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A NOTE ON THE SPECTRA OF TRIDIAGONAL MATRICES

C. M. DA FONSECA and J. PETRONILHO

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Using orthogonal polynomials, we give a different approach to some recent results on tridiagonal matrices.

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1. Introduction and preliminaries. Most of the results on tridiagonal matrices can be seen in the context of orthogonal polynomials. Here, we will see two examples. The aim of this note is to give simplified proofs of some recent results on tridiagonal matrices, using arguments from the theory of orthogonal polynomials.

One of the most important tools in the study of orthogonal polynomials is the spectral theorem for orthogonal polynomials, which states that any orthogonal polynomial sequence (OPS) $\{P_n\}_{n\geq 0}$ is characterized by a three-term recurrence relation

$$xP_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

with initial conditions $P_{-1}(x) = 0$ and $P_0(x) = \text{const} \neq 0$, where $\{\alpha_n\}_{n\geq 0}$, $\{\beta_n\}_{n\geq 0}$, and $\{\gamma_n\}_{n\geq 0}$ are sequences of complex numbers such that $\alpha_n\gamma_{n+1} \neq 0$ for all $n = 0, 1, 2, \ldots$.

The next proposition is known as the separation theorem for zeros and tells us that the zeros of P_n and P_{n+1} mutually separate each other.

Theorem 1.1 (see, e.g., [1, page 28]). *In (1.1), let $\beta_n \in \mathbb{R}$ and $\gamma_{n+1} > 0$ for all $n = 0, 1, 2, \ldots$. Then, for each n, P_n has n real and distinct zeros, denoted in increasing order by $x_{n1} < x_{n2} < \cdots < x_{nn}$. Furthermore, the interlacing inequalities $x_{n+1,i} < x_{ni} < x_{n+1,i+1}$ $(i = 1, \ldots, n)$ hold for every $n = 1, 2, \ldots$.*

In 1961, Wendroff showed that some reciprocal-like of the separation theorem is also true.

Theorem 1.2 [7]. *Let $a < x_1 < x_2 < \cdots < x_n < b$ and $x_i < y_i < x_{i+1}$, for $i = 1, \ldots, n-1$. Then there exists a monic orthogonal polynomial sequence (MOPS) $\{P_n\}_{n\geq 0}$ such that $P_n(x) = (x - x_1)\cdots(x - x_n)$ and $P_{n-1}(x) = (x - y_1)\cdots(x - y_{n-1})$.*

Notice that the three-term recurrence relation (1.1) can be written in matrix form as

$$x\begin{pmatrix} P_0(x) \\ P_1(x) \\ \vdots \\ P_n(x) \end{pmatrix} = J_{n+1}\begin{pmatrix} P_0(x) \\ P_1(x) \\ \vdots \\ P_n(x) \end{pmatrix} + \alpha_n P_{n+1}(x)\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}, \tag{1.2}$$

where J_{n+1} is a tridiagonal Jacobi matrix of order $n+1$, defined by

$$J_{n+1} := \begin{pmatrix} \beta_0 & \alpha_0 & & & \\ \gamma_1 & \beta_1 & \alpha_1 & & \\ & \gamma_2 & \ddots & \ddots & \\ & & \ddots & \beta_{n-1} & \alpha_{n-1} \\ & & & \gamma_n & \beta_n \end{pmatrix} \quad (n = 0,1,2,\ldots). \tag{1.3}$$

It follows that if $\{x_{nj}\}_{j=1}^{n}$ is the set of zeros of the polynomial P_n, then each x_{nj} is an eigenvalue of the corresponding Jacobi matrix J_n of order n, and an associated eigenvector is $[P_0(x_{nj}), P_1(x_{nj}), \ldots, P_{n-1}(x_{nj})]^t$.

Given a family of orthogonal polynomials $\{P_n\}_{n\ge 0}$ defined by (1.1) with $\alpha_n = 1$ for all n and $\gamma_n > 0$ for all $n = 1,2,\ldots$ (so that $\{P_n\}_{n\ge 0}$ is an MOPS), we may define the associated monic polynomials of order r (r a positive integer) $\{P_n^{(r)}\}_{n\ge 0}$, $r = 0,1,2,\ldots,$ by the shifted recurrence

$$P_{n+1}^{(r)}(x) = (x - \beta_{n+r}) P_n^{(r)}(x) - \gamma_{n+r} P_{n-1}^{(r)}(x), \quad n = 0,1,2,\ldots, \tag{1.4}$$

with $P_{-1}^{(r)} = 0$ and $P_{-1}^{(0)} = 1$. The *anti-associated polynomials* for the family $\{P_n\}_{n\ge 0}$, denoted by $\{P_{n+r}^{(-r)}\}_{n\ge 0}$, are obtained by pushing down a given Jacobi matrix and by introducing, in the empty upper left corner, new coefficients β_{-i}, $i = r, r-1, \ldots, 1$, on the subdiagonal and new coefficients $\gamma_{-i} > 0$, $i = r-1, r-2, \ldots, 0$, on the lower subdiagonal (see, e.g., [6]). The new Jacobi matrix is then of the form

