

DERIVATIONS OF QUASI *-ALGEBRAS

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The spatiality of derivations of quasi *-algebras is investigated by means of representation theory. Moreover, in view of physical applications, the spatiality of the limit of a family of spatial derivations is considered.

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1. Introduction. In the so-called algebraic approach to quantum systems, one of the basic problems to be solved consists in the rigorous definition of the algebraic dynamics, that is, the time evolution of observables and/or states. For instance, in quantum statistical mechanics or in quantum field theory, one tries to recover the dynamics by performing a certain limit of the strictly *local* dynamics. However, this can be successfully done only for few models and under quite strong topological assumptions (see, e.g., [22] and the references therein). In many physical models, the use of local observables corresponds, roughly speaking, to the introduction of some *cutoff* (and to its successive removal) and this is in a sense a general and frequently used procedure; see [8, 10, 21, 23] for conservative systems and [1, 9] for dissipative ones.

Introducing a cutoff means that in the description of some physical system, we know a *regularized* Hamiltonian H_L , where L is a certain parameter closely depending on the nature of the system under consideration. The role of the commutator $[H_L, A]$, A being an observable of the physical system (in a sense that will be made clearer in the following), is crucial in the analysis of the dynamics of the system. We have discussed several properties of this map in a recent paper, [14], focusing our attention mainly on the existence of the algebraic dynamics α_t given a family of operators H_L as above. Here, in a certain sense, we reverse the point of view. We start with a (generalized) derivation δ and we first consider the following problem: under which conditions is the map δ spatial (i.e., implemented by a certain operator)? The spatiality of derivations is a very classical problem when formulated in *-algebras and it has been extensively studied in the literature in a large variety of situations, mostly depending on the topological structure of the *-algebras under consideration (C^* -algebras, von Neumann algebras, O^* -algebras, etc.; see [2, 3, 15, 22]). In this paper, we consider a more general setup, turning our attention to derivations taking their values in a quasi *-algebra. This choice is motivated by possible applications to the physical situations described above. Indeed, if \mathcal{A}_0 denotes the *-algebra of local observables of the system, in order to perform the so-called thermodynamical limits of certain local observables, one endows \mathcal{A}_0 with a locally convex topology τ , conveniently chosen for this aim (the so-called *physical topology*). The

completion \mathcal{A} of $\mathcal{A}_0[\tau]$, where thermodynamical limits mostly live, may fail to be an algebra, but it is in general a *quasi *-algebra* [3, 21, 24]. For these reasons, we start with considering, given a quasi *-algebra $(\mathcal{A}, \mathcal{A}_0)$, a derivation δ defined in \mathcal{A}_0 taking its values in \mathcal{A} , and investigate its spatiality. In particular, we consider the case where δ is the limit of a net $\{\delta_L\}$ of spatial derivations of \mathcal{A}_0 , and give conditions for its spatiality and for the implementing operator to be the limit, in some sense, of the operators H_L implementing the $\{\delta_L\}$'s.

The paper is organized as follows. In Section 2, we give the essential definitions of the algebraic structures needed in the sequel. In Section 3, the possibility of extending δ beyond \mathcal{A}_0 , through a notion of τ -closability, is investigated. Section 4 is devoted to the analysis of the spatiality of *-derivations which are induced by *-representations, and of the spatiality of the limit of a net of spatial *-derivations. We also extend our results to the situation where the *-representation, instead of living in Hilbert space, takes its values in a quasi *-algebra of operators in rigged Hilbert space (qu*-representation).

2. The mathematical framework. Let \mathcal{A} be a linear space and \mathcal{A}_0 a *-algebra contained in \mathcal{A} as a subspace. We say that \mathcal{A} is a quasi *-algebra with distinguished *-algebra \mathcal{A}_0 (or, simply, over \mathcal{A}_0) if

- (i) the left multiplication ax and the right multiplication xa of an element a of \mathcal{A} and an element x of \mathcal{A}_0 which extend the multiplication of \mathcal{A}_0 are always defined and bilinear;
- (ii) $x_1(x_2a) = (x_1x_2)a$ and $x_1(ax_2) = (x_1a)x_2$ for each $x_1, x_2 \in \mathcal{A}_0$ and $a \in \mathcal{A}$;
- (iii) an involution $*$ which extends the involution of \mathcal{A}_0 is defined in \mathcal{A} with the property $(ax)^* = x^*a^*$ and $(xa)^* = a^*x^*$ for each $x \in \mathcal{A}_0$ and $a \in \mathcal{A}$.

A quasi *-algebra $(\mathcal{A}, \mathcal{A}_0)$ is said to have a unit $\mathbb{1}$ if there exists an element $\mathbb{1} \in \mathcal{A}_0$ such that $a\mathbb{1} = \mathbb{1}a = a$, for all $a \in \mathcal{A}$. In this paper, we will always assume that the quasi *-algebra under consideration has an identity.

Let $\mathcal{A}_0[\tau]$ be a locally convex *-algebra. Then the completion $\overline{\mathcal{A}_0[\tau]}$ of $\mathcal{A}_0[\tau]$ is a quasi *-algebra over \mathcal{A}_0 equipped with the following left and right multiplications: for any $x \in \mathcal{A}_0$ and $a \in \mathcal{A}$,

$$ax \equiv \lim_{\alpha} x_{\alpha}x, \quad xa \equiv \lim_{\alpha} x x_{\alpha}, \tag{2.1}$$

where $\{x_{\alpha}\}$ is a net in \mathcal{A}_0 which converges to a with respect to the topology τ . Furthermore, the left and right multiplications are separately continuous. A *-invariant subspace \mathcal{A} of $\overline{\mathcal{A}_0[\tau]}$ containing \mathcal{A}_0 is said to be a (quasi)-*-subalgebra of $\overline{\mathcal{A}_0[\tau]}$ if ax and xa are in \mathcal{A} for any $x \in \mathcal{A}_0$ and $a \in \mathcal{A}$. Then we have

$$x_1(x_2a) = \lim_{\alpha} x_1(x_2x_{\alpha}) = \lim_{\alpha} (x_1x_2)x_{\alpha} = (x_1x_2)a \tag{2.2}$$

and similarly,

$$(ax_1)x_2 = a(x_1x_2), \quad x_1(ax_2) = (x_1a)x_2, \tag{2.3}$$

for each $x_1, x_2 \in \mathcal{A}_0$ and $a \in \mathcal{A}$, which implies that \mathcal{A} is a quasi *-algebra over \mathcal{A}_0 , and furthermore, $\mathcal{A}[\tau]$ is a locally convex space containing \mathcal{A}_0 as dense subspace and the right and left multiplications are separately continuous. Hence, \mathcal{A} is said to be a *locally convex quasi *-algebra* over \mathcal{A}_0 .

If $(\mathcal{A}[\tau], \mathcal{A}_0)$ is a locally convex quasi *-algebra, we indicate with $\{p_\alpha, \alpha \in I\}$ a directed set of seminorms which defines τ .

In a series of papers [7, 11, 12, 13], we have considered a special class of quasi *-algebras, called CQ^* -algebras, which arises as completions of C^* -algebras. They can be introduced in the following way.

Let \mathcal{A} be a right Banach module over the C^* -algebra \mathcal{A}_b with involution \flat and C^* -norm $\|\cdot\|_b$, and further with isometric involution $*$, such that $\mathcal{A}_b \subset \mathcal{A}$. Set $\mathcal{A}_\sharp = (\mathcal{A}_b)^*$. We say that $\{\mathcal{A}, *, \mathcal{A}_b, \flat\}$ is a CQ^* -algebra if

- (i) \mathcal{A}_b is dense in \mathcal{A} with respect to its norm $\|\cdot\|$,
- (ii) $\mathcal{A}_0 := \mathcal{A}_b \cap \mathcal{A}_\sharp$ is dense in \mathcal{A}_b with respect to its norm $\|\cdot\|_b$,
- (iii) $(ab)^* = b^*a^*$, for all $a, b \in \mathcal{A}_0$,
- (iv) $\|\mathcal{Y}\|_b = \sup_{a \in \mathcal{A}, \|a\| \leq 1} \|a\mathcal{Y}\|, \mathcal{Y} \in \mathcal{A}_b$.

Since $*$ is isometric, the space \mathcal{A}_\sharp is itself, as it is easily seen, a C^* -algebra with respect to the involution $x^\sharp := (x^*)^\flat$ and the norm $\|x\|_\sharp := \|x^*\|_b$.

A CQ^* -algebra is called *proper* if $\mathcal{A}_\sharp = \mathcal{A}_b$. When also $\flat = \sharp$, we indicate a proper CQ^* -algebra with the notation $(\mathcal{A}, *, \mathcal{A}_0)$, since $*$ is the only relevant involution and $\mathcal{A}_0 = \mathcal{A}_\sharp = \mathcal{A}_b$.

An example of CQ^* -algebra is provided by certain subspaces of $\mathcal{B}(\mathcal{H}_{+1}, \mathcal{H}_{-1}), \mathcal{B}(\mathcal{H}_{+1})$, and $\mathcal{B}(\mathcal{H}_{-1})$, the spaces of operators acting on a triplet (scale) of Hilbert spaces generated in a canonical way by an unbounded operator $S \geq 1$. For details, see [3, 11, 12]. From a purely algebraic point of view, each CQ^* -algebra can be considered as an example of partial *-algebra, [3, 4, 5], by which we mean a vector space \mathcal{A} with involution $a \rightarrow a^*$ (i.e., $(a + \lambda b)^* = a^* + \bar{\lambda}b^*$; $a = a^{**}$) and a subset $\Gamma \subset \mathcal{A} \times \mathcal{A}$ such that (i) $(a, b) \in \Gamma$ implies $(b^*, a^*) \in \Gamma$; (ii) $(a, b), (a, c) \in \Gamma$ imply $(a, b + \lambda c) \in \Gamma$; and (iii) if $(a, b) \in \Gamma$, then there exists an element $ab \in \mathcal{A}$ and for this multiplication (which is not supposed to be associative) the following properties hold: if $(a, b) \in \Gamma$ and $(a, c) \in \Gamma$, then $ab + ac = a(b + c)$ and $(ab)^* = b^*a^*$.

In the following, we also need the concept of **-representation*.

