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MINIMAL SEQUENTIAL HAUSDORFF SPACES

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To my teachers

A sequential space (X,T) is called minimal sequential if no sequential topology on X is strictly weaker than T. This paper begins the study of minimal sequential Hausdorff spaces. Characterizations of minimal sequential Hausdorff spaces are obtained using filter bases, sequences, and functions satisfying certain graph conditions. Relationships between this class of spaces and other classes of spaces, for example, minimal Hausdorff spaces, countably compact spaces, H-closed spaces, SQ-closed spaces, and subspaces of minimal sequential spaces, are investigated. While the property of being sequential is not (in general) preserved by products, some information is provided on the question of when the product of minimal sequential spaces is minimal sequential.

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1. Introduction. All hypothesized spaces are Hausdorff topological spaces. If (X,T) is a space and $Q \subset X$, $\{x \in X :$ some sequence in Q converges to $x\}$, that is, the *T-sequential closure of Q* will be denoted by $\Delta_T(Q)$. A subset $Q \subset X$ is *T-sequentially closed* if $\Delta_T(Q) = Q$, and is *T-sequentially open* if $X - Q$ is T-sequentially closed. It is not difficult to show that (1) the collection of T-sequentially open subsets, which we denote by S_T, is a topology on X, (2) $T \subset S_T$, and (3) $Q \in S_T$ if and only if each sequence in X which is T-convergent to an element of Q is ultimately in Q. The space (X,T) is called *sequential* if $T = S_T$. In this case, the phrase "T is sequential" is often used. It is obvious that first countable spaces are sequential and it is known that a sequential space might fail to be first countable [4]. In this paper, a space (X,T) is *minimal sequential* if T is sequential and no sequential topology on X is strictly weaker (= smaller) than T. Such a space will be called *minimal Hausdorff (sq)*. This terminology parallels the following from [1].

If P is a property of topological spaces, then $P(1)$ will mean a space which is first countable and has property P, thus a space is Hausdorff (1) provided it is Hausdorff and first countable. It is proved in [4] that sequential spaces are characterized as quotients of first countable (metric) spaces. Hence, the class of minimal Hausdorff (sq) spaces coincides with the class of spaces which are minimal in the class of quotients of first countable (metric) spaces. Two proper subclasses of this class of spaces have been recently investigated in [9].

In Section 2, a number of characterizations of minimal Hausdorff (sq) spaces are established in terms of filter bases, sequences, and functions into such spaces satisfying certain graph conditions. These characterizations include parallels of those of minimal Hausdorff spaces by Bourbaki [2] in terms of open filter bases, and of those by Herrington and Long [6] in terms of arbitrary filter bases. These characterizations reveal that a number of spaces which have been the object of study are minimal Hausdorff (sq). Indeed, minimal P spaces, where P is either of the properties semimetrizable, symmetrizable, neighborhood $\mathcal{F}$, $\mathcal{F}$, weakly first countable, are minimal Hausdorff (sq) (see [11, 12]). In Section 3, some relationships between the class of minimal Hausdorff (sq) spaces and other classes of spaces, for example, minimal Hausdorff spaces, countably compact spaces, H-closed spaces, and SQ-closed spaces in the sense of Thompson [14], are determined. It is established that minimal Hausdorff sequential spaces, as well as sequentially compact sequential spaces, are minimal Hausdorff (sq) and that every minimal Hausdorff (sq) space is sequentially H-closed. Sequentially H-closed spaces were studied by Thompson [14] and Espelie et al. [3] under the name of SQ-closed spaces. Examples are provided to show that a minimal Hausdorff (sq) space need not be H-closed and that a sequential H-closed space might fail to be minimal Hausdorff (sq). Other examples are given to distinguish this class of spaces from other classes of spaces. In Section 3, subspaces and products of minimal Hausdorff (sq) spaces are studied. Parallel to the result of Katětov [10] and Stone [13] that a space is compact if and only if every closed subspace is H-closed, we prove that a space is countably compact if and only if every closed subspace is SQ-closed. Although the property of a space being minimal Hausdorff (sq) is not preserved by products, we apply results from [3, 5] to provide some information on products of such spaces in this section.

