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SOBOLEV SPACE ESTIMATES FOR SOLUTIONS OF THE EQUATION $\overline{\partial}u = f$ ON POLYCYLINDERS

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In trying to improve Weinstock's results on approximation by holomorphic functions on certain product domains, we are led to estimates in Sobolev spaces for the $\overline{\partial}$-operator on polycylinders for (γ,q)-forms. This generalizes our results for the same operator on polycylinders previously obtained, and can be applied to a number of other problems such as the Corona problem.

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1. Introduction. Had we the uniform estimates of Grauert and Lieb for the Cauchy-Riemann equation [5] on polycylinders, we would have improved Weinstock's result [6, Theorem 1.1] on polycylinders a long time ago. On the other hand, we have had several estimates for the $\overline{\partial}$-operator in Sobolev spaces [2, 3, 4] on polycylinders, which we have been trying to improve. At the same time, we noticed that because some Sobolev norms dominate the uniform norm, if we did the approximation in those Sobolev norms, we would get the desired improvement of the above-mentioned theorem of Weinstock on polycylinders. We therefore went ahead here and jazzed up our previous estimates to get new estimates. We also applied our results to solve the Sobolev-Corona problem.

2. Delta-bar in $W^{k,p}_{(\gamma,q+1)}(\Omega)$. Let $L^p_{(\gamma,q)}(U)$ denote the space of forms of type (γ,q) with coefficients in $L^p(U)$. Then,

$$f = {\sum_{|J|=\gamma}}' {\sum_{|J|=q}}' f_{I,J}\, dz^I \wedge d\overline{z}^J, \tag{2.1}$$

where $\sum'$ means that the summation is performed only over strictly increasing multi-indices, $I = (i_1,\dots,i_\gamma)$, $J = (j_1,\dots,j_q)$, $dz^I = dz_{i_1} \wedge \cdots \wedge dz_{i_\gamma}$, $d\overline{z}^J = d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}$, U is open in $\mathbb{C}^n$, and $1 \le p \le \infty$.

The norm of the (γ,q)-form in (2.1) is defined by

$$\begin{gathered} \|f\|_{L^p_{(\gamma,q)}(U)} = \left\{ {\sum_I}' {\sum_J}' \left|\left|f_{I,J}\right|\right|_{L^p(U)} \right\}^{1/p}, \quad \text{for } 1 \le p < \infty, \\ \|f\|_{L^\infty_{(\gamma,q)}(U)} = \max_{I,J} \left|\left|f_{I,J}\right|\right|_{L^\infty(U)}. \end{gathered} \tag{2.2}$$

Let $W^{k,p}(U)$, $1 \le p \le \infty$, $k = 1,2,\dots$, be the space of functions which together with their distributional derivatives of order through k are in $L^p(U)$, with the usual norm, and $W^{k,p}_{(\gamma,q)}(U)$ the space of (γ,q)-forms with coefficients in $W^{k,p}(U)$, with the norm defined by

$$\|f\|_{W^{k,p}_{(\gamma,q)}(U)} = \left\{ \sum_I{}' \sum_J{}' \|f_{I,J}\|_{W^{k,p}(U)} \right\}^{1/p}, \quad 1 \le p < \infty,$$
$$\|f\|_{W^{k,p}_{(\gamma,q)}(U)} = \max_{I,J} \|f_{I,J}\|_{W^{k,p}(U)}. \tag{2.3}$$

A bounded open set Ω in $\mathbb{C}^n$ is called a polycylinder if Ω can be expressed as a product of n bounded open sets in $\mathbb{C}$, that is, $\Omega = U_1 \times U_2 \times \cdots \times U_n$, where each U_j is open and bounded in $\mathbb{C}$. And Ω is called admissible if each U_j has boundary with plane measure zero.

Our main result is the following theorem.