Let \mathcal{D} be a dense domain in Hilbert space \mathcal{H} . As usual, we denote with $\mathcal{L}^\dagger(\mathcal{D})$ the space of all closable operators A with domain \mathcal{D} such that $D(A^*) \supset \mathcal{D}$ and both A and A^* leave \mathcal{D} invariant. As it is known, \mathcal{D} is a *-algebra with the usual operations $A + B, \lambda A, AB$, and the involution $A^\dagger = A^*|_{\mathcal{D}}$. Now let \mathcal{A} be a locally convex quasi *-algebra over \mathcal{A}_0 and π_o a *-representation of \mathcal{A}_0 , that is, a *-homomorphism from \mathcal{A}_0 into the *-algebra $\mathcal{L}^\dagger(\mathcal{D})$, for some dense domain \mathcal{D} . In general, extending π_o beyond \mathcal{A}_0 will force us to abandon the invariance of the domain \mathcal{D} . That is, for $A \in \mathcal{A} \setminus \mathcal{A}_0$, the extended representative $\pi(A)$ will only belong to $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ which is defined as the set of all closable operators X in \mathcal{H} such that $D(X) = \mathcal{D}$ and $D(X^*) \supset \mathcal{D}$ and it is a partial *-algebra (called partial O^* -algebra on \mathcal{D}) with the usual operations $X + Y, \lambda X$, the involution $X^\dagger = X^*|_{\mathcal{D}}$, and the weak product $X \square Y \equiv X^\dagger Y$ whenever $Y \mathcal{D} \subset D(X^\dagger)$ and $X^\dagger \mathcal{D} \subset D(Y^*)$.

It is also known that, defining on \mathfrak{D} a suitable (graph) topology, one can build up the rigged Hilbert space $\mathfrak{D} \subset \mathcal{H} \subset \mathfrak{D}'$, where \mathfrak{D}' is the dual of \mathfrak{D} [18] and one has

$$\mathcal{L}^\dagger(\mathfrak{D}) \subset \mathcal{L}(\mathfrak{D}, \mathfrak{D}'), \tag{2.4}$$

where $\mathcal{L}(\mathfrak{D}, \mathfrak{D}')$ denotes the space of all continuous linear maps from \mathfrak{D} into \mathfrak{D}' . Moreover, under additional topological assumptions, the following inclusions hold: $\mathcal{L}^\dagger(\mathfrak{D}) \subset \mathcal{L}^\dagger(\mathfrak{D}, \mathcal{H}) \subset \mathcal{L}(\mathfrak{D}, \mathfrak{D}')$. A more complete definition will be given in Section 4.

Let $(\mathcal{A}, \mathcal{A}_0)$ be a quasi $*$ -algebra, \mathfrak{D}_π a dense domain in a certain Hilbert space \mathcal{H}_π , and π a linear map from \mathcal{A} into $\mathcal{L}^\dagger(\mathfrak{D}_\pi, \mathcal{H}_\pi)$ such that

- (i) $\pi(a^*) = \pi(a)^\dagger$ for all $a \in \mathcal{A}$;
- (ii) if $a \in \mathcal{A}$, $x \in \mathcal{A}_0$, then $\pi(a) \square \pi(x)$ is well defined and $\pi(ax) = \pi(a) \square \pi(x)$.

We say that such a map π is a $*$ -representation of \mathcal{A} ;

- (iii) if $\pi(\mathcal{A}_0) \subset \mathcal{L}^\dagger(\mathfrak{D}_\pi)$, then π is a $*$ -representation of the quasi $*$ -algebra $(\mathcal{A}, \mathcal{A}_0)$.

Let π be a $*$ -representation of \mathcal{A} . The strong topology τ_s on $\pi(\mathcal{A})$ is the locally convex topology defined by the following family of seminorms: $\{p_\xi(\cdot); \xi \in \mathfrak{D}_\pi\}$, where $p_\xi(\pi(a)) \equiv \|\pi(a)\xi\|$, where $a \in \mathcal{A}$, $\xi \in \mathfrak{D}_\pi$.

For an overview on partial $*$ -algebras and related topics, we refer to [3].

3. $*$ -Derivations and their closability. Let $(\mathcal{A}, \mathcal{A}_0)$ be a quasi $*$ -algebra.

DEFINITION 3.1. A $*$ -derivation of \mathcal{A}_0 is a map $\delta : \mathcal{A}_0 \rightarrow \mathcal{A}$ with the following properties:

- (i) $\delta(x^*) = \delta(x)^*$ for all $x \in \mathcal{A}_0$;
- (ii) $\delta(\alpha x + \beta y) = \alpha\delta(x) + \beta\delta(y)$ for all $x, y \in \mathcal{A}_0$ and for all $\alpha, \beta \in \mathbb{C}$;
- (iii) $\delta(xy) = x\delta(y) + \delta(x)y$ for all $x, y \in \mathcal{A}_0$.

As we see, the $*$ -derivation is originally defined only on \mathcal{A}_0 . Nevertheless, it is clear that this is not the unique possibility at hand: δ could also be defined on the whole \mathcal{A} , or in a subset of \mathcal{A} containing \mathcal{A}_0 , under some continuity or closability assumption. Since the continuity of δ is a rather strong requirement, we consider here a weaker condition.

DEFINITION 3.2. A $*$ -derivation δ of \mathcal{A}_0 is said to be τ -closable if for any net $\{x_\alpha\} \subset \mathcal{A}_0$ such that $x_\alpha \xrightarrow{\tau} 0$ and $\delta(x_\alpha) \xrightarrow{\tau} b \in \mathcal{A}$, $b = 0$ results.

If δ is a τ -closable $*$ -derivation, then we define

$$D(\overline{\delta}) = \{a \in \mathcal{A} : \exists \{x_\alpha\} \subset \mathcal{A}_0 \text{ s.t. } \tau\text{-}\lim_\alpha x_\alpha = a, \delta(x_\alpha) \text{ converges in } \mathcal{A}\}. \tag{3.1}$$

Now, for any $a \in D(\overline{\delta})$, we put

$$\overline{\delta}(a) = \tau\text{-}\lim_\alpha \delta(x_\alpha), \tag{3.2}$$

and the following lemma holds.

LEMMA 3.3. *If $\delta(\mathcal{A}_0) \subset \mathcal{A}_0$, then $D(\overline{\delta})$ is a quasi $*$ -algebra over \mathcal{A}_0 .*

PROOF. First we observe that $D(\bar{\delta})$ is a complex vector space. In particular, it is closed under involution. In fact, from the definition itself, if $a \in D(\bar{\delta})$, then there exists a net $\{x_\alpha\}$ τ -converging to a . But, since the involution is τ -continuous, the net $\{x_\alpha^*\}$ is τ -converging to $a^* \in \mathcal{A}$. We conclude that whenever $a \in D(\bar{\delta})$, $a^* \in D(\bar{\delta})$.

Next we show that the multiplication of an element $a \in D(\bar{\delta})$ and $x \in \mathcal{A}_0$ is well defined. We consider here the product ax . The proof of the existence of xa is similar.

Since $a \in D(\bar{\delta})$, then there exists $\{x_\alpha\} \subset \mathcal{A}_0$ such that $x_\alpha \xrightarrow{\tau} a$. Moreover, the net $\delta(x_\alpha)$ τ -converges to an element $b \in \mathcal{A} : \delta(x_\alpha) \xrightarrow{\tau} b = \bar{\delta}(a)$. Recalling now that the multiplication is separately continuous and since, by assumptions, $\delta(x) \in \mathcal{A}_0$, we deduce that $\delta(x_\alpha x) = \delta(x_\alpha)x + x_\alpha\delta(x) \xrightarrow{\tau} \bar{\delta}(a)x + a\delta(x)$, which shows that ax belongs to $D(\bar{\delta})$ and that $\bar{\delta}(ax) = \tau\text{-}\lim_\alpha \delta(x_\alpha x)$. \square

This lemma shows that, under some assumptions, it is possible to extend δ to a set larger than \mathcal{A}_0 which, also if it is different from \mathcal{A} , is a quasi *-algebra over \mathcal{A}_0 itself. This result suggests the following rather general definition.

DEFINITION 3.4. Let $(\mathcal{A}, \mathcal{A}_0)$ be a quasi *-algebra and \mathcal{D} a vector subspace of \mathcal{A} such that $(\mathcal{D}, \mathcal{A}_0)$ is a quasi *-algebra. A map $\delta : \mathcal{D} \rightarrow \mathcal{A}$ is called a *-derivation if

- (i) $\delta(\mathcal{A}_0) \subset \mathcal{A}_0$ and $\delta_0 \equiv \delta|_{\mathcal{A}_0}$ is a *-derivation of \mathcal{A}_0 ;
- (ii) δ is linear;
- (iii) $\delta(ax) = a\delta(x) + \delta(a)x = a\delta_0(x) + \delta(a)x$ for all $a \in \mathcal{D}$ and for all $x \in \mathcal{A}_0$.

REMARK 3.5. Because of the previous results, if δ_0 is τ -closable, then its closure $\bar{\delta}_0$ is a *-derivation defined on $D(\bar{\delta}_0)$.

Now we look for conditions for a *-derivation δ to be closable, making use of some duality result. For that, we first recall that if $(\mathcal{A}[\tau], \mathcal{A}_0)$ is a locally convex quasi *-algebra and δ is a *-derivation of \mathcal{A}_0 , we can define the adjoint derivation δ' acting on a subspace $D(\delta')$ of the dual space \mathcal{A}' of \mathcal{A} . The derivation δ' is first defined, for $\omega \in \mathcal{A}'$ and $x \in \mathcal{A}_0$, by $(\delta'\omega)(x) = \omega(\delta(x))$ and then extended to the domain

$$D(\delta') = \{\omega \in \mathcal{A}' : \delta'\omega \text{ has a continuous extension to } \mathcal{A}\}. \tag{3.3}$$

A classical result, [20], states that δ is τ -closable if and only if $D(\delta')$ is $\sigma(\mathcal{A}', \mathcal{A})$ -dense in \mathcal{A}' . We now prove the following result.

PROPOSITION 3.6. Let $\delta : \mathcal{A}_0 \rightarrow \mathcal{A}$ be a *-derivation. Assume that there exists $\omega \in \mathcal{A}'$ such that $\omega|_{\mathcal{A}_0}$ is a positive linear functional on \mathcal{A}_0 and

- (1) $\omega \circ \delta$ is τ -continuous on \mathcal{A}_0 ;
- (2) the GNS representation π_ω of \mathcal{A}_0 is faithful.

Then δ is τ -closable.