2. Characterizations of minimal Hausdorff (sq) spaces. Preliminary to our first theorem, we introduce some additional concepts and notations which are utilized throughout the paper. If (X,T) is a space and $Q \subset X$, we use the notation $\Sigma_T(Q)$ ($\Sigma_T(x)$ if $Q = \{x\}$) for the collection of elements of T which contain Q (simply $\Sigma(Q)$ when confusion is unlikely); we let $\overline{Q}$ (Q°) represent the *closure* (*interior*) of Q. If Ω is a filter base on a space, $\operatorname{adh}\Omega$ will represent its adherence ($\operatorname{adh}\Omega = \bigcap_{F\in\Omega}\overline{F}$). If Q is a subset of a space X, Veličko [15] called $\{x \in X : Q \cap \overline{V} \neq \varnothing$ is satisfied for each $V \in \Sigma(x)\}$ the *θ-closure of* Q and denoted this set by $[Q]_\theta$ ($[x]_\theta$ if $Q = \{x\}$). He called Q *θ-closed* if $[Q]_\theta = Q$ and showed that $[Q]_\theta$ might fail to be θ-closed. Indeed, a subset Q of a countable H-closed space can satisfy $[Q]_\theta^n \subsetneq [Q]_\theta^{n+1}$ for every positive integer n [8]. It is known that $[Q]_\theta = \bigcap_{\Sigma(Q)}\overline{V}$, that for $(x,y \in X)$, $x \in [y]_\theta$ if and only if $y \in [x]_\theta$ [3], and that X is Hausdorff if and only if $[x]_\theta = \{x\}$ for each $x \in X$. The *θ-adherence of a filter base* on X denoted by $[\Omega]_\theta$ is $\bigcap_\Omega [F]_\theta$, and Ω *θ-converges to x* denoted by $\Omega \underset{\theta}{\rightarrow} x$, if for each $V \in \Sigma(x)$, there is an $F \in \Omega$ satisfying $F \subset \overline{V}$ [15]. The notions θ-cluster point and θ-convergence of nets are similarly defined. If $\mathcal{F}$ is a filter on a space (X,T), the filter base $T \cap \mathcal{F}$ will be called the *open part* of $\mathcal{F}$ and will be denoted by $\mathcal{O}(\mathcal{F})$, $\{A \in \mathcal{O}(\mathcal{F}) : A$ is regular open$\}$ will be called the *regular-open part* of $\mathcal{F}$ and can be denoted by $\mathcal{R}(\mathcal{F})$. Recall that an open subset V is *regular open* if $V = \overline{V}^\circ$, and a topological space is *semiregular* if the regular-open subsets of the space are a base for its topology. Since it is an elementary fact of General Topology that $\overline{\overline{A^\circ}^\circ} = \overline{A^\circ}$ for any subset A of a topological space, we see that $\operatorname{adh}\mathcal{O}(\mathcal{F}) = \operatorname{adh}\mathcal{R}(\mathcal{F}) = [\Omega]_\theta$.

Theorem 2.1. *The following statements are equivalent for a space (X,T):*

(i) *if x_n is a sequence in X with at most one $(T-\theta)$-cluster point, x_n converges;*

(ii) *if $T^* \subset T$ is a topology on X, then $\Delta_{T^*}(Q) = \Delta_T(Q)$ is satisfied for each $Q \subset X$;*

(iii) *if T^* is a topology on X and $T^* \subset T$, then each T^*-sequentially closed subset of X is T-sequentially closed;*

(iv) *if T^* is a topology on X and $T^* \subset T$, then $S_T = S_{T^*}$;*

(v) *each countable filter base on X with at most one $(T-\theta)$-adherent point T-converges;*

(vi) *all topologies on X which are smaller than T have the same convergent sequences as T;*

(vii) *if $\mathcal{F}$ is a filter on X with a countable base and $\mathbb{O}(\mathcal{F})$ has at most one T-adherent point, then $\mathbb{O}(\mathcal{F})$ converges;*

(viii) *if $\mathcal{F}$ is a filter on X with a countable base and $\mathcal{R}(\mathcal{F})$ has at most one T-adherent point, then $\mathcal{R}(\mathcal{F})$ converges.*

