IJMMS 2004:27, 1447–1453
PII. S016117120430801X
http://ijmms.hindawi.com

ESTIMATES FOR THE NORMS OF SOLUTIONS OF DIFFERENCE SYSTEMS WITH SEVERAL DELAYS

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Received 1 August 2003

We derive explicit stability conditions for time-dependent difference equations with several delays in C^n (the set of n complex vectors) and estimates for the size of the solutions. The growth rates obtained here are not necessarily decay rates.

2000 Mathematics Subject Classification: 39A10.

1. Introduction. Stability of systems of difference equations with delays has been discussed by many authors, for example, see Gil' and Cheng [6], Zhang [11], Elaydi and Zhang [5], Pituk [10], Agarwal [1], and the references therein.

In the stability literature, one can find two major trends; stability using the first approximation Lyapunov method and the direct Lyapunov functional method. For this latter trend, see Zhang and Chen [12], Crisci et al. [4], Lakshmikantham and Trigiante [7], and Agarwal and Wong [2]. By this method many very strong results are obtained. But finding Lyapunov functionals is usually a difficult task.

In this note, we consider a class of perturbed difference equations with several delays and, by means of a Gronwall inequality and the recent estimates for the powers A^k of a constant matrix A established in Corduneanu [3], we derive explicit stability conditions. Further, we suppose that the unperturbed linear difference equations have a bounded growth. Actually, this work is an extension of Medina [8] to time-dependent difference equations with several delays.

2. Preliminary facts. Let C^n be the set of n complex vectors endowed with a norm $\|\cdot\|$. Let A_k $(k=1,2,\dots)$ be $n\times n$-complex matrices.

Consider in C^n the equation

$$u_{j+1}=A_ju_j+f_j(u_{j-\sigma_1},\dots,u_{j-\sigma_p}),\quad j=0,1,\dots,\tag{2.1}$$

where $p\geq 1$, $\sigma_1,\sigma_2,\dots,\sigma_p$ are nonnegative integers such that $0=\sigma_1<\sigma_2<\cdots<\sigma_p,\sigma_i\in Z^+$, and Z^+ is the set of nonnegative integers, f_j, $j=0,1,2,\dots,$ maps C^{np} into C^n.

We will consider (2.1) subject to the initial conditions

$$u_j=\tau_j,\quad j=-\sigma_p,-\sigma_p+1,\dots,0.\tag{2.2}$$

It is assumed that there are nonnegative sequences q_l $(l = 1,2,3,\dots,p)$ such that

$$\|f_j(u_{j-\sigma_1},\dots,u_{j-\sigma_p})\| \le \sum_{l=1}^{p} q_l(j)\|u_{j-\sigma_l}\|^m, \quad j = 0,1,\dots, \tag{2.3}$$

and m is a fixed positive real number.

Unlike differential equations, discrete equations with the given initial conditions always have a solution.

In order to establish our main result, we will use the following discrete Gronwall-type inequality.

Theorem 2.1 [9]. *Assume that*

$$z(k) \le C + \sum_{i=0}^{k-1}\sum_{j=1}^{p} a_j(i)z(i-\sigma_j)^m, \quad k \in Z^+, \tag{2.4}$$

where $m > 0$, $0 = \sigma_1 < \sigma_2 < \cdots < \sigma_p$, $p \ge 1$, $C > 0$, $a_j(k) \ge 0$ *for* $j = 1,2,\dots,p$ *and* $k \in Z^+$, *and* $z(k) \le C$ *for* $k = -\sigma_p, -\sigma_p + 1,\dots,0$.

(a) *If* $0 < m < 1$ *and* $C \le 1$, *then*

$$z(k) \le C^{m^k}\prod_{i=0}^{k-1}\left[1 + \sum_{j=1}^{p} a_j(i)\right], \quad k \in Z^+. \tag{2.5}$$

(b) *If* $m = 1$, *then*

$$z(k) \le C\prod_{i=0}^{k-1}\left[1 + \sum_{j=1}^{p} a_j(i)\right], \quad k \in Z^+. \tag{2.6}$$

(c) *If* $m > 1$, *then*

$$z(k) \le \frac{C}{\left\{1-(m-1)C^{m-1}\cdot\sum_{i=0}^{k-1}\sum_{j=1}^{p} a_j(i)\right\}^{1/(m-1)}}, \quad k \in Z^+, \tag{2.7}$$

provided that

$$1-(m-1)C^{m-1}\sum_{i=0}^{k-1}\sum_{j=1}^{p} a_j(i) > 0, \quad k \in Z^+. \tag{2.8}$$

Assumption 2.2. It is assumed that the unperturbed linear difference equation

$$u_{j+1} = A_j u_j, \quad j = 0,1,\dots, \tag{2.9}$$

has a bounded growth, that is, there exist real constants $\gamma \ge 1$ and $\alpha > 0$ such that

$$\|\Phi(j,i)\| \le \gamma\alpha^{j-i}, \quad \forall j \ge i \ge 0, \tag{2.10}$$

where $\Phi(k,l) = \prod_{j=l}^{k-1} A_j$, $k > l$, is the fundamental matrix solution of (2.9).