PROOF. First we notice that condition (3.1) above implies that $\omega \in D(\delta')$. Secondly, let $x, y, z \in \mathcal{A}_0$. Since $\omega(x\delta(y)z) = \omega(\delta(xyz)) - \omega(\delta(x)yz) - \omega(xy\delta(z))$, we have, as a consequence of the continuity of $\omega \circ \delta$ and of ω itself,

$$|\omega(x\delta(y)z)| \leq p_\alpha(xyz) + p_\beta(\delta(x)yz) + p_\gamma(xy\delta(z)) \leq C_{x,z}p_\sigma(y), \tag{3.4}$$

where we have also used the continuity of the multiplication. $C_{x,z}$ is a suitable positive constant depending on both x and z . We further define a new linear functional $\omega_{x,z}(y) = \omega(xy z)$. Of course we have $|\omega(xy z)| \leq D_{x,z} p_\alpha(y)$ for some seminorm p_α and a positive constant $D_{x,z}$. It follows that $\omega_{x,z}$ has a continuous extension to \mathcal{A} , which we still denote with the same symbol. Moreover, since $(\delta' \omega_{x,z})(y) = \omega_{x,z}(\delta(y)) = \omega(x\delta(y)z)$, we have $|(\delta' \omega_{x,z})(y)| \leq C_{x,z} p_\sigma(y)$ for every $y \in \mathcal{A}_0$. This implies that $\omega_{x,z}$ belongs to $D(\delta')$ or, in other words, that $\omega_{x,z}$ has a continuous extension to \mathcal{A} . For this reason, we have $D(\delta') \supset \text{linear span}\{\omega_{x,z} : x, z \in \mathcal{A}_0\}$, and this set is dense in \mathcal{A}' . Were it not so, then there would exist a nonzero element $y \in \mathcal{A}_0$ such that $\omega_{x,z}(y) = 0$ for all $x, z \in \mathcal{A}_0$. But this is in contrast with the faithfulness of the GNS-representation π_ω since we would also have $\omega(xy z) = \langle \pi_\omega(y) \lambda_\omega(z), \lambda_\omega(x^*) \rangle = 0$ for all $x, z \in \mathcal{A}_0$, which, in turn, would imply that $\pi_\omega(y) = 0$. \square

4. Spatiality of *-derivations induced by *-representations. Let $(\mathcal{A}, \mathcal{A}_0)$ be a quasi *-algebra and δ a *-derivation of \mathcal{A}_0 as defined in Section 3. Let π be a *-representation of $(\mathcal{A}, \mathcal{A}_0)$. We always assume that whenever $x \in \mathcal{A}_0$ is such that $\pi(x) = 0$, $\pi(\delta(x)) = 0$ as well. Under this assumption, the linear map

$$\delta_\pi(\pi(x)) = \pi(\delta(x)), \quad x \in \mathcal{A}_0, \tag{4.1}$$

is well defined on $\pi(\mathcal{A}_0)$ with values in $\pi(\mathcal{A})$ and it is a *-derivation of $\pi(\mathcal{A}_0)$. We call δ_π the *-derivation induced by π .

Given such a representation π and its dense domain \mathcal{D}_π , we consider the usual graph topology t_\dagger generated by the seminorms

$$\xi \in \mathcal{D}_\pi \rightarrow \|A\xi\|, \quad A \in \mathcal{L}^\dagger(\mathcal{D}_\pi). \tag{4.2}$$

Calling \mathcal{D}'_π the conjugate dual of \mathcal{D}_π , we get the usual rigged Hilbert space $\mathcal{D}_\pi[t_\dagger] \subset \mathcal{H}_\pi \subset \mathcal{D}'_\pi[t'_\dagger]$, where t'_\dagger denotes the strong dual topology of \mathcal{D}'_π . As usual, we denote with $\mathcal{L}(\mathcal{D}_\pi, \mathcal{D}'_\pi)$ the space of all continuous linear maps from $\mathcal{D}_\pi[t_\dagger]$ into $\mathcal{D}'_\pi[t'_\dagger]$, and with $\mathcal{L}^\dagger(\mathcal{D}_\pi)$ the *-algebra of all operators A in \mathcal{H}_π such that both A and its adjoint A^* map \mathcal{D}_π into itself. In this case, $\mathcal{L}^\dagger(\mathcal{D}_\pi) \subset \mathcal{L}(\mathcal{D}_\pi, \mathcal{D}'_\pi)$. Each operator $A \in \mathcal{L}^\dagger(\mathcal{D}_\pi)$ can be extended to all of \mathcal{D}'_π in the following way:

$$\langle \hat{A}\xi', \eta \rangle = \langle \xi', A^\dagger \eta \rangle \quad \forall \xi' \in \mathcal{D}'_\pi, \eta \in \mathcal{D}_\pi. \tag{4.3}$$

Therefore, the multiplication of $X \in \mathcal{L}(\mathcal{D}_\pi, \mathcal{D}'_\pi)$ and $A \in \mathcal{L}^\dagger(\mathcal{D}_\pi)$ can always be defined:

$$(X \circ A)\xi = X(A\xi), \quad (A \circ X)\xi = \hat{A}(X\xi) \quad \forall \xi \in \mathcal{D}_\pi. \tag{4.4}$$

With these definitions, it is known that $(\mathcal{L}(\mathcal{D}_\pi, \mathcal{D}'_\pi), \mathcal{L}^\dagger(\mathcal{D}_\pi))$ is a quasi *-algebra.

We can now prove the following theorem.

THEOREM 4.1. *Let $(\mathcal{A}, \mathcal{A}_0)$ be a locally convex quasi *-algebra with identity and δ a *-derivation of \mathcal{A}_0 . Then the following statements are equivalent.*

(i) *There exists a $(\tau - \tau_s)$ -continuous, ultra-cyclic *-representation π of \mathcal{A} , with ultra-cyclic vector ξ_0 , such that the *-derivation δ_π induced by π is spatial, that is, there exists*

$H = H^\dagger \in \mathcal{L}(\mathcal{D}_\pi, \mathcal{D}'_\pi)$ such that $H\xi_0 \in \mathcal{H}_\pi$ and

$$\delta_\pi(\pi(x)) = i\{H \circ \pi(x) - \pi(x) \circ H\} \quad \forall x \in \mathcal{A}_0. \tag{4.5}$$

(ii) There exists a positive linear functional f on \mathcal{A}_0 such that

$$f(x^*x) \leq p(x)^2 \quad \forall x \in \mathcal{A}_0, \tag{4.6}$$

for some continuous seminorm p of τ and, denoting with \tilde{f} the continuous extension of f to \mathcal{A} , the following inequality holds:

$$|\tilde{f}(\delta(x))| \leq C(\sqrt{f(x^*x)} + \sqrt{f(xx^*)}) \quad \forall x \in \mathcal{A}_0 \tag{4.7}$$

and for some positive constant C .

(iii) There exists a positive sesquilinear form φ on $\mathcal{A} \times \mathcal{A}$ such that φ is invariant, that is,

$$\varphi(ax, y) = \varphi(x, a^*y) \quad \forall a \in \mathcal{A}, x, y \in \mathcal{A}_0; \tag{4.8}$$

φ is τ -continuous, that is,

$$|\varphi(a, b)| \leq p(a)p(b) \quad \forall a, b \in \mathcal{A} \tag{4.9}$$

and for some continuous seminorm p of τ ; and φ satisfies the following inequality:

$$|\varphi(\delta(x), \mathbf{1})| \leq C(\sqrt{\varphi(x, x)} + \sqrt{\varphi(x^*, x^*)}) \quad \forall x \in \mathcal{A}_0 \tag{4.10}$$

and for some positive constant C .

PROOF. First we show that (i) implies (ii).

We recall that the ultra-cyclicity of the vector ξ_0 means that $\mathcal{D}_\pi = \pi(\mathcal{A}_0)\xi_0$. Therefore, the map defined as

$$f(x) = \langle \pi(x)\xi_0, \xi_0 \rangle, \quad x \in \mathcal{A}_0, \tag{4.11}$$

is a positive linear functional on \mathcal{A}_0 . Moreover, since $f(x^*x) = \|\pi(x)\xi_0\|^2$, (4.6) follows because of the $(\tau - \tau_s)$ -continuity of π . As for (4.7), it is clear first of all that f has a unique extension to \mathcal{A} defined as

$$\tilde{f}(a) = \langle \pi(a)\xi_0, \xi_0 \rangle, \quad a \in \mathcal{A}, \tag{4.12}$$

due to the $(\tau - \tau_s)$ -continuity of π . Therefore, we have, using (4.5),

$$\begin{aligned} |\tilde{f}(\delta(x))| &= |\langle H \circ \pi(x)\xi_0, \xi_0 \rangle - \langle H\xi_0, \pi(x^*)\xi_0 \rangle| \\ &\leq \|H\xi_0\| \left(\langle \pi(x)\xi_0, \pi(x)\xi_0 \rangle^{1/2} + \langle \pi(x^*)\xi_0, \pi(x^*)\xi_0 \rangle^{1/2} \right) \end{aligned} \tag{4.13}$$

so that inequality (4.7) follows with $C = \|H\xi_0\|$.

We now prove that (ii) implies (iii). For that, we define a sesquilinear form φ in the following way: let a, b be in \mathcal{A} and let $\{x_\alpha\}, \{y_\beta\}$ be two nets in \mathcal{A}_0 , τ -converging, respectively, to a and b . We put

$$\varphi(a, b) = \lim_{\alpha, \beta} f(y_\beta^* x_\alpha). \tag{4.14}$$

It is readily checked that φ is well defined. The proofs of (4.8), (4.9), and (4.10) are easy consequences of definition (4.14) together with the properties of f .

To conclude the proof, we still have to check that (iii) implies (i).

Given φ as in (iii) above, we consider the GNS-construction generated by φ .