Proof. (i)⇒(ii). Let $x \in \Delta_{T^*}(Q)$. Then there is a sequence x_n in Q such that $x_n \xrightarrow[T^*]{} x$. If h is a subnet of x_n and $h \xrightarrow[\theta]{} z$ with respect to T, then $h \xrightarrow[\theta]{} z$ with respect to T^* since $T^* \subset T$. Since T^* is Hausdorff, it follows that $z = x$. Hence, x_n has at most one $(T-\theta)$-cluster point. From (i), $x_n \xrightarrow[T]{} x$. So, $x \in \Delta_T(Q)$ and $\Delta_{T^*}(Q) \subset \Delta_T(Q)$. It follows from $T^* \subset T$ that $\Delta_T(Q) \subset \Delta_{T^*}(Q)$.

(ii)⇒(iii)⇒(iv). The proof is obvious.

(iv)⇒(v). Let $\mathbb{N}$ be the set of positive integers, let $\Omega = \{F_n : n \in \mathbb{N}\}$ be a filter base, and let $v \in X$ such that $F_{n+1} \subset F_n$, $[\Omega]_\theta \subset \{v\}$, and suppose Ω does not converge. Choose a $V_0 \in \Sigma_T(v)$ satisfying $F_n - V_0 \neq \varnothing$. Let x_n be a sequence such that $x_n \in F_n - V_0$. Clearly, $x_n \not\xrightarrow[T]{} v$. Let

$$T^* = \big(T - \Sigma_T(v)\big) \cup \big\{V \in \Sigma_T(v) : x_n \in V \text{ ultimately}\big\}, \tag{2.1}$$

let $x, y \in X - \{v\}$, $x \neq y$. Since T is a Hausdorff topology, there exist $V \in \Sigma_T(x) - (\Sigma_T(v) \cup \Sigma_T(y))$, $W \in \Sigma_T(y) - (\Sigma_T(v) \cup \Sigma_T(x))$ such that $V \cap W = \varnothing$. Let $x \in X$, $x \neq v$. Then $x \notin [F_n]_\theta$ for some n. For such an n, there exist $V \in \Sigma_T(x)$, $W \in \Sigma_T(F_n)$ such that $V \cap W = \varnothing$. Thus, T^* is a Hausdorff topology on X and clearly $T^* \subset T$. Obviously, $T^* \neq T$ since $x_n \xrightarrow[T^*]{} v$; the range of x_n, call it $R(x_n)$, is T-sequentially closed since the only T-convergent sequences in $R(x_n)$ are those which are ultimately constant. On the other hand, $v \in \Delta_{T^*}(R(x_n)) - R(x_n)$. Hence, $R(x_n)$ is not T^*-sequentially closed and this contradicts (iv).

(v)⇒(vi). Let x_n be a sequence in X and suppose $x_n \xrightarrow[T^*]{} x$. Using the same argument as in the proof of (i)⇒(ii) above, x is the only possible $(T-\theta)$-adherent point of the elementary filter generated by x_n. Hence, $x_n \xrightarrow[T]{} x$.

(vi)⇒(i). Let x_n be a sequence in X with at most one $(T-\theta)$-cluster point. Without loss of generality, choose $v \in X$ such that no cluster point of x_n is in $X - \{v\}$. Employ the same construction as in the proof of (iv)⇒(v) to get a Hausdorff topology T^* on X such that $T^* \subset T$, and $x_n \xrightarrow[T^*]{} v$. Then, $x_n \xrightarrow[T]{} v$ in view of (vi).

(vii)⇒(v). Let Ω be a countable filter base on X, let $v \in X$, and suppose $[\Omega]_\theta \subset \{v\}$. Then $\mathbb{O}(\mathcal{F}) = \bigcup_\Omega \Sigma(F)$ for the filter $\mathcal{F}$ with base Ω and $\operatorname{adh} \mathbb{O}(\mathcal{F}) = [\Omega]_\theta$. Hence, $\mathbb{O}(\mathcal{F}) \underset{T}{\rightarrow} v$, and consequently $\Omega \underset{T}{\rightarrow} v$.