3. Main results. Now, we are in a position to establish our main results pertaining to the bounded growth and the zero convergence properties of the solutions of (2.1) subject to the conditions (2.2).

Theorem 3.1. *Assume that conditions (2.3) and (2.10) hold. In addition, assume that*

$$\sum_{i=0}^{\infty}\sum_{l=1}^{p}\alpha^{(m-1)i-m\sigma_l}q_l(i)<\infty. \tag{3.1}$$

Then,

(a) *if $0<m\le 1$ and $C=\gamma\|\tau_0\|\le 1$, every solution u_j of (2.1), (2.2), such that $\|u_j\|\le C\alpha^j$ for $j=-\sigma_p,-\sigma_p+1,\dots,0$, satisfies*

$$\|u_j\|\le \alpha^j(\gamma\|\tau_0\|)^{m^j}\exp\left(\gamma\sum_{i=0}^{\infty}\sum_{l=1}^{p}\beta_l(i)\right),\quad j=1,2,\dots, \tag{3.2}$$

where $\beta_l(i)=\alpha^{(m-1)i-m\sigma_l-1}q_l(i)$,

(b) *if $m>1$ and*

$$\|\tau_0\|\le\left\{\frac{\eta}{(m-1)\lambda\gamma^m}\right\}^{1/(m-1)},\quad \lambda=\sum_{i=0}^{\infty}\sum_{l=1}^{p}\beta_l(i), \tag{3.3}$$

for some $\eta\in(0,1)$ and $C=\gamma\|\tau_0\|$, every solution u_j of (2.1), (2.2) such that $\|u_j\|\le C\alpha^j$ for $j=-\sigma_p,-\sigma_p+1,\dots,0$, satisfies

$$\|u_j\|\le\frac{\gamma\alpha^j}{(1-\eta)^{1/(m-1)}}\|\tau_0\|,\quad j=0,1,2,\dots. \tag{3.4}$$

Proof. By inductive arguments, we can prove that the unique solution $\{u_j\}_{j=-\sigma_p}^{\infty}$ of (2.1), subject to given initial values $u_0=\tau_0$, $u_{-1},\dots,u_{-\sigma_p}$, satisfies

$$u_j=\Phi(j,0)\tau_0+\sum_{i=0}^{j-1}\Phi(j,i+1)f_i\left(u_{(i-\sigma_1)},\dots,u_{(i-\sigma_p)}\right),\quad j\in Z^+. \tag{3.5}$$

Hence, by conditions (2.3) and (2.10),

$$\|u_j\|\le\gamma\alpha^j\|\tau_0\|+\gamma\sum_{i=0}^{j-1}\sum_{l=1}^{p}\alpha^{j-i-1}q_l(i)\|u_{i-\sigma_l}\|^m. \tag{3.6}$$

This yields

$$\alpha^{-j}\|u_j\|\le\gamma\|\tau_0\|+\gamma\sum_{i=0}^{j-1}\sum_{l=1}^{p}\alpha^{-i-1}q_l(i)\|u_{i-\sigma_l}\|^m. \tag{3.7}$$

By setting $z(j)=\alpha^{-j}\|u_j\|$ and $\beta_l(i)=\alpha^{(m-1)i-m\sigma_l-1}q_l(i)$, it follows that

$$z(j)\le C+\gamma\sum_{l=1}^{p}\sum_{i=0}^{j-1}\beta_l(i)z^m(i-\sigma_l), \tag{3.8}$$

where $C=\gamma\|\tau_0\|$ and $z(j)\le C$ for $j=-\sigma_p,-\sigma_p+1,\dots,0$.

Case 1. If $0 < m \le 1$ and $C \le 1$, then by Theorem 2.1(a) it follows that

$$z(j) \le C^{m^j} \prod_{i=0}^{j-1} \left[1 + \gamma \sum_{l=1}^{p} \beta_l(i) \right] \le C^{m^j} \exp\left(\gamma \sum_{i=0}^{\infty} \sum_{l=1}^{p} \beta_l(i) \right), \tag{3.9}$$

and the proof of Case 1 is complete.