Let $\mathcal{N}_\varphi = \{a \in \mathcal{A}; \varphi(a, a) = 0\}$; then $\mathcal{A}/\mathcal{N}_\varphi = \{\lambda_\varphi(a) = a + \mathcal{N}_\varphi, a \in \mathcal{A}\}$ is a pre-Hilbert space with inner product $\langle \lambda_\varphi(a), \lambda_\varphi(b) \rangle = \varphi(a, b)$, $a, b \in \lambda_\varphi(\mathcal{A})$. We call \mathcal{H}_φ the completion of $\lambda_\varphi(\mathcal{A})$ in the norm $\|\cdot\|_\varphi$ given by this inner product. It is easy to check that $\lambda_\varphi(\mathcal{A}_0)$ is $\|\cdot\|_\varphi$ -dense in \mathcal{H}_φ . In fact, due to the definition of locally convex quasi $*$ -algebra, given $a \in \mathcal{A}$, there exists a net $x_\alpha \subset \mathcal{A}_0$ such that $x_\alpha \xrightarrow{\tau} a$. Therefore, we have, using the continuity of φ ,

$$\|\lambda_\varphi(a) - \lambda_\varphi(x_\alpha)\|_\varphi^2 = \|\lambda_\varphi(a - x_\alpha)\|_\varphi^2 = \varphi(a - x_\alpha, a - x_\alpha) \leq p(a - x_\alpha)^2 \rightarrow 0. \tag{4.15}$$

We can now define a $*$ -representation π_φ with ultra-cyclic vector $\lambda_\varphi(\mathbf{1})$ as follows:

$$\pi_\varphi(a)\lambda_\varphi(x) = \lambda_\varphi(ax), \quad a \in \mathcal{A}, x \in \mathcal{A}_0. \tag{4.16}$$

In particular, the fact that $\lambda_\varphi(\mathbf{1})$ is ultra-cyclic follows from the fact that $\pi_\varphi(\mathcal{A}_0)\lambda_\varphi(\mathbf{1}) = \lambda_\varphi(\mathcal{A}_0)$ is dense in \mathcal{H}_φ . Moreover, the representation π_φ is also $(\tau - \tau_s)$ -continuous; in fact, taking $a \in \mathcal{A}$ and $x \in \mathcal{A}_0$, we have

$$\|\pi_\varphi(a)\lambda_\varphi(x)\|_\varphi^2 = \|\lambda_\varphi(ax)\|_\varphi^2 = \varphi(ax, ax) \leq (p(ax))^2 \leq y_x(p'(a))^2. \tag{4.17}$$

The last inequality follows from the continuity of the multiplication. This inequality shows that whenever $\tau - \lim_\alpha x_\alpha = a$, then $\tau_s - \lim_\alpha \pi_\varphi(x_\alpha) = \pi_\varphi(a)$.

This construction produces a $*$ -representation π_φ with all the properties required for π in (i). As a consequence, we can define a $*$ -derivation δ_{π_φ} induced by π_φ as in (4.1): $\delta_{\pi_\varphi}(\pi_\varphi(x)) = \pi_\varphi(\delta(x))$ for $x \in \mathcal{A}_0$. The proof of the spatiality of δ_{π_φ} generalizes the proof of the analogous statement for C^* -algebras (see, e.g., [15]).

Let $\overline{\mathcal{H}_\varphi}$ be the conjugate space of \mathcal{H}_φ , with inner product

$$\langle \lambda_\varphi(x), \lambda_\varphi(y) \rangle_{\overline{\mathcal{H}_\varphi}} = \langle \lambda_\varphi(y), \lambda_\varphi(x) \rangle_{\mathcal{H}_\varphi}. \tag{4.18}$$

From now on, we will indicate with the same symbol $\langle \cdot, \cdot \rangle$ all the inner products whenever no possibility of confusion arises.

Let \mathcal{M}_φ be the subspace of $\mathcal{H}_\varphi \oplus \overline{\mathcal{H}_\varphi}$ spanned by the vectors $\{\lambda_\varphi(x), \lambda_\varphi(x^*)\}$, $x \in \mathcal{A}_0$. We define a linear functional F_φ on \mathcal{M}_φ by

$$F_\varphi(\{\lambda_\varphi(x), \lambda_\varphi(x^*)\}) = i\varphi(\delta(x), \mathbf{1}), \quad x \in \mathcal{A}_0. \tag{4.19}$$

Inequality (4.10), together with the equality $\|\{\lambda_\varphi(x), \lambda_\varphi(x^*)\}\|^2 = \varphi(x, x) + \varphi(x^*, x^*)$, shows that f_φ is indeed continuous so that by Riesz's lemma, there exists a vector $\{\xi_1, \xi_2\} \in \mathcal{H}_\varphi \oplus \overline{\mathcal{H}_\varphi}$ such that

$$\begin{aligned} F_\varphi(\{\lambda_\varphi(x), \lambda_\varphi(x^*)\}) &= \langle \{\lambda_\varphi(x), \lambda_\varphi(x^*)\}, \{\xi_1, \xi_2\} \rangle \\ &= \langle \lambda_\varphi(x), \xi_1 \rangle + \langle \xi_2, \lambda_\varphi(x^*) \rangle. \end{aligned} \tag{4.20}$$

Using the invariance of φ , we also deduce that

$$F_\varphi(\{\lambda_\varphi(x), \lambda_\varphi(x^*)\}) = i\varphi(\delta(x), \mathbf{1}) = -\overline{i\varphi(\delta(x^*), \mathbf{1})}, \tag{4.21}$$

which, together with (4.20), gives

$$\frac{1}{i}\varphi(\delta(x), \mathbf{1}) = \langle \lambda_\varphi(x), \eta \rangle - \langle \eta, \lambda_\varphi(x^*) \rangle, \quad x \in \mathcal{A}_0, \tag{4.22}$$

where we have introduced the vector η as

$$\eta = \frac{\xi_2 - \xi_1}{2}. \tag{4.23}$$

Now we define the operator H in the following way:

$$H\lambda_\varphi(x) = \frac{1}{i}\lambda_\varphi(\delta(x)) + \hat{\pi}_\varphi(x)\eta, \quad x \in \mathcal{A}_0, \tag{4.24}$$

where $\hat{\pi}_\varphi$ indicates the extension of π_φ , defined in the usual way, which we need to introduce since η belongs to \mathcal{H}_φ and not to $\mathcal{D}_{\pi_\varphi}$ in general.

First of all, we notice from (4.24) that $H\lambda_\varphi(\mathbf{1}) = \eta \in \mathcal{H}_\varphi$, as stated in (i). Moreover, H is also well defined and symmetric since for all $x, y \in \mathcal{A}_0$,

$$\begin{aligned} &\langle H\pi_\varphi(x)\lambda_\varphi(\mathbf{1}), \pi_\varphi(y)\lambda_\varphi(\mathbf{1}) \rangle - \langle \pi_\varphi(x)\lambda_\varphi(\mathbf{1}), H\pi_\varphi(y)\lambda_\varphi(\mathbf{1}) \rangle \\ &= \langle H\lambda_\varphi(x), \lambda_\varphi(y) \rangle - \langle \lambda_\varphi(x), H\lambda_\varphi(y) \rangle \\ &= \left\langle \left(\frac{1}{i}\lambda_\varphi(\delta(x)) + \hat{\pi}_\varphi(x)\eta \right), \lambda_\varphi(y) \right\rangle - \left\langle \lambda_\varphi(x), \left(\frac{1}{i}\lambda_\varphi(\delta(y)) + \hat{\pi}_\varphi(y)\eta \right) \right\rangle \tag{4.25} \\ &= \frac{1}{i}(\varphi(\delta(x), y) + \varphi(x, \delta(y))) + \langle \hat{\pi}_\varphi(x)\eta, \lambda_\varphi(y) \rangle - \langle \lambda_\varphi(x), \hat{\pi}_\varphi(y)\eta \rangle \\ &= \frac{1}{i}\varphi(\delta(y^*x), \mathbf{1}) + \langle \eta, \lambda_\varphi(x^*y) \rangle - \langle \lambda_\varphi(y^*x), \eta \rangle = 0. \end{aligned}$$

This last equality follows from (4.22). We finally have to prove that H implements the derivation δ_{π_φ} . For this, let $x, y, z \in \mathcal{A}_0$. Then we have

$$\begin{aligned}
 & i(\langle H \circ \pi_\varphi(x) \lambda_\varphi(y), \lambda_\varphi(z) \rangle - \langle \pi_\varphi(x) \circ H \lambda_\varphi(y), \lambda_\varphi(z) \rangle) \\
 &= i(\langle H \lambda_\varphi(xy), \lambda_\varphi(z) \rangle - \langle H \lambda_\varphi(y), \lambda_\varphi(x^*y) \rangle) \\
 &= i \left(\left\langle \frac{1}{i} \lambda_\varphi(\delta(xy)) + \hat{\pi}_\varphi(xy) \eta, \lambda_\varphi(z) \right\rangle \right. \\
 &\quad \left. - \left\langle \frac{1}{i} \lambda_\varphi(\delta(y)) + \hat{\pi}_\varphi(y) \eta, \lambda_\varphi(x^*z) \right\rangle \right) \\
 &= \varphi(\delta(x)y, z) = \langle \pi_\varphi(\delta(x)) \lambda_\varphi(y), \lambda_\varphi(z) \rangle.
 \end{aligned} \tag{4.26}$$

Again, we made use of (4.22). □

REMARK 4.2. If we add to a spatial *-derivation δ_0 a *perturbation* δ_p such that $\delta = \delta_0 + \delta_p$ is again a *-derivation, it is reasonable to consider the question as to whether δ is still spatial. The answer is positive under very general (and natural) assumptions: since δ_0 is spatial, the above proposition states that there exists a positive linear functional f on \mathcal{A}_0 whose extension \tilde{f} satisfies, among the others, inequality (4.7): $|\tilde{f}(\delta_0(x))| \leq C(\sqrt{f(x^*x)} + \sqrt{f(xx^*)})$ for all $x \in \mathcal{A}_0$. If we require that δ_p satisfies the inequality $|\tilde{f}(\delta_p(x))| \leq |\tilde{f}(\delta_0(x))|$ for all $x \in \mathcal{A}_0$, which is exactly what we expect since δ_p is *small compared to* δ_0 , we first deduce that δ_p is spatial and, since for all $x \in \mathcal{A}_0$, $|\tilde{f}(\delta(x))| \leq 2C(\sqrt{f(x^*x)} + \sqrt{f(xx^*)})$, using the same proposition, we deduce that δ is spatial too. If H, H_0 , and H_p denote the operators that implement, respectively, δ, δ_0 , and δ_p , we also get the equality $i[H, A]\psi = i[H_0 + H_p, A]\psi$ for all $A \in \mathcal{L}^+(\mathcal{D}_\pi)$ and $\psi \in \mathcal{D}_\pi$.