(v)⇒(vii). Let $\mathcal{F}$ be a filter on X with countable base Ω, $v \in X$, and suppose $\operatorname{adh} \mathbb{O}(\mathcal{F}) \subset \{v\}$. Then $[\Omega]_\theta \subset \operatorname{adh} \mathbb{O}(\mathcal{F})$, so $\Omega \underset{T}{\rightarrow} v$ and consequently, $\Sigma(v) \subset \mathbb{O}(\mathcal{F})$.

(v)⇔(viii). We see that $\operatorname{adh} \mathbb{O}(\mathcal{F}) = \operatorname{adh} \mathcal{R}(\mathcal{F}) = [\Omega]_\theta$. □

In view of Theorem 2.1, we have the following theorem.

Theorem 2.2. *The following statements are equivalent for a sequential space (X,T):*

(i) *(X,T) is minimal Hausdorff (sq);*

(ii) *if x_n is a sequence in X with at most one $(T-\theta)$-cluster point, then x_n converges;*

(iii) *if $T^* \subset T$ is a topology on X, then $\Delta_{T^*}(Q) = \Delta_T(Q)$ is satisfied for each $Q \subset X$;*

(iv) *if T^* is a topology on X and $T^* \subset T$, then each T^*-sequentially closed subset of X is T-sequentially closed;*

(v) *if T^* is a topology on X and $T^* \subset T$, then $S_T = S_{T^*}$;*

(vi) *each countable filter base on X with at most one $(T-\theta)$-adherent point T-converges;*

(vii) *all topologies on X which are smaller than T have the same convergent sequences as T;*

(viii) *if $\mathcal{F}$ is a filter on X with a countable base and $\mathbb{O}(\mathcal{F})$ has at most one T-adherent point, then $\mathbb{O}(\mathcal{F})$ converges;*

(ix) *if $\mathcal{F}$ is a filter on X with a countable base and $\mathcal{R}(\mathcal{F})$ has at most one T-adherent point, then $\mathcal{R}(\mathcal{F})$ converges.*

Corollary 2.3 comes easily from equivalence (ix) of Theorem 2.2.

Corollary 2.3. *A minimal Hausdorff (sq) space is semiregular.*

In [14], Thompson introduced the class of SQ-closed spaces. A space is *SQ-closed* if its continuous image in any Hausdorff space is sequentially closed. In [3], it is proved that a space is SQ-closed if and only if every countable filter base on the space with at most one θ-adherent point θ-converges.

Corollary 2.4 is immediate in view of this result and equivalence (vi) of Theorem 2.2.

Corollary 2.4. *A minimal Hausdorff (sq) space is SQ-closed.*

Corollary 2.5. *A minimal Hausdorff (sq) space is H-closed if and only if it is minimal Hausdorff.*

Next we present characterizations of minimal Hausdorff (sq) spaces in terms of functions into such spaces satisfying certain graph conditions. Let X, Y be spaces and let $f : X \to Y$. We will say that f has a *subclosed (strongly subclosed)* graph if $\operatorname{adh}(f(\Omega)) \subset \{f(x)\}$ $([f(\Omega)]_\theta \subset \{f(x)\})$ for each $x \in X$ and filter base Ω on $X - \{x\}$ satisfying $\Omega \to x$. When f has a subclosed (strongly subclosed) graph and $\{f(x)\}$ is closed (θ-closed) in Y for each $x \in X$, we say that f has a *closed (strongly closed) graph*. The notion of strongly closed graph was introduced by Herrington and Long in [6], while subclosed and strongly subclosed graphs were introduced by Joseph [7]. The function f is said to be *sequentially continuous at* $x \in X$ if $f(x_n) \to f(x)$ for each sequence x_n in X satisfying $x_n \to x$, and is said to be *sequentially continuous* if it is sequentially continuous at each $x \in X$. If X is a nonempty set, $v \in X$, and Ω is a filter base with empty intersection on $X - \{v\}$, $X(v,\Omega)$ denotes X with the topology $T = \{Q \subset X : v \notin Q \text{ or } F \subset Q \text{ for some } F \in \Omega\}$. Clearly, $X(v,\Omega)$ is Hausdorff and is first countable if Ω is countable. Let X, Y, and Z be nonempty sets, $f : X \to Z$, $g : Y \to Z$. Denote $\{(x,y) \in X \times Y : f(x) = g(y)\}$ ($\{x \in X : f(x) = g(x)\}$) by $\mathscr{E}(f,g,X\times Y,Z)$ ($\mathscr{E}(f,g,X,Z)$).