Case 2. If $m > 1$, proceeding in a similar way to Case 1, we arrive at the inequality (3.8). Hence, by Theorem 2.1(b), it follows that

$$z(j) \le \frac{C}{\left\{ 1 - (m-1)\gamma C^{m-1} \sum_{i=0}^{j-1} \sum_{l=1}^{p} \beta_l(i) \right\}^{1/(m-1)}}, \tag{3.10}$$

provided that

$$1 - (m-1)\gamma C^{m-1} \sum_{i=0}^{j-1} \sum_{l=1}^{p} \beta_l(i) > 0. \tag{3.11}$$

Let $\eta \in (0,1)$ be an arbitrary number. We will prove that the condition (3.11) holds for all τ_0 satisfying

$$||\tau_0|| \le \left\{ \frac{\eta}{(m-1)\lambda\gamma^m} \right\}^{1/(m-1)} =: R, \tag{3.12}$$

where $\lambda = \sum_{i=0}^{\infty} \sum_{l=1}^{p} \beta_l(i) < \infty$.

Indeed, for all such a τ_0, we have

$$(m-1)\gamma^m ||\tau_0||^{m-1} \sum_{i=0}^{j-1} \sum_{l=1}^{p} \beta_l(i) \le (m-1)\gamma^m ||\tau_0||^{m-1} \sum_{i=0}^{\infty} \sum_{l=1}^{p} \beta_l(i) \le \eta. \tag{3.13}$$

Thus,

$$1 - (m-1)\gamma^m ||\tau_0||^{m-1} \sum_{i=0}^{j-1} \sum_{l=1}^{p} \beta_l(i) \ge 1 - \eta > 0. \tag{3.14}$$

Consequently, for all τ_0 such that $\|\tau_0\| \le R$, we have

$$\begin{aligned} ||u_j|| &\le \frac{C\alpha^j ||\tau_0||}{\left\{ 1 - (m-1)\gamma C^{m-1} \sum_{i=0}^{j-1} \sum_{l=1}^{p} \beta_l(i) \right\}^{1/(m-1)}} \\ &\le \frac{\gamma\alpha^j}{(1-\eta)^{1/(m-1)}} ||\tau_0||, \quad j \in Z^+. \end{aligned} \tag{3.15}$$

Hence the proof of Case 2 is complete. □

Remark 3.2. We want to point out the explicit dependence of the growth constants of the perturbed equation (2.1) upon the growth constants of the unperturbed equation (2.9) and the estimate for the perturbation f. Further, the growth rates obtained here are not necessarily decay rates.

Corollary 3.3. *Under the assumptions of Theorem 3.1, with α in the open interval $(0,1)$, we have*

(i) *if $0 < m \le 1$ and $C = \gamma\|\tau_0\| \le 1$, every solution u_j with sufficiently small initial data tends to zero as $j \to \infty$,*

(ii) *if $m > 1$ and $\|\tau_0\| \le R$, then the zero solution u_j of (2.1), (2.2) is asymptotically stable.*

Indeed, the inequality $\|u_j\| \le \gamma\alpha^j/(1-\eta)^{1/(m-1)}\|\tau_0\| := K\alpha^j\|\tau_0\|$, $j \in Z^+$, shows that for any $\varepsilon > 0$, we can choose a suitable number $0 < \delta < \min\{R, \varepsilon/K\}$ and a number $N > 0$ such that for all $k > N$ and $\|\tau_0\| < \delta$, we have $\|u_j\| < \varepsilon$.

Remark 3.4. If $A_k \equiv A$ is a constant matrix, whose spectral radius is less than 1, then the zero solution of (2.9) is uniformly asymptotically stable. However, this result cannot be extended to nonautonomous equations (see [7, Theorem 4.4.1]).

4. Special cases. If the system (2.9),

$$u_{j+1} = A_j u_j, \tag{4.1}$$

has slowly varying coefficients, then the condition (2.10) concerning growth of the solutions can be avoided in the case

$$||A_k - A_j|| \le q_{k-j} \quad (q_k = q_{-k} = \text{const} > 0,\ q_0 = 0;\ j,k = 1,2,\dots). \tag{4.2}$$

On the other hand, Corduneanu [3] established that for any constant matrix A there exists a constant $\Gamma \ge 1$, independent of the integers $j = 0,1,2,\dots$ such that

$$||A^j|| \le \Gamma\rho^j(A), \quad j = 0,1,\dots, \tag{4.3}$$

where $\rho(A)$ is the spectral radius of A.