The problem of the spatiality of a derivation is particularly interesting when dealing with quantum systems with infinite degrees of freedom. The reason is that for these systems, we need to introduce a regularizing cutoff in their descriptions and remove this cutoff only at the very end. Specifically, something like this can happen: the physical system \mathcal{S} is associated to, say, the whole space \mathbb{R}^3 . In order to describe the dynamics of \mathcal{S} , the canonical approach (see [15] and the references therein) consists in considering a subspace $V \subset \mathbb{R}^3$, the physical system \mathcal{S}_V which naturally lives in this region, and writing down the so-called local Hamiltonian H_V for \mathcal{S}_V . This Hamiltonian is a selfadjoint bounded operator which implements the infinitesimal dynamics δ_V of \mathcal{S}_V . To obtain information about the dynamics for \mathcal{S} , we need to compute a limit (in V) to *remove the cutoff*. This can be a problem already at this infinitesimal level (see also [14] and the references therein) and becomes harder and harder, in general, when the interest is moved to the finite form of the algebraic dynamics, that is, when we try to integrate the derivation. Among the other things, for instance, it may happen that the net H_V or the related net δ_V (or both) does not converge in any reasonable topology, or that δ_V is not spatial. Another possibility that may occur is the following: H_V converges (in some topology) to a certain operator H and δ_V converges (in some other topology) to a certain *-derivation δ , but δ is not spatial or, even if it is, H is not the operator which implements δ .

However, under some reasonable conditions, all these possibilities can be controlled. The situation is governed by the next proposition which is based on the assumption that there exist a $(\tau - \tau_s)$ -continuous *-representation π in the Hilbert space \mathcal{H}_π , which is ultra-cyclic with ultra-cyclic vector ξ_0 , and a family of *-derivations (in the sense of Definition 3.1) $\{\delta_n : n \in \mathbb{N}\}$ of the *-algebra \mathcal{A}_0 with identity. We define a related family of *-derivations $\delta_\pi^{(n)}$ induced by π , defined on $\pi(\mathcal{A}_0)$, and with values in $\pi(\mathcal{A})$:

$$\delta_\pi^{(n)}(\pi(x)) = \pi(\delta_n(x)), \quad x \in \mathcal{A}_0. \tag{4.27}$$

PROPOSITION 4.3. *Suppose that*

- (i) $\{\delta_n(x)\}$ is τ -Cauchy for all $x \in \mathcal{A}_0$;
- (ii) for each $n \in \mathbb{N}$, $\delta_\pi^{(n)}$ is spatial, that is, there exists an operator H_n such that

$$H_n = H_n^\dagger \in \mathcal{L}(\mathcal{D}_\pi, \mathcal{D}'_\pi),$$

$$H_n \xi_0 \in \mathcal{H}_\pi, \quad \delta_\pi^{(n)}(\pi(x)) = i\{H_n \circ \pi(x) - \pi(x) \circ H_n\} \quad \forall x \in \mathcal{A}_0; \tag{4.28}$$

- (iii) $\sup_n \|H_n \xi_0\| =: L < \infty$.

Then

- (a) there exists $\delta(x) = \tau - \lim \delta_n(x)$ for all $x \in \mathcal{A}_0$, which is a *-derivation of \mathcal{A}_0 ;
- (b) δ_π , the *-derivation induced by π , is well defined and spatial;
- (c) if H is the selfadjoint operator which implements δ_π , and if $\langle (H_n - H)\xi_0, \xi \rangle \rightarrow 0$ for all $\xi \in \mathcal{D}_\pi$, then H_n converges weakly to H .

PROOF. (a) This first statement is trivial.

(b) For $a, b \in \mathcal{A}$, we put $\varphi(a, b) = \langle \pi(a)\xi_0, \pi(b)\xi_0 \rangle$. Then φ is an invariant positive sesquilinear form on $\mathcal{A} \times \mathcal{A}$ since

$$\begin{aligned} \varphi(ax, y) &= \langle \pi(ax)\xi_0, \pi(y)\xi_0 \rangle = \langle \pi(a)\pi(x)\xi_0, \pi(y)\xi_0 \rangle \\ &= \langle \pi(x)\xi_0, \pi(a^*)\pi(y)\xi_0 \rangle = \varphi(x, a^*y) \end{aligned} \tag{4.29}$$

for all $a \in \mathcal{A}$ and $x, y \in \mathcal{A}_0$. φ is τ -continuous: if $a, b \in \mathcal{A}$,

$$|\varphi(a, b)| = |\langle \pi(a)\xi_0, \pi(b)\xi_0 \rangle| \leq \|\pi(a)\xi_0\| \|\pi(b)\xi_0\| \leq p_\alpha(a)p_\alpha(b), \tag{4.30}$$

for some continuous seminorm p_α on \mathcal{A} , because of the $(\tau - \tau_s)$ -continuity of π .

From this inequality, we deduce that for $x \in \mathcal{A}_0$,

$$\begin{aligned} |\varphi(\delta(x), \mathbf{1})| &= \lim_n |\varphi(\delta_n(x), \mathbf{1})| \\ &= \lim_n |\langle H_n \circ \pi(x)\xi_0, \xi_0 \rangle - \langle \pi(x) \circ H_n \xi_0, \xi_0 \rangle| \\ &= \limsup_n |\langle H_n \circ \pi(x)\xi_0, \xi_0 \rangle - \langle \pi(x) \circ H_n \xi_0, \xi_0 \rangle| \\ &\leq \limsup_n \|H_n \xi_0\| (\|\pi(x)\xi_0\| + \|\pi(x^*)\xi_0\|) \\ &\leq L \left(\sqrt{\varphi(x, x)} + \sqrt{\varphi(x^*, x^*)} \right). \end{aligned} \tag{4.31}$$

This sesquilinear form φ satisfies all the conditions required in [Theorem 4.1\(iii\)](#). Then, following the same steps as in the proof of [Theorem 4.1](#), (iii) \Rightarrow (i), we construct the GNS-representation π_φ associated to φ . We call \mathcal{H}_φ , ξ_φ , and H_φ , respectively, the Hilbert space, the ultra-cyclic vector, and the symmetric operator implementing the derivation associated to π_φ . Among others, the following equality must be satisfied:

$$\varphi(a, b) = \langle \pi(a)\xi_0, \pi(b)\xi_0 \rangle = \langle \pi_\varphi(a)\xi_\varphi, \pi_\varphi(b)\xi_\varphi \rangle \quad \forall a, b \in \mathcal{A}, \tag{4.32}$$

which implies that π_φ and π are unitarily equivalent, that is, there exists a unitary operator $U : \mathcal{H}_\pi \rightarrow \mathcal{H}_\varphi$ such that $U\xi_0 = \xi_\varphi$, $U\pi(a)U^{-1} = \pi_\varphi(a)$, for all $a \in \mathcal{A}$, and U is continuous from $D_\pi[t_\pi]$ into $D_\varphi[t_\varphi]$. We prove here only this last property. Let $x, y \in \mathcal{A}_0$; we have

$$\|\pi_\varphi(y)U\pi(x)\xi_0\|_\varphi = \|U\pi(y)\pi(x)\xi_0\|_\varphi = \|\pi(y)\pi(x)\xi_0\|, \tag{4.33}$$

which implies that U^* can be extended to an operator $U^\dagger : \mathcal{D}'_\varphi \rightarrow \mathcal{D}'_\pi$. We have now

$$\delta_{\pi_\varphi}(\pi_\varphi(x)) = \pi_\varphi(\delta(x)) = U\pi(\delta(x))U^{-1} = U\delta_\pi(\pi(x))U^{-1}, \tag{4.34}$$

which implies that $\delta_\pi(\pi(x)) = U^{-1}\delta_{\pi_\varphi}(\pi_\varphi(x))U$. Since δ_{π_φ} is well defined, this equality implies that also δ_π is well defined. Indeed, we have

$$\pi(x) = 0 \implies \pi_\varphi(x) = 0 \implies \delta_{\pi_\varphi}(\pi_\varphi(x)) = 0 \implies \delta_\pi(\pi(x)) = 0. \tag{4.35}$$

Now we define $H = U^{-1}H_\varphi U|_{\mathcal{D}_\pi}$. Then

$$\begin{aligned} \delta_\pi(\pi(x)) &= U^{-1}\delta_{\pi_\varphi}(\pi_\varphi(x))U \\ &= iU^{-1}(H_\varphi \circ \pi_\varphi(x) - \pi_\varphi(x) \circ H_\varphi)U \\ &= i(U^{-1}H_\varphi U \circ U^{-1}\pi_\varphi(x)U - U^{-1}\pi_\varphi(x)U \circ U^{-1}H_\varphi U) \\ &= i(H \circ \pi(x) - \pi(x) \circ H), \end{aligned} \tag{4.36}$$

which allows us to conclude.

(c) For $x, y, z \in \mathcal{A}_0$, we have, using the definition of φ and its τ -continuity,

$$\begin{aligned} \varphi(\delta_n(x)y, z) &= \langle \delta_\pi^{(n)}(\pi(x))\pi(y)\xi_0, \pi(z)\xi_0 \rangle \\ &= i(\langle (H_n \circ \pi(x))\pi(y)\xi_0, \pi(z)\xi_0 \rangle \\ &\quad - \langle (\pi(x) \circ H_n)\pi(y)\xi_0, \pi(z)\xi_0 \rangle) \\ &\rightarrow \varphi(\delta(x)y, z). \end{aligned} \tag{4.37}$$

Since $\langle (\pi(x) \circ H_n)\pi(y)\xi_0, \pi(z)\xi_0 \rangle = \langle H_n\pi(y)\xi_0, \pi(x^*z)\xi_0 \rangle$, we deduce that, taking $y = \mathbf{1}$, $\langle (\pi(x) \circ H_n)\xi_0, \pi(z)\xi_0 \rangle = \langle H_n\xi_0, \pi(x^*z)\xi_0 \rangle \rightarrow \langle H\xi_0, \pi(x^*z)\xi_0 \rangle$ because of the assumption on H_n . Then, since

$$\begin{aligned} \varphi(\delta(x), z) &= i\langle (H \circ \pi(x))\xi_0, \pi(z)\xi_0 \rangle \\ &\quad - \langle (\pi(x) \circ H)\xi_0, \pi(z)\xi_0 \rangle, \end{aligned} \tag{4.38}$$

we get, by (4.37), that $\langle (H_n\pi(x))\xi_0, \pi(z)\xi_0 \rangle \rightarrow \langle (H\pi(x))\xi_0, \pi(z)\xi_0 \rangle$ for all $x, z \in \mathcal{A}_0$. Then H_n converges weakly to H . \square

EXAMPLE 4.4 (a radiation model). In this example, the representation π is just the identity map. We consider a model of n free bosons, [6], whose dynamics is given by the Hamiltonian, $H = \sum_{i=1}^n a_i^\dagger a_i$. Here a_i and a_i^\dagger are, respectively, the annihilation and creation operators for the i th mode. They satisfy the following CCR:

$$[a_i, a_j^\dagger] = \mathbf{1}\delta_{i,j}. \tag{4.39}$$

Let Q_L be the projection operator on the subspace of \mathcal{H} with at most L bosons. This operator can be written considering the spectral decomposition of $H_{(i)} = a_i^\dagger a_i = \sum_{l=0}^\infty lE_l^{(i)}$. We have $Q_L = \sum_{i=1}^n \sum_{l=0}^L E_l^{(i)}$. We now define a bounded operator H_L in \mathcal{H} by $H_L = Q_L H Q_L$. It is easy to check that, for any vector Φ_M with M bosons (i.e., an eigenstate of the number operator $N = H = \sum_{i=1}^n a_i^\dagger a_i$ with eigenvalue M), the condition $\sup_L \|H_L \Phi_M\| < \infty$ is satisfied. In particular, for instance, $\sup_L \|H_L \Phi_0\| = 0$. It may be worth remarking that all the vectors Φ_M are cyclic. Denoting with δ_L the derivation implemented by H_L and by δ the one implemented by H , it is clear that all the assumptions of Proposition 4.3 are satisfied so that, in particular, the weak convergence of H_L to H follows. This is not surprising since it is known that H_L converges to H strongly on a dense domain [6].