Theorem 2.6. *The following statements are equivalent for a sequential space Z:*
(i) *Z is minimal Hausdorff (sq);*
(ii) *for each space X, each $f : X \to Z$ with a strongly closed graph is sequentially continuous;*
(iii) *for all spaces X, Y and $f : X \to Z$, $g : Y \to Z$ with strongly closed and closed graphs, respectively, $\mathscr{E}(f,g,X\times Y,Z)$ is a sequentially closed subset of $X \times Y$;*
(iv) *for each space X and all $f,g : X \to Z$ with strongly closed and closed graphs, respectively, $\mathscr{E}(f,g,X,Z)$ is a sequentially closed subset of X;*
(v) *for each space X and all $f,g : X \to Z$ with strongly closed and closed graphs, respectively, $\mathscr{E}(f,g,X,Z) = X$ whenever $\Delta(\mathscr{E}(f,g,X,Z)) = X$.*

Proof. (i)⇒(ii). Let X be a space, let $f : X \to Z$ have a strongly closed graph, let $x \in X$ be a point which is not isolated, and let x_n be a sequence in $X - \{x\}$ such that $x_n \to x$. Let Ω be the usual base for the elementary filter generated by $\{x_n\}$. Then $[f(\Omega)]_\theta = \{f(x)\}$, since Z is minimal Hausdorff (sq), it follows by equivalence (vi) of Theorem 2.2 that $f(\Omega) \to f(x)$ and the proof that f is sequentially continuous is complete.

(ii)⇒(iii). Suppose X, Y are spaces, $f : X \to Z$, $g : Y \to Z$ are functions with strongly closed and closed graphs, respectively, and let (x_n,y_n) be a sequence in $\mathscr{E}(f,g,X\times Y,Z)$ such that $(x_n,y_n) \to (x,y)$. Then $x_n \to x$, $y_n \to y$, and $f(x_n) \to f(x)$ from (ii). Hence, $g(y_n) \to f(x)$ and $g(y) = f(x)$ since g has a closed graph. Therefore, $(x,y) \in \mathscr{E}(f,g,X\times Y,Z)$.

(iii)⇒(iv). Let X be a space and let $f,g : X \to Z$ have strongly closed and closed graphs, respectively. Then $\mathscr{E}(f,g,X,Z) = \pi(\mathscr{E}(f,g,X\times X,Z) \cap \mathscr{D})$, where π is the projection of $X \times X$ onto X and $\mathscr{D}$ is the diagonal of $X \times X$. From (iii), $\mathscr{E}(f,g,X\times X,Z) \cap \mathscr{D}$ is a sequentially closed subset of $\mathscr{D}$, and the restriction of π to $\mathscr{D}$ is a homeomorphism, so (iv) holds.

(iv)⇒(v). The proof is obvious.

(v)⇒(i). Suppose Ω is a countable filter base on Z and $v \in Z$ such that $[\Omega]_\theta \subset \{v\}$. Assuming that $\Omega \not\to v$, there is a $V_0 \in \Sigma(v)$ such that $\Gamma = \{F - V_0 : F \in \Omega\}$ is a filter base on Z such that $[\Gamma]_\theta \subset [\Omega]_\theta$. Choose $w \in Z - \{v\}$ and define $f,g : Z(v,\Gamma) \to Z$ by $f(x) = x$ for all x, $g(v) = w$, $g(x) = x$ on $Z - \{v\}$. We see that $\mathscr{E}(f,g,Z(v,\Gamma),Z) = Z - \{v\}$, while $\Delta(\mathscr{E}(f,g,Z(v,\Gamma),Z)) = Z$. To establish that f and g have strongly closed and closed graphs, respectively, we need to check only at v. If Λ is a filter base on $Z - \{v\}$ and $\Lambda \to v$, then $[f(\Lambda)]_\theta = [\Lambda]_\theta \subset [\Omega]_\theta \subset \{v\} = \{f(v)\}$, while $\operatorname{adh}(g(\Lambda)) \subset \operatorname{adh}\Gamma = \varnothing$. $\square$

3. Relationships between the class of minimal Hausdorff (sq) spaces and the classes of countably compact spaces, minimal Hausdorff spaces, SQ-closed spaces, and H-closed spaces. Thompson [14] has proved that every countably compact space is SQ-closed. Theorem 3.1 parallels the well-known result that a space is compact if and only if each of its closed subspaces is H-closed [10, 13].