In particular, if $A = (a_{ij})$ is a triangular constant matrix, then $\Gamma = 1$.

Consider in C^n the equation

$$u_{j+1} = A_j u_j + g_j, \quad j \in Z^+, \tag{4.4}$$

where A_j $(j = 0,1,\dots)$ are $n \times n$-complex matrices and g_j, u_j are vectors in C^n.

Theorem 4.1. *Under condition (4.2), assume that*

$$\begin{aligned} \rho_0 &= \sup_{l=0,1,\dots} (\Gamma_l \rho(A_l)) < 1, \\ S_0(A) &= \sum_{k=0}^{\infty} q_k \rho_0^k < 1, \\ S_1(A;g) &= \sum_{k=0}^{\infty} \rho_0^k ||g_k|| < \infty, \end{aligned} \tag{4.5}$$

where Γ_l *and* $\rho(A_l)$ *have the same meaning as the quantities in (4.3) referring to A. Then any solution* $\{u_j\}_{j=0}^{\infty}$ *of (4.4) satisfies the inequality*

$$\sup_{j=1,2,\dots} ||u_j|| \le \frac{\rho_0||u_0|| + S_1(A;g)}{1 - S_0(A)}. \tag{4.6}$$

Proof. Rewrite (4.4) as

$$u_{j+1} - A_l u_j = (A_j - A_l)u_j + g_j \tag{4.7}$$

with a fixed integer l. The variation of parameters formula yields

$$u_{l+1} = A_l^{l+1}u_0 + \sum_{j=0}^{l} A_l^{l-j}[(A_j - A_l)u_j + g_j]. \tag{4.8}$$

It follows from (4.2) and (4.3) that

$$\begin{aligned} ||u_{l+1}|| &\le \rho_0^{l+1}||u_0|| + \sum_{j=0}^{l} ||A_l^{l-j}||[q_{l-j}||u_j|| + ||g_j||] \\ &\le \rho_0||u_0|| + \max_{j=0,1,2,\dots,l} ||u_j|| \sum_{j=0}^{l} ||A_l^j||[q_j + ||g_{l-j}||] \\ &\le \rho_0||u_0|| + \max_{j=0,1,2,\dots,l} ||u_j|| \cdot \sum_{j=0}^{\infty} q_j \sup_{l=0,1,2,\dots} ||A_l^j|| + \sum_{j=0}^{\infty} ||g_j|| \sup_{l=0,1,2,\dots} ||A_l^j||. \end{aligned} \tag{4.9}$$

Consequently,

$$\max_{j=0,1,2,\dots,l+1} ||u_j|| \le \rho_0||u_0|| + S_0(A) \cdot \max_{j=0,1,\dots,l+1} ||u_j|| + S_1(A;g), \tag{4.10}$$

and we infer that

$$\sup_{j=1,2,\dots} ||u_j|| \le \frac{\rho_0||u_0|| + S_1(A;g)}{1 - S_0(A)}, \tag{4.11}$$

concluding the proof. □

Consider the equation

$$u_{j+1} = A_j u_j + f_j(u_j), \quad j = 0,1,2,\dots. \tag{4.12}$$

Assume that there are constants ν, $\mu \ge 0$ such that

$$||f_j(u)|| \le \nu\|u\| + \mu, \quad j = 0,1,2,\dots. \tag{4.13}$$

Denote

$$\theta_0 = \frac{\mu}{1 - \rho_0}, \quad \rho_0 = \sup_{l=0,1,\dots} (\Gamma_l \rho(A_l)). \tag{4.14}$$

Now we are in a position to formulate the next result of this paper.

Theorem 4.2. *Under conditions (4.2) and (4.13), assume that*

$$\rho_0 < 1, \quad \psi(A;f) = \sum_{j=0}^{\infty} (q_j + \nu)\rho_0^j < 1. \tag{4.15}$$

Then any solution $\{u_j\}_{j=0}^{\infty}$ *of (4.12) satisfies the inequality*

$$\sup_{j=1,2,\dots} \|u_j\| \le \frac{\rho_0\|u_0\| + \theta_0}{1 - \psi(A;f)}. \tag{4.16}$$

Proof. It can be proved in a similar way to Theorem 4.1, so we will omit the proof. □

Acknowledgments. I thank the referees of this paper for their careful and insightful critique. This research was supported by Fondecyt Chile under Grant 1.030.460 and by Dirección de Investigación, Universidad de Los Lagos.

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