EXAMPLE 4.5 (a mean-field spin model). The situation described here is quite different from the one in the previous example. First of all (see [8, 10]) there exists no Hamiltonian for the whole physical system but only for a finite volume subsystem: $H_V = (1/|V|) \sum_{i,j \in V} \sigma_3^i \sigma_3^j$, where i and j are the indices of the lattice site, σ_3^i is the third component of the Pauli matrices, V is the volume cutoff, and $|V|$ is the number of the lattice sites in V . It is convenient to introduce the *mean magnetization* operator $\sigma_3^V = (1/|V|) \sum_{i \in V} \sigma_3^i$. We indicate with \uparrow_i and \downarrow_i the eigenstates of σ_3^i with eigenvalues $+1$ and -1 , respectively. We define $\Phi_1 = \otimes_{i \in V} \uparrow_i$. It is clear that $\sigma_3^V \Phi_1 = \Phi_1$, which implies that $H_V \Phi_1 = |V| \Phi_1$, which in turn implies that $\sup_V \|H_V \Phi_1\| = \infty$. This means that the cyclic vector Φ_1 does not satisfy the main assumption of Proposition 4.3, and for this reason, nothing can be said about the convergence of H_V . However, it is possible to consider a different cyclic vector

$$\Phi_0 = \cdots \otimes \uparrow_{j-1} \otimes \downarrow_j \otimes \uparrow_{j+1} \otimes \downarrow_{j+2} \otimes \cdots \tag{4.40}$$

which is again an eigenstate of σ_3^V . Its eigenvalue depends on the volume V . However, it is clear that $\|\sigma_3^V \Phi_0\| = (1/|V|)\|\Phi_0\|\epsilon_V$, where ϵ_V can take only the values 0, 1. Analogously, we have $\|H_V \Phi_0\| = (1/|V|)\|\Phi_0\|\epsilon_V^2 \rightarrow 0$. This means that this vector satisfies the assumptions of [Proposition 4.3](#) so that the derivation $\delta_V(\cdot) = i[H_V, \cdot]$ converges to a derivation δ which is spatial and implemented by H , and that H_V is weakly convergent to H .

As we see, contrary to [Example 4.4](#), the choice of the cyclic vector which we take as our starting point is very important in order to be able to prove the existence of δ , its spatiality, and convergence of H_V to a limit operator. It is also worth remarking that the same conclusions could also be found replacing Φ_0 with any vector which can be obtained as a local perturbation of Φ_0 itself.

REMARK 4.6. All the results we have proved above can be specialized to CQ^* -algebras, which can be considered as a particular example of locally convex quasi $*$ -algebras. The main difference in this case concerns statement (c) of [Proposition 4.3](#): the weak convergence of H_n to H , in this case, is replaced by a strong convergence. In more details, referring to the example of [Section 2](#) and calling $\Omega \in \mathcal{H}_{+1}$ a cyclic vector, we can prove that if $\|(H_n - H)\Omega\|_{-1} \rightarrow 0$, then $\|(H_n - H)A\Omega\|_{-1} \rightarrow 0$ for all $A \in B(\mathcal{H}_{+1})$.

The following result gives an interplay between the results of this section and of the previous sections. In particular, we consider now the possibility of extending the domain of definition of the derivation δ (as we did in [Section 3](#)) defined as a limit of a net of derivations δ_n (as we have done in this section). For this, we first need the following definition.

DEFINITION 4.7. Let $(\mathcal{A}[\tau], \mathcal{A}_0)$ be a locally convex quasi $*$ -algebra. A sequence $\{\delta_n\}$ of $*$ -derivations is called uniformly τ -continuous if for any continuous seminorm p on \mathcal{A} , there exists a continuous seminorm q on \mathcal{A} such that

$$p(\delta_n(x)) \leq q(x) \quad \forall x \in \mathcal{A}_0, \forall n \in \mathbb{N}. \tag{4.41}$$

We can now prove the following.

PROPOSITION 4.8. *Let δ be the τ -limit of a uniformly τ -continuous sequence $\{\delta_n\}$ of $*$ -derivations such that the set*

$$\mathfrak{D}(\delta) = \left\{ x \in \mathcal{A}_0 : \exists \tau - \lim_n \delta_n(x) \right\} \tag{4.42}$$

is τ -dense in \mathcal{A}_0 . Then, δ is a $$ -derivation and, denoting with $\tilde{\delta}_n$ the continuous extension of δ_n to \mathcal{A} , we have $\{x \in \mathcal{A} : \exists \tau - \lim_n \tilde{\delta}_n(x)\} = \mathcal{A}$.*

PROOF. The proof that δ is a $*$ -derivation is trivial.

Let a be a generic element in \mathcal{A} . Since, by assumption, $\mathfrak{D}(\delta)$ is τ -dense in \mathcal{A}_0 , and therefore in \mathcal{A} , there exists a net $\{x_\alpha\} \subset \mathfrak{D}(\delta)$ τ -converging to a . This means that for any continuous seminorms p and for any $\epsilon > 0$, there exists $\alpha_{p,\epsilon}$ such that $p(a - x_\alpha) < \epsilon$ for all $\alpha > \alpha_{p,\epsilon}$.

Take an arbitrary continuous seminorm p on \mathcal{A} . Let q be the continuous seminorm on \mathcal{A} satisfying (4.41). Then,

$$\begin{aligned} p(\tilde{\delta}_n(a) - \tilde{\delta}_m(a)) &\leq p(\tilde{\delta}_n(a - x_\alpha)) + p((\tilde{\delta}_n - \tilde{\delta}_m)(x_\alpha)) + p(\tilde{\delta}_m(a - x_\alpha)) \\ &\leq 2q(a - x_\alpha) + p((\tilde{\delta}_n - \tilde{\delta}_m)(x_\alpha)) \\ &\leq 2\epsilon + p((\tilde{\delta}_n - \tilde{\delta}_m)(x_\alpha)) \leq \epsilon' \end{aligned} \tag{4.43}$$

for all fixed $\alpha > \alpha_{q,\epsilon}$ and n, m large enough. This completes the proof. □

All the results obtained in this section rely on the fact that there exists one underlying Hilbert space related to the representation, in the case of locally convex quasi *-algebras, or to triplets of Hilbert spaces for CQ^* -algebras. However, it is known that in some physically relevant situation like in quantum field theory, the relevant operators are the quantum fields and these operators belong to $\mathcal{L}(\mathfrak{D}, \mathfrak{D}')$ for suitable \mathfrak{D} , instead of being in some $\mathcal{L}^\dagger(\mathfrak{D}, \mathcal{H})$. This motivates our interest for the next result, which extends in a nontrivial way Proposition 4.3. Before stating the main proposition (Theorem 4.10), we need to introduce some definitions.

Let $(\mathcal{A}, \mathcal{A}_0)$ be a quasi *-algebra and π_0 a *-representation of \mathcal{A}_0 on the domain $\mathfrak{D}_{\pi_0} \subset \mathcal{H}_{\pi_0}$. This means that π_0 maps \mathcal{A}_0 into $\mathcal{L}^\dagger(\mathfrak{D}_{\pi_0})$ and that π_0 is a *-homomorphism of *-algebras. As usual, we endow \mathfrak{D}_{π_0} with the topology t_\dagger , the graph topology generated by $\mathcal{L}^\dagger(\mathfrak{D}_{\pi_0})$. In this way, we get the rigged Hilbert space $\mathfrak{D}_{\pi_0} \subset \mathcal{H}_{\pi_0} \subset \mathfrak{D}'_{\pi_0}$, where \mathfrak{D}'_{π_0} is the dual of $\mathfrak{D}_{\pi_0}[t_\dagger]$. On \mathfrak{D}'_{π_0} , we consider the strong dual topology t'_\dagger defined by the seminorms

$$\|F\|_{\mathcal{M}} = \sup_{\xi \in \mathcal{M}} |\langle F, \xi \rangle|, \quad \mathcal{M} \text{ bounded in } \mathfrak{D}_{\pi_0}[t_\dagger]. \tag{4.44}$$

In $\mathcal{L}(\mathfrak{D}_{\pi_0}, \mathfrak{D}'_{\pi_0})$, we consider the *quasi-strong topology* τ_{qs} defined by the seminorms

$$\mathcal{L}(\mathfrak{D}_{\pi_0}, \mathfrak{D}'_{\pi_0}) \ni X \rightarrow \|X\xi\|_{\mathcal{M}}, \quad \xi \in \mathfrak{D}_{\pi_0}, \mathcal{M} \text{ bounded in } \mathfrak{D}_{\pi_0}[t_\dagger], \tag{4.45}$$

and the *uniform topology* $\tau_{\mathfrak{D}}$ defined by the seminorms

$$\mathcal{L}(\mathfrak{D}_{\pi_0}, \mathfrak{D}'_{\pi_0}) \ni X \rightarrow \|X\|_{\mathcal{M}} = \sup_{\xi, \eta \in \mathcal{M}} |\langle X\xi, \eta \rangle|, \quad \mathcal{M} \text{ bounded in } \mathfrak{D}_{\pi_0}[t_\dagger]. \tag{4.46}$$

DEFINITION 4.9. Let $(\mathcal{A}, \mathcal{A}_0)$ and π_0 be as above. A linear map $\pi : \mathcal{A} \rightarrow \mathcal{L}(\mathfrak{D}_\pi, \mathfrak{D}'_\pi)$ is called a *qu*-representation* of \mathcal{A} associated with π_0 if π extends π_0 and

$$\begin{aligned} \pi(a^*) &= \pi(a)^\dagger \quad \forall a \in \mathcal{A}; \\ \pi(ax) &= \pi(a)\pi_0(x) \quad \forall a \in \mathcal{A}, x \in \mathcal{A}_0. \end{aligned} \tag{4.47}$$

THEOREM 4.10. Let $(\mathcal{A}, \mathcal{A}_0)$ be a locally convex quasi *-algebra with identity and with topology τ and δ a *-derivation of \mathcal{A}_0 .