Theorem 3.1. *A space X is countably compact if and only if each closed subspace of X is SQ-closed.*

Proof. If (X,T) is countably compact, then each closed subspace is countably compact, and hence is SQ-closed. Conversely, suppose that each closed subspace of X is SQ-closed and that $A \subset X$ is infinite and has no limit points in X. Then A is a closed subspace of X which is discrete in its relative topology, and is therefore not SQ-closed. □

We have the following result for minimal Hausdorff (sq) spaces.

Theorem 3.2. *A sequentially compact sequential space is minimal Hausdorff (sq).*

Proof. Let (X,T) be a sequentially compact sequential space, let $v \in X$, and let x_n be a sequence in X with no θ-cluster point in $X-\{v\}$. If $V_0 \in \Sigma(v)$ and x_n is frequently in $X-V_0$, then some subsequence of x_n converges to some point in $X-V_0$, a contradiction to the assumption that x_n has no θ-cluster point in $X-\{v\}$. Therefore $x_n \underset{T}{\to} v$. □

Corollary 3.3. *A countably compact sequential space is minimal Hausdorff (sq).*

Proof. A countably compact sequential space is sequentially compact. □

Corollary 3.4. *If each closed subspace of a sequential space (X,T) is SQ-closed, then X is minimal Hausdorff (sq).*

In [6], Herrington and Long proved that a Hausdorff space is minimal Hausdorff if and only if each filter base on the space with at most one θ-adherent point converges. In view of this result, it is obvious that every sequential minimal Hausdorff space is minimal Hausdorff (sq). But, as can be seen from the following example, a minimal Hausdorff (sq) space need not be minimal Hausdorff.

Example 3.5. The space $[0,\Omega)$ of ordinals less than the first uncountable ordinal endowed with the order topology is not minimal Hausdorff although it is countably compact and first countable.

Example 3.5 also establishes that a minimal Hausdorff (sq) space need not be H-closed since the space there is regular and is not compact.

The following example shows that a first countable minimal Hausdorff space need not be countably compact. The space is the classical example of a countable minimal Hausdorff space which is not compact.

Example 3.6. Let $X = \{0\} \cup \mathbb{N} \cup \{j+1/n : j,n \in \mathbb{N}-\{1\}\}$ and define $V \subset X$ to be open if V satisfies the following properties:

(i) if $j \in (V \cap \mathbb{N})-\{1\}$, then $j+1/n \in V$ ultimately;

(ii) if $0 \in V$, then, ultimately, $j+1/2n \in V$ for all n;
(iii) if $1 \in V$, then, ultimately, $j+1/(2n+1) \in V$ for all n.

The next example shows that a sequential H-closed space need not be minimal Hausdorff (sq).

Example 3.7. Let $Y = \{0\} \cup (\mathbb{N} - \{1\}) \cup \{j+1/2n : j,n \in \mathbb{N} - \{1\}\}$ with the subspace topology T from X in Example 3.6. Then 0 is the only θ-cluster point of x_n defined by $x_n = n+1$, but $x_n \not\to 0$.

It is well known that both compactness and sequential compactness imply countable compactness. We have the following implication diagram for the class of sequential spaces; none of the implications is reversible:

$$\begin{tikzcd}
\text{Compact} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{Minimal Hausdorff} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{H-closed} \arrow[d, Rightarrow] \\
\text{Countably compact} \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & \text{Minimal Hausdorff (sq)} \arrow[r, Rightarrow] & \text{SQ-closed} \\
\text{Sequentially compact.} & &
\end{tikzcd} \tag{3.1}$$

Theorem 3.8. *Every closed subspace of a sequential space is minimal Hausdorff (sq) if and only if the space is sequentially compact.*

