n) + 47^2\sigma_3\left(\dfrac{n}{47}\right) & \text{if } 47 \mid n. \end{cases}
\tag{2.19}
$$

Proof. By Lemma 1.5, F_4, G_4, $F_3\oplus G_1$, $F_2\oplus G_2$, $F_1\oplus G_3$ are quadratic forms of type $(-4,\Gamma_0(47),1)$. We know from Lemma 1.1 that there exist Eisenstein series which correspond to each other. For $k=4$, we have $\alpha = 24/221$ for $\rho_4 = 1/240$. Thus we get

$$
\begin{aligned}
E(\tau;F_4) &= E(\tau;G_4) = E(\tau;F_3\oplus G_1) = E(\tau;F_2\oplus G_2) = E(\tau;F_1\oplus G_3) \\
&= 1 + \sum_{n=1}^{\infty}\left(\alpha\sigma_3(n)z^n + \beta\sigma_3(n)z^{qn}\right) \\
&= 1 + \frac{24}{221}z + \frac{24.9}{221}z^2 + \frac{24.28}{221}z^3 + \frac{24.73}{221}z^4 + \frac{24.126}{221}z^5 + \cdots.
\end{aligned}
\tag{2.20}
$$

By Lemma 1.2, the difference $\wp(\tau;F_4) - E(\tau;F_4)$ is a cusp form of type $(-4,\Gamma_0(47),1)$. On the other hand from Theorem 2.4, $\wp(\tau;F_2,\varphi_{11})$, $\wp(\tau;G_2,\varphi_{11})$, $\wp(\tau;G_2,\varphi_{22})$, $\wp(\tau;F_1\oplus G_1,\varphi_{11})$, $\wp(\tau;F_1\oplus G_1,\varphi_{22})$ are bases of the cusp space $S_4(\Gamma_0(47),1)$. Therefore, we can find integers $c_1,\dots,c_5$ such that

$$
\begin{aligned}
\wp(\tau;F_4) - E(\tau;F_4) &= c_1\wp(\tau;F_2,\varphi_{11}) + c_2\wp(\tau;G_2,\varphi_{11}) + c_3\wp(\tau;G_2,\varphi_{22}) \\
&\quad + c_4\wp(\tau;F_1\oplus G_1,\varphi_{11}) + c_5\wp(\tau;F_1\oplus G_1,\varphi_{22}).
\end{aligned}
\tag{2.21}
$$

From (2.8) and (2.20), we have

$$
\wp(\tau;F_4) - E(\tau;F_4) = \frac{1744}{221}z + \frac{5088}{221}z^2 + \frac{6400}{221}z^3 + \frac{3552}{221}z^4 + \frac{7584}{221}z^5 + \cdots.
\tag{2.22}
$$

From (2.21) and (2.22), we get

$$
\begin{aligned}
\wp(\tau;F_4) &= E(\tau;F_4) + \frac{1272}{29.221}\wp(\tau;F_2,\varphi_{11}) - \frac{6498200}{29.221}\wp(\tau;G_2,\varphi_{11}) \\
&\quad - \frac{8716584}{29.221}\wp(\tau;G_2,\varphi_{22}) + \frac{17864}{29.221}\wp(\tau;F_1\oplus G_1,\varphi_{11}) \\
&\quad + \frac{811736}{29.221}\wp(\tau;F_1\oplus G_1,\varphi_{22}).
\end{aligned}
\tag{2.23}
$$

Similarly, we obtain

$$
\begin{aligned}
\wp(\tau;G_4) &= E(\tau;G_4) - \frac{54}{29.221}\wp(\tau;F_2,\varphi_{11}) - \frac{1305584}{29.221}\wp(\tau;G_2,\varphi_{11}) \\
&\quad - \frac{1742266}{29.221}\wp(\tau;G_2,\varphi_{22}) + \frac{5046}{29.221}\wp(\tau;F_1\oplus G_1,\varphi_{11}) \\
&\quad + \frac{134592}{29.221}\wp(\tau;F_1\oplus G_1,\varphi_{22}), \\
\wp(\tau;F_3\oplus G_1) &= E(\tau;F_3\oplus G_1) + \frac{609}{29.221}\wp(\tau;F_2,\varphi_{11}) + \frac{372911}{29.221}\wp(\tau;G_2,\varphi_{11}) \\
&\quad + \frac{416904}{29.221}\wp(\tau;G_2,\varphi_{22}) + \frac{5046}{29.221}\wp(\tau;F_1\oplus G_1,\varphi_{11}) \\
&\quad - \frac{52374}{29.221}\wp(\tau;F_1\oplus G_1,\varphi_{22}), \\
\wp(\tau;F_2\oplus G_2) &= E(\tau;F_2\oplus G_2) + \frac{167}{29.221}\wp(\tau;F_2,\varphi_{11}) + \frac{1193705}{29.221}\wp(\tau;G_2,\varphi_{11}) \\
&\quad + \frac{1545330}{29.221}\wp(\tau;G_2,\varphi_{22}) + \frac{5046}{29.221}\wp(\tau;F_1\oplus G_1,\varphi_{11}) \\
&\quad - \frac{228953}{29.221}\wp(\tau;F_1\oplus G_1,\varphi_{22}), \\
\wp(\tau;F_1\oplus G_3) &= E(\tau;F_1\oplus G_3) - \frac{54}{29.221}\wp(\tau;F_2,\varphi_{11}) - \frac{302460}{29.221}\wp(\tau;G_2,\varphi_{11}) \\
&\quad - \frac{418656}{29.221}\wp(\tau;G_2,\varphi_{22}) + \frac{8250}{29.221}\wp(\tau;F_1\oplus G_1,\varphi_{11}) \\
&\quad - \frac{54491}{29.221}\wp(\tau;F_1\oplus G_1,\varphi_{22})
\end{aligned}
\tag{2.24}
$$

as desired. □

References

[1] G. Lomadze, *On the number of representations of positive integers by a direct sum of binary quadratic forms with discriminant* −23, Georgian Math. J. **4** (1997), no. 6, 523–532.

Ahmet Tekcan: Department of Mathematics, Faculty of Science, University of Uludag, Görükle 16059, Bursa, Turkey

E-mail address: fahmet@uludag.edu.tr

Osman Bizim: Department of Mathematics, Faculty of Science, University of Uludag, Görükle 16059, Bursa, Turkey

E-mail address: obizim@uludag.edu.tr