Theorem 2.1. *Let Ω be an admissible polycylinder in $\mathbb{C}^n$ and let $f \in W^{k,p}_{(\gamma,q+1)}(\Omega)$ be $\bar\partial$-closed, $1 \le p \le \infty$, $k = 1,2,\dots$, then there is a $u \in W^{k,p}_{(\gamma,q)}(\Omega)$ such that $\bar\partial u = f$ and*

$$\|u\|_{W^{k,p}_{(\gamma,q)}(\Omega)} \le \delta \|f\|_{W^{k,p}_{(\gamma,q+1)}(\Omega)}, \tag{2.4}$$

where δ depends on Ω.

3. The Corona problem. The Corona problem is stated in [1]. Let X be a relatively compact domain in a topological space Y. Let $f_0,\dots,f_N$ be complex-valued continuous functions on X; $f_0,\dots,f_N$ verify the Corona assumption if there is $\delta > 0$ such that

$$\sum_S |f_S| \ge \delta > 0 \quad \text{on } X. \tag{3.1}$$

Let A be a function algebra on X. The Corona problem is solvable in A (on X) when each set $f_0,\dots,f_N \in A$, which verifies the Corona assumption, represents 1 in A, that is, there are $g_0,\dots,g_N$ in A such that

$$f_0 g_0 + \cdots + f_N g_n = 1 \quad \text{on } X. \tag{3.2}$$

From Theorem 2.1 through cohomology with bounds (see [4, Theorem 2.3]), we have the following theorem.

Theorem 3.1. *Let Ω be an admissible polycylinder and suppose that Ω has a Lipschitz boundary, and $pk > n$ (in which case $\Gamma(\Omega,\mathbb{O}) \cap W^{k,p}(\Omega)$ is an algebra with members extending continuously to the boundary of Ω). Then the Corona problem is solvable in $\Gamma(\Omega,\mathbb{O}) \cap W^{k,p}(\Omega)$ (here $\mathbb{O}$ is the structure sheaf of $\mathbb{C}^n$, and $\Gamma(\Omega,\mathbb{O})$ is the set of sections of $\mathbb{O}$ over Ω).*

4. Approximation. Let $K = \overline{\Omega}$ be the closure in $\mathbb{C}^n$ of an admissible polycylinder, let $C(K)$ denote the Banach space of continuous complex-valued functions on K with the uniform norm, and let $H(K)$ denote the closure in $C(K)$ of the space of functions which are holomorphic in some neighborhood of K.

Our last result is then the following theorem.

Theorem 4.1. *If U is a neighborhood of K, $f \in C^2(U)$, and $\partial f/\partial \bar z_j = 0$ on K, $1 \le j \le n$, then $f \in H(K)$.*

In this paper, we prove the $(0,1)$ version of Theorems 2.1 and 4.1 only. Theorem 3.1 follows when considered as the weak Corona theorem as in [1]. The general version of Theorem 2.1 can be proved using the induction process in [3].

5. Solution of $\bar\partial u = f((\gamma,q) = (0,1))$. For all f satisfying the hypothesis of Theorem 2.1 in this case, extend f to all of $\mathbb{C}^n$ by zero outside Ω and call it again f. Then $\bar\partial f = 0$ in the distribution sense in $\mathbb{C}^n$. Then the following is true.

Lemma 5.1. *If $u(z) = (2\pi i)^{-1}\int_{\mathbb{C}}(\xi - z_1)^{-1} f_1(\xi, z_2,\dots,z_n)\,d\xi\wedge d\bar\xi$, where $f = \sum_{j=1}^n f_j d\bar z_j$, with $f_1 \not\equiv 0$, then $\bar\partial u = f$ and*

$$\|u\|_{W^{k,p}(\Omega)} \le \delta \|f\|_{W^{k,p}_{(0,1)}(\Omega)}, \quad 1 \le p \le \infty,\ k = 1,2,\dots, \tag{5.1}$$

where δ depends only on Ω.