Proof. If every closed subspace of a space is minimal Hausdorff (sq), then each closed subspace is SQ-closed and the space is countably compact from Theorem 3.1. Since a countably compact sequential space is sequentially compact, the necessity is established. The sufficiency follows from Theorem 3.2. □

Before moving to some results on products of minimal sequential spaces, we note that quotients of sequential spaces are sequential [4] and quotients of SQ-closed spaces are SQ-closed [3]. However, the space in Example 3.7 is a quotient of the space in Example 3.6 under the partition $\{\{x\} : x \in \mathbb{N} - \{1\}\} \cup \{\{0,1\}\} \cup \{\{1/2n, 1/(2n+1)\} : n \in \mathbb{N} - \{1\}\}$. Hence, quotients of minimal Hausdorff (sq) spaces might fail to be minimal Hausdorff (sq).

Theorem 3.9. *If X_μ is a family of spaces such that the product $X = \Pi X_\mu$ is sequential, then each X_μ is minimal Hausdorff (sq).*

Proof. From [4], X_μ is sequential. Now, let y_n be a sequence in X_μ and $y \in X_\mu$ such that the filter base Ω defined by $F_n = \{y_k : k \geq n\}$ satisfies $[\Omega]_\theta \subset \{y\}$. Choose $v \in X$ and let v_n be the point in X with μ-coordinate y_n and every other coordinate the same as that of v. Then the point $x \in X$ with μ-coordinate y and all other coordinates the same as those of v is the only possible θ-cluster point of the sequence v_n. Hence, $v_n \to x$ and $y_n \to y$. □

Theorem 3.10. *If X_μ is a family of H-closed minimal Hausdorff (sq) spaces such that the product $X = \Pi X_\mu$ is sequential, then X is minimal Hausdorff (sq).*

Proof. Each X_μ is H-closed and minimal Hausdorff (sq), and thus minimal Hausdorff from Corollary 2.5. Hence, X is minimal Hausdorff. □

Theorem 3.11 shows that if a sequence of minimal Hausdorff (sq) spaces has a sequential product, the product is minimal Hausdorff (sq).

Theorem 3.11. *If X_n is a sequence of sequential spaces such that the product $X = \Pi X_n$ is sequential, then X is minimal Hausdorff (sq) if and only if each X_n is minimal Hausdorff (sq).*

Proof

Necessity. This follows from Theorem 3.9.

Sufficiency. Let Ω be a countable filter base on X and suppose $[\Omega]_\theta \subset \{x\}$. From [3], X is SQ-closed and $[\Omega] \underset{\theta}{\to} x$. The projection of the filter base Ω is $\pi_k(\Omega) \underset{\theta}{\to} \pi_k(x)$. So, $[\pi_k(\Omega)]_\theta \subset \{\pi_k(x)\}$. Otherwise, $z \in X$ defined by $\pi_n(z) = \pi_n(x)$ if $n \neq k$, $\pi_k(z) \in [\pi_k(\Omega)]_\theta - \{\pi_k(x)\}$ satisfies $z \in [\Omega]_\theta$, $z \neq x$. Therefore, $\pi_k(\Omega) \to \pi_k(x)$ and $\Omega \to x$. □

Since the product of a sequence of first countable spaces is first countable, we have Corollary 3.12.

Corollary 3.12. *The product ΠX_n of a sequence of first countable minimal Hausdorff (sq) spaces is minimal Hausdorff (sq) if and only if each X_n is first countable minimal Hausdorff (sq).*

Corollary 3.13 comes as a consequence of Theorem 3.11 and a necessary and sufficient condition for the product of two sequential spaces to be sequential, given in [5], in terms of X and Y as quotients (under the quotient maps φ_X, φ_Y) of the topological sums of the convergent sequences (see [4, Proposition 1.12]).

Corollary 3.13. *The product of spaces X, Y is minimal Hausdorff (sq) if and only if X, Y are sequential and $\varphi_X \times \varphi_Y$ is a quotient map.*

Proof

Sufficiency. If X, Y are minimal Hausdorff (sq) spaces and $\varphi_X \times \varphi_Y$ is a quotient map, then $X \times Y$ is minimal Hausdorff (sq) from Theorem 3.11 since, from [5], $X \times Y$ is sequential.

Necessity. This follows from [5] and Theorem 3.9. □

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