Then the following statements are equivalent.

- (i) There exists a $(\tau - \tau_{qs})$ -continuous, ultra-cyclic qu^* -representation π of $(\mathcal{A}, \mathcal{A}_0)$, with ultra-cyclic vector ξ_0 such that the $*$ -derivation δ_π induced by π is spatial, that is, there exists $H = H^\dagger \in \mathcal{L}(\mathcal{D}_\pi, \mathcal{D}'_\pi)$ such that

$$\delta_\pi(\pi(x)) = i\{H \circ \pi(x) - \pi(x) \circ H\} \quad \forall x \in \mathcal{A}_0. \tag{4.48}$$

- (ii) There exist a positive linear functional f on \mathcal{A}_0 and a sesquilinear positive form Ω on $\mathcal{A}_0 \times \mathcal{A}_0$ such that

- (a) for some continuous seminorm p on $\mathcal{A}[\tau]$,

$$f(x^*x) \leq p(x)^2 \quad \forall x \in \mathcal{A}_0; \tag{4.49}$$

- (b) if \tilde{f} is the continuous extension of f to \mathcal{A} , then the inequalities

$$|\tilde{f}(y^*x)| \leq p(x)\Omega(y, y)^{1/2} \quad \forall x, y \in \mathcal{A}_0, \tag{4.50}$$

hold for some continuous seminorm p ;

- (c)

$$|\tilde{f}(y^*ax)| \leq \gamma_a\Omega(x, x)^{1/2}\Omega(y, y)^{1/2} \quad \forall x, y \in \mathcal{A}_0, a \in \mathcal{A} \tag{4.51}$$

and for some positive constant γ_a ;

- (d)

$$|\tilde{f}(\delta(x))| \leq C(\Omega(x, x)^{1/2} + \Omega(x^*, x^*)^{1/2}) \quad \forall x \in \mathcal{A}_0 \tag{4.52}$$

and for some positive constant C ;

- (e) for any ultra-cyclic $*$ -representation Θ of \mathcal{A}_0 , with ultra-cyclic vector ξ_θ , satisfying

$$f(x) = \langle \Theta(x)\xi_\theta, \xi_\theta \rangle \tag{4.53}$$

for all $x \in \mathcal{A}_0$, the sesquilinear form on $\mathcal{D}_\theta \times \mathcal{D}_\theta$, $\mathcal{D}_\theta = \Theta(\mathcal{A}_0)\xi_\theta$, defined by

$$\varphi_\theta(\Theta(x)\xi_\theta, \Theta(y)\xi_\theta) = \Omega(x, y), \tag{4.54}$$

is jointly continuous on $\mathcal{D}_\theta[t_+]$.

PROOF. We prove that (i) implies (ii). For this, let π be a $(\tau - \tau_{qs})$ -continuous, ultra-cyclic qu^* -representation of \mathcal{A} associated with π_0 , with ultra-cyclic vector ξ_0 : $\pi_0(\mathcal{A}_0)\xi_0 = \mathcal{D}_\pi$. For all $x \in \mathcal{A}_0$, we define $f(x) = \langle \pi_0(x)\xi_0, \xi_0 \rangle$. Then, since π coincides with π_0 on \mathcal{A}_0 and since π is $(\tau - \tau_{qs})$ -continuous, we have

$$f(x^*x) = \langle \pi_0(x^*x)\xi_0, \xi_0 \rangle = \langle \pi(x^*x)\xi_0, \xi_0 \rangle = \|\pi(x)\xi_0\|^2 \leq p(x)^2 \tag{4.55}$$

for some continuous seminorm p of $\mathcal{A}[\tau]$. In fact, $\|\pi(x)\xi_0\|$ is one of the seminorms defining τ_{qs} . If \tilde{f} is called the continuous extension of f , it is clear that for any $a \in \mathcal{A}$,

we have $\tilde{f}(a) = \langle \pi(a)\xi_0, \xi_0 \rangle$. Therefore, for $x, y \in \mathcal{A}_0$ and $a \in \mathcal{A}$, we have

$$\begin{aligned} \tilde{f}(y^*ax) &= \langle \pi(y^*ax)\xi_0, \xi_0 \rangle = \langle \pi(ax)\xi_0, \pi_0(y)\xi_0 \rangle \\ &= \langle \pi(a)\pi_0(x)\xi_0, \pi_0(y)\xi_0 \rangle, \end{aligned} \tag{4.56}$$

and since, by assumption, $\pi(a)\pi_0(x)\xi_0$ is a continuous functional on $\mathcal{D}[t_+]$, there exist a positive constant γ and a continuous seminorm on $\mathcal{D}[t_+]$ such that

$$|\tilde{f}(y^*ax)| \leq \gamma \|T\pi_0(y)\xi_0\|, \tag{4.57}$$

where $T \in \mathcal{L}^\dagger(\mathcal{D}_\pi)$ labels the seminorm. The best value of γ can be found considering the following bounded subset \mathcal{M} of $\mathcal{D}_\pi[t_+]$: $\mathcal{M} = \{\xi \in \mathcal{D}_\pi : \|T\xi\| = 1\}$. In this way, we get

$$|\tilde{f}(y^*ax)| \leq \|\pi(a)\pi_0(x)\xi_0\|_{\mathcal{M}} \|T\pi_0(y)\xi_0\| \leq p_x(a) \|T\pi_0(y)\xi_0\|. \tag{4.58}$$

The last inequality follows from the $(\tau - \tau_{q_s})$ -continuity of π . Furthermore, since $\pi(a)$ belongs to $\mathcal{L}(\mathcal{D}_\pi, \mathcal{D}'_\pi)$, the following inequality also holds:

$$\|\pi(a)\pi_0(x)\xi_0\|_{\mathcal{M}} \leq \gamma_2 \|C\pi(x)\xi_0\| \tag{4.59}$$

for a certain positive constant γ_2 and an operator $C \in \mathcal{L}^\dagger(\mathcal{D}_\pi)$.

Moreover, since $\tilde{f}(\delta(x)) = i\{\langle \pi(x)\xi_0, H\xi_0 \rangle - \langle H\xi_0, \pi(x^*)\xi_0 \rangle\}$, and since, as a functional, $H\xi_0$ is continuous, there exist a $B \in \mathcal{L}^\dagger(\mathcal{D}_\pi)$ and a positive constant γ_1 such that

$$|\tilde{f}(\delta(x))| \leq \gamma_1 (\|B\pi(x)\xi_0\| + \|B\pi(x^*)\xi_0\|). \tag{4.60}$$

Inequalities (4.57), (4.59), and (4.60) refer to three elements of $\mathcal{L}^\dagger(\mathcal{D}_\pi)$: B , C , and T . It is always possible to find another element $A \in \mathcal{L}^\dagger(\mathcal{D}_\pi)$ such that

$$\|A\eta\| \geq \|B\eta\|, \quad \|A\eta\| \geq \|T\eta\|, \quad \|A\eta\| \geq \|C\eta\| \quad \forall \eta \in \mathcal{D}_\pi. \tag{4.61}$$

We now define the positive sesquilinear form Ω on $\mathcal{A}_0 \times \mathcal{A}_0$ as

$$\Omega(x, y) = \langle A\pi(x)\xi_0, A\pi(y)\xi_0 \rangle, \quad x, y \in \mathcal{A}_0. \tag{4.62}$$

Then, because of (4.61), inequalities (4.49), (4.50), (4.51), and (4.52) easily follow. As for the joint continuity of φ_θ , we start noticing that since $f(x) = \langle \pi_0(x)\xi_0, \xi_0 \rangle = \langle \Theta(x)\xi_\theta, \xi_\theta \rangle$, then Θ is unitarily equivalent to π_0 since they are both unitarily equivalent to the GNS-representation π_f defined by f on \mathcal{A}_0 because of the essential uniqueness of the latter. Thus, there exists a unitary operator $U : \mathcal{H}_\theta \rightarrow \mathcal{H}_{\pi_0}$, with $\xi_0 = U\xi_\theta$ and such that $\Theta(x) = U^{-1}\pi_0(x)U$.

By the definition itself,

$$\varphi_{\pi_0}(\pi_0(x)\xi_0, \pi_0(y)\xi_0) = \Omega(x, y) = \langle A\pi_0(x)\xi_0, A\pi_0(y)\xi_0 \rangle, \tag{4.63}$$

then φ_{π_0} is jointly continuous on $\mathfrak{D}_{\pi_0}[t_{\dagger}]$. Therefore,

$$\begin{aligned} \varphi_{\theta}(\Theta(x)\xi_{\theta}, \Theta(y)\xi_{\theta}) &= \Omega(x, y) = \langle A\pi_0(x)\xi_0, A\pi_0(y)\xi_0 \rangle \\ &= \langle U^{-1}AU\Theta(x)\xi_{\theta}, U^{-1}AU\Theta(y)\xi_{\theta} \rangle \end{aligned} \tag{4.64}$$

and φ_{θ} is jointly continuous on $\mathfrak{D}_{\theta}[t_{\dagger}]$ too.

We prove now the converse implication, that is, (ii) implies (i).