Proof. We regularize f coefficientwise:

$$f_m = \sum_{j=1}^n (f_j)_m d\bar z_j,$$
$$(f_j)_m(z) = \int f_j\left(z - \frac{\xi}{m}\right)\phi(\xi)\,d\lambda(\xi) = m^{2n}\int f_j(\xi)\phi(m(z-\xi))\,d\lambda(\xi), \tag{5.2}$$

where $\phi \in C_0^\infty(\mathbb{C}^n)$, $\int \phi\, d\lambda = 1$, $\phi \ge 0$, $\operatorname{supp}\phi = \{z : |z| \le 1\}$, and λ is a Lebesgue measure. Then $\|f_m\|_{L^p_{(0,1)}} \le \|f\|_{L^p_{(0,1)}}$ for $1 \le p \le \infty$, $f_m \to f$ in

$$L^1_{(0,1)}(\Omega) \tag{5.3}$$

as $m \to \infty$ for $1 \le p \le \infty$, and f_m is $\bar\partial$-closed in $\mathbb{C}^n$.

Now, let

$$u_m(z) = (2\pi i)^{-1}\int_{\mathbb{C}} (\xi - z_1)^{-1}(f_1)_m(\xi, z_2,\dots,z_n)\,d\xi\wedge d\bar\xi. \tag{5.4}$$

Then

$$u_m(z) = -(2\pi i)^{-1}\int_{\mathbb{C}^n} \xi^{-1}(f_1)(z_1 - \xi, z_2,\dots,z_n)\,d\xi\wedge\bar\xi, \tag{5.5}$$

and from (5.4) and (5.5),

$$\frac{\partial u_m(z)}{\partial z_l} = (2\pi i)^{-1}\int_{\mathbb{C}}(\xi - z_1)^{-1}\frac{\partial (f_l)_m(\xi, z_2,\dots,z_n)}{\partial \xi}d\xi\wedge d\overline{\xi} = (f_l)_m(z). \tag{5.6}$$

Therefore, $\overline{\partial}u_m = f_m$, and since $f_m \to f$ in $L^1_{(0,1)}(\Omega)$ for $1 \le p \le \infty$, $u_m \to u$ in $L^1(\Omega)$ if $1 \le p \le \infty$, and we have $\overline{\partial}u = f$.

For $1 \le p \le \infty$, it is clear from [2] that

$$\|u\|_{L^p(\Omega)} \le \delta\|f\|_{L^p_{(0,1)}(\Omega)}, \tag{5.7}$$

with δ depending Ω.

Now, let $\partial^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1}\partial y_1^{\alpha_2}\cdots\partial x_n^{\partial_{2n-1}}\partial y_n^{\alpha_{2n}}$, $\alpha = (\alpha_1,\alpha_2,\dots,\alpha_{2n})$, $z = (x_1 + iy_1,\dots, x_n + iy_n)$, then [2, 3] we see, where $\gamma(\alpha)$ is a power of -1, that

$$\gamma^{(\alpha)}\partial^\alpha u_m(z) = (2\pi i)^{-1}\int_{\mathbb{C}}(\xi - z_1)^{-1}\partial^\alpha (f_1)_m(\xi, z_2,\dots,z_n)d\xi\wedge d\overline{\xi}. \tag{5.8}$$

Since it is clear that for $1 \le p \le \infty$, when $k \ge |\alpha|$ and $f \in W^{k,p}_{(0,1)}(\Omega)$, $\partial^\alpha(f_1)m \to \partial^\alpha f$ in $L^1(\Omega)$ when $m \to \infty$, it follows that, for $k = 1,2,\dots,$

$$\gamma^{(\alpha)}\partial^\alpha u(z) = (2\pi i)^{-1}\int_{U_1}(\xi - z_1)^{-1}\partial^\alpha f_1(\xi, z_2,\dots,z_n)d\xi\wedge\overline{\xi}. \tag{5.9}$$

Therefore, for $k = 1,2,\dots$, $1 \le p \le \infty$,

$$\|u\|_{W^{k,p}}(\Omega) \le \delta\|f\|_{W^{k,p}_{(0,1)}(\Omega)}. \tag{5.10}$$

If $f_1 \equiv 0$ and $f \not\equiv 0$, there is an f_{j_0} which we can use in place of f_1. □

6. Proof of Theorem 4.1. Let f satisfy the hypothesis of Theorem 4.1. We may suppose that f has compact support in U. Regularize f as we did in Section 5:

$$f_m = \int f\left(z - \frac{\xi}{m}\right)\phi(\xi)d\lambda(\xi) = m^{2n}\int f(\xi)\phi(m(z-\xi))d\lambda(\xi). \tag{6.1}$$

Then, $f_m \in C^\infty(\mathbb{C}^n)$, $f_m \to f$ in $W^{1,\infty}(\mathbb{C}^n)$ as $m \to \infty$, and for each j,

$$\frac{\partial f_m}{\partial \overline{z}_j}(z) = \int \frac{\partial f}{\partial \overline{z}j}\left(z - \frac{\xi}{m}\right)\phi(\xi)d\lambda(\xi), \tag{6.2}$$

so if G is open in $\mathbb{C}^n$,

$$||f_m||_{W^{1,\infty}(G)} \le \|f\|_{W^{1,\infty}(G^\nu)}, \qquad \left\|\frac{\partial f_m}{\partial \overline{z}_j}\right\|_{W^{1,\infty}(G)} \le \left\|\frac{\partial f}{\partial \overline{z}_j}\right\|_{W^{1,\infty}(G^\nu)}, \tag{6.3}$$

where $G^\nu = \{z - \nu\xi : z \in G,\ |\xi| \le 1\}$.

Now, we can find a sequence $\{\Omega_\lambda\}$ of decreasing admissible polycylinders such that $K = \overline{\Omega} = \cap\Omega_\lambda$ and such that δ in Theorem 2.1 is the same for each Ω_λ. Note also that

$$||\bar\partial f||_{W^{1,\infty}(\Omega_\lambda)} \longrightarrow 0 \quad \text{as } \lambda \longrightarrow \infty. \tag{6.4}$$

Let $\epsilon > 0$ be given. Choose m_0 such that

$$||f - f_m||_{W^{1,\infty}(K)} < \frac{\epsilon}{2} \quad \text{for } m > m_0. \tag{6.5}$$

Choose λ_0 such that

$$||\bar\partial f||_{W^{1,\infty}_{(0,1)}(\Omega_{\lambda_0})} < \frac{\delta^{-1}\epsilon}{4}. \tag{6.6}$$

Then, for small ν,

$$||\bar\partial f||_{W^{1,\infty}_{(0,1)}(\Omega^{\nu}_{\lambda_0})} < \frac{\delta^{-1}\epsilon}{2}. \tag{6.7}$$

By Theorem 2.1, we can choose $u \in W^{1,\infty}(\Omega_{\lambda_0})$ such that $\bar\partial u = \bar\partial f_m$, $m > m_0$ (fixed), and

$$\begin{aligned} \|u\|_{W^{1,\infty}(\Omega_{\lambda_0})} &\le \delta||\bar\partial f_m||_{W^{1,\infty}_{(0,1)}(\Omega_{\lambda_0})} \\ &\le \delta||\bar\partial f||_{W^{1,\infty}_{(0,1)}(\Omega^{\nu}_{\lambda_0})} \quad \text{(by (6.3))} \\ &< \frac{\epsilon}{2} \quad \text{(by (6.7))}. \end{aligned} \tag{6.8}$$

Then $h = f_m - u$ is holomorphic in a neighborhood of K and with $\|\cdot\|_K$ the uniform norm, we have

$$\begin{aligned} \|f - h\|_K &\le ||f - f_m||_K + ||f_m - h||_K \\ &\le ||f - f_m||_{W^{1,\infty}(K)} + \|u\|_{W^{1,\infty}(K)} \\ &< \frac{\epsilon}{2} + \|u\|_{W^{1,\infty}(\Omega_{\lambda_0})} \\ &< \epsilon. \end{aligned} \tag{6.9}$$

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