We assume that there exist f and Ω satisfying all the properties we have required in (ii). We define the following vector space: $\mathcal{N}_f = \{a \in \mathcal{A} : \tilde{f}(a^*x) = 0 \ \forall x \in \mathcal{A}_0\}$. It is clear that if $a \in \mathcal{N}_f$ and $y \in \mathcal{A}_0$, then $ya \in \mathcal{N}_f$. We denote with $\lambda_f(a)$, for $a \in \mathcal{A}$, the element of the vector space $\mathcal{A}/\mathcal{N}_f$ containing a . The subspace $\lambda_f(\mathcal{A}_0) = \{\lambda_f(x), x \in \mathcal{A}_0\}$ is a pre-Hilbert space with inner product

$$\langle \lambda_f(x), \lambda_f(y) \rangle = f(y^*x), \quad x, y \in \mathcal{A}_0, \tag{4.65}$$

and the form $\langle \lambda_f(x), \lambda_f(a) \rangle = \tilde{f}(a^*x)$, $x \in \mathcal{A}_0$, $a \in \mathcal{A}$, puts $\mathcal{A}/\mathcal{N}_f$ and $\lambda_f(\mathcal{A}_0)$ in separating duality. Now we can define an ultra-cyclic *-representation π_0 of \mathcal{A}_0 in the following way: its domain \mathfrak{D}_{π_0} coincides with $\lambda_f(\mathcal{A}_0)$, and $\pi_0(x)\lambda_f(y) = \lambda_f(xy)$ for $x, y \in \mathcal{A}_0$. The vector $\lambda_f(\mathbf{1})$ is ultra-cyclic and $f(x) = \langle \pi_0(x)\lambda_f(\mathbf{1}), \lambda_f(\mathbf{1}) \rangle$, for all $x \in \mathcal{A}_0$. Therefore, the sesquilinear form $\varphi_{\pi_0}(\pi_0(x)\lambda_f(\mathbf{1}), \pi_0(y)\lambda_f(\mathbf{1})) = \Omega(x, y)$ is jointly continuous in $\mathfrak{D}_{\pi_0}[t_{\dagger}]$.

We now claim that $\mathcal{A}/\mathcal{N}_f \subset \mathfrak{D}'_{\pi_0}$, the dual space of $\mathfrak{D}_{\pi_0}[t_{\dagger}]$. This follows from the joint continuity of φ_{π_0} , which gives the following estimate:

$$|\Omega(x, y)| \leq y \|A'\pi_0(x)\lambda_f(\mathbf{1})\| \|A'\pi_0(y)\lambda_f(\mathbf{1})\|, \tag{4.66}$$

which holds for all $x, y \in \mathcal{A}_0$, for suitable $y > 0$, and $A' \in \mathcal{L}^{\dagger}(\mathfrak{D}_{\pi_0})$. Using the extension of (4.50) to $\mathcal{A}_0 \times \mathcal{A}$ and (4.66), we find

$$|\langle \lambda_f(x), \lambda_f(a) \rangle| = |\tilde{f}(a^*x)| \leq p(a)\Omega(x, x)^{1/2} \leq y^{1/2}p(a) \|A'\pi_0(x)\lambda_f(\mathbf{1})\|, \tag{4.67}$$

which implies that $\lambda_f(a) \in \mathfrak{D}'_{\pi_0}$.

We can now extend π_0 to \mathcal{A} in a natural way: for $a \in \mathcal{A}$, we put $\pi(a)\lambda_f(x) = \lambda_f(ax)$ for all $x \in \mathcal{A}_0$. For each $a \in \mathcal{A}$, $\pi(a)$ is well defined and maps $\mathfrak{D}_{\pi_0}[t_{\dagger}]$ into $\mathfrak{D}'_{\pi_0}[t'_{\dagger}]$ continuously. Moreover, π is $(\tau - \tau_{qs})$ -continuous. The induced derivation δ_{π} is well defined, as is easily checked, and its spatiality can be proven by repeating essentially the same steps as in Theorem 4.1. □

REMARK 4.11. In the so-called Wightman formulation of quantum field theory (see, e.g., [19]), the point-like $A(x)$, $x \in \mathbb{R}^4$, can be a very singular mathematical object such as a sesquilinear form depending on x and defined on $\mathfrak{D} \times \mathfrak{D}$, where \mathfrak{D} is a dense domain in Hilbert space \mathcal{H} . The *smearred field* is an operator-valued distribution $f \in \mathcal{S}(\mathbb{R}^4) \rightarrow \mathcal{L}^{\dagger}(\mathfrak{D})$, $\mathcal{S}(\mathbb{R}^4)$ being the space of Schwartz test functions. If f has support contained in a bounded region \mathbb{O} of \mathbb{R}^4 , then $\overline{A}(f)$ is affiliated with the local von Neumann algebra $\mathcal{A}(\mathbb{O})$ of all observables in \mathbb{O} .

A reasonable approach [16, 17] consists in considering the point-like field $A(x)$, for each $x \in \mathbb{R}^4$, as an element of $\mathcal{L}(\mathcal{D}, \mathcal{D}')$, once a locally convex topology on \mathcal{D} has been defined. A crucial physical prescription is that the field must be covariant under the action of a unitary representation $U(g)$ of some transformation group (such as the Poincaré or Lorentz group) and, as it is known, the infinitesimal generator H of time translations gives the energy operator of the system which defines in a natural way a spatial *-derivation of the quasi *-algebra $(\mathcal{A}, \mathcal{A}_0)$ of observables.

There could be however a different approach. This occurs when a field $x \mapsto A(x)$ is defined on the basis of some heuristic considerations. In order that $A(x)$ represents a reasonable physical solution of the problem under consideration, covariance under some Lie algebra of infinitesimal transformation must be imposed. For infinitesimal time translations, this amounts to find some *-derivation δ of the quasi *-algebra obtained by taking the weak completion of the *-algebra \mathcal{A}_0 generated by the local von Neumann algebras $\mathcal{A}(\mathbb{O})$, with \mathbb{O} a bounded region of \mathbb{R}^4 . But, of course, a number of problems arises.

The first one consists in finding an appropriate domain \mathcal{D} for the family of operators $\{A(f); f \in \mathcal{S}(\mathbb{R}^4)\}$ and an appropriate topology on \mathcal{D} in such a way that $A(x) \in \mathcal{L}(\mathcal{D}, \mathcal{D}')$ for every $x \in \mathbb{R}^4$. Once this is done, if the identical representation has the properties required in Theorem 4.10, then a symmetric operator H implementing δ can be found and one expects H to be the energy operator of the system. But, as it is well known, the problem of integrating δ is far to be solved even in much more regular situations than those considered here. We hope to discuss these problems in a future paper.

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REFERENCES

- [1] G. Alli and G. L. Sewell, *New methods and structures in the theory of the multimode Dicke laser model*, J. Math. Phys. **36** (1995), no. 10, 5598–5626.
- [2] J.-P. Antoine, A. Inoue, and C. Trapani, *O*-dynamical systems and *-derivations of unbounded operator algebras*, Math. Nachr. **204** (1999), 5–28.
- [3] ———, *Partial *-Algebras and Their Operator Realizations*, Mathematics and Its Applications, vol. 553, Kluwer Academic Publishers, Dordrecht, 2002.
- [4] J.-P. Antoine and W. Karwowski, *Partial *-algebras of closed linear operators in Hilbert space*, Publ. Res. Inst. Math. Sci. **21** (1985), no. 1, 205–236.
- [5] ———, *Addendum/erratum: "Partial *-algebras of closed linear operators in Hilbert space"*, Publ. Res. Inst. Math. Sci. **22** (1986), no. 3, 507–511.
- [6] F. Bagarello, *Applications of topological *-algebras of unbounded operators*, J. Math. Phys. **39** (1998), no. 11, 6091–6105.
- [7] F. Bagarello, A. Inoue, and C. Trapani, *Some classes of topological quasi *-algebras*, Proc. Amer. Math. Soc. **129** (2001), no. 10, 2973–2980.
- [8] F. Bagarello and G. Morchio, *Dynamics of mean-field spin models from basic results in abstract differential equations*, J. Statist. Phys. **66** (1992), no. 3–4, 849–866.
- [9] F. Bagarello and G. L. Sewell, *New structures in the theory of the laser model. II. Microscopic dynamics and a nonequilibrium entropy principle*, J. Math. Phys. **39** (1998), no. 5, 2730–2747.

- [10] F. Bagarello and C. Trapani, "Almost" mean-field Ising model: an algebraic approach, *J. Statist. Phys.* **65** (1991), no. 3-4, 469-482.
- [11] ———, *States and representations of CQ^* -algebras*, *Ann. Inst. H. Poincaré Phys. Théor.* **61** (1994), no. 1, 103-133.
- [12] ———, *CQ^* -algebras: structure properties*, *Publ. Res. Inst. Math. Sci.* **32** (1996), no. 1, 85-116.
- [13] ———, *Morphisms of certain Banach C^* -modules*, *Publ. Res. Inst. Math. Sci.* **36** (2000), no. 6, 681-705.
- [14] ———, *Algebraic dynamics in O^* -algebras: a perturbative approach*, *J. Math. Phys.* **43** (2002), no. 6, 3280-3292.
- [15] O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics*, Texts and Monographs in Physics, vol. 1, Springer-Verlag, New York, 1979.
- [16] G. Epifanio and C. Trapani, *Quasi $*$ -algebras valued quantized fields*, *Ann. Inst. H. Poincaré Phys. Théor.* **46** (1987), no. 2, 175-185.
- [17] K. Fredenhagen and J. Hertel, *Local algebras of observables and pointlike localized fields*, *Comm. Math. Phys.* **80** (1981), no. 4, 555-561.
- [18] I. M. Gelfand and N. Ya. Vilenkin, *Generalized Functions. Vol. 4: Applications of Harmonic Analysis*, Academic Press, New York, 1964.
- [19] R. Haag, *Local Quantum Physics*, Texts and Monographs in Physics, Springer-Verlag, Berlin, 1992.
- [20] G. Köthe, *Topological Vector Spaces. II*, Grundlehren der Mathematischen Wissenschaften, vol. 237, Springer-Verlag, New York, 1979.
- [21] G. Lassner, *Topological algebras and their applications in quantum statistics*, *Wiss. Z. Karl-Marx-Univ. Leipzig Math.-Natur. Reihe* **30** (1981), no. 6, 572-595.
- [22] S. Sakai, *Operator Algebras in Dynamical Systems*, Encyclopedia of Mathematics and Its Applications, vol. 41, Cambridge University Press, Cambridge, 1991.
- [23] W. Thirring and A. Wehrl, *On the mathematical structure of the B.C.S.-model*, *Comm. Math. Phys.* **4** (1967), 303-314.
- [24] C. Trapani, *Quasi $*$ -algebras of operators and their applications*, *Rev. Math. Phys.* **7** (1995), no. 8, 1303-1332.

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