$$\left(\begin{array}{cccc|ccc} \beta_{-r} & 1 & & & & & \\ \gamma_{-r+1} & \beta_{-r+1} & 1 & & & & \\ & \gamma_{-r+2} & \ddots & \ddots & & & \\ & & \ddots & \beta_{-1} & 1 & & \\ \cline{5-7} & & & \gamma_0 & \beta_0 & 1 & \\ & & & & \gamma_1 & \ddots & \ddots \\ & & & & & \ddots & \ddots \end{array}\right). \tag{1.5}$$

If $\{Q_n\}_{n\ge 0}$ satisfies $Q_{-1} = 0$, $Q_0 = 1$, and

$$Q_{n+1}(x) = (x - \beta_{-r+n}) Q_n(x) - \gamma_{-r+n} Q_{n-1}(x), \quad n = 0,1,\ldots,r-1, \tag{1.6}$$

then, clearly,

$$P_n^{(-r)}(x) = Q_n(x), \quad n = 0,1,\ldots,r. \tag{1.7}$$

For $n > r$, the anti-associated polynomials satisfy the three-term recurrence relation

$$P_{n+r+1}^{(-r)}(x) = (x - \beta_n) P_{n+r}^{(-r)}(x) - \gamma_n P_{n+r-1}^{(-r)}(x), \quad n = 0,1,2,\ldots. \tag{1.8}$$

The anti-associated polynomial $P_{n+r}^{(-r)}$ was represented in [6] as a linear combination of the original family P_n and the associated polynomials $P_{n-1}^{(1)}$ in the following way:

$$P_{n+r}^{(-r)}(x) = Q_r(x)P_n(x) - \gamma_0 Q_{r-1}(x)P_{n-1}^{(1)}(x), \quad n = 0,1,2,\dots. \tag{1.9}$$

2. Inverse eigenvalue problems. For an $n \times n$ matrix A, we denote by $\sigma(A)$ the spectrum of A and by A' the $(n-1)$th leading principal submatrix of A. In [5], Gray and Wilson stated the following theorem.

Theorem 2.1 [5]. *Let $\{\mu_1,\dots,\mu_n\}$ and $\{\nu_1,\dots,\nu_{n-1}\}$ be sets of real numbers satisfying*

$$\mu_1 < \nu_1 < \mu_2 < \nu_2 < \cdots < \mu_{n-1} < \nu_{n-1} < \mu_n. \tag{2.1}$$

Then there exists a unique symmetric tridiagonal $n \times n$ matrix A with positive super- and subdiagonals such that $\sigma(A) = \{\mu_1,\dots,\mu_n\}$ and $\sigma(A') = \{\nu_1,\dots,\nu_{n-1}\}$.

This result is a consequence of Theorem 1.2. The positiveness of the two mentioned off-diagonals is a direct consequence of Wendroff's proof.

One says that a set of numbers S is symmetric if $S = -S$. If $\{P_k\}_{k\geq 0}$ is the MOPS given by Wendroff's theorem, then it is possible to prove that $\beta_k = 0$ for all k in the three-term recurrence relation (1.1). As a consequence, we have the following proposition.

Theorem 2.2 [3]. *Let $S_1 = \{\mu_1,\dots,\mu_n\}$ and $S_2 = \{\nu_1,\dots,\nu_{n-1}\}$ be symmetric sets of real numbers satisfying*

$$\mu_1 < \nu_1 < \mu_2 < \nu_2 < \cdots < \mu_{n-1} < \nu_{n-1} < \mu_n. \tag{2.2}$$

Then there exists a unique symmetric tridiagonal zero main diagonal $n \times n$ matrix A with positive super- and subdiagonals such that $\sigma(A) = \{\mu_1,\dots,\mu_n\}$ and $\sigma(A') = \{\nu_1,\dots,\nu_{n-1}\}$.

3. Antipodal tridiagonal pattern. An $n \times n$ (sign) *pattern* is a matrix, where each entry is $+$, $-$, or 0. A pattern $S = (s_{ij})$ defines a *pattern class* of real matrices

$$Q(S) = \{A = (a_{ij}) \mid \operatorname{sign}(a_{ij}) = s_{ij},\ \forall i,j\}. \tag{3.1}$$

For $n \geq 2$, we consider the *antipodal tridiagonal pattern* T_n defined as

$$T_n = \begin{pmatrix} - & + & & & \\ - & 0 & + & & \\ & - & \ddots & \ddots & \\ & & \ddots & 0 & + \\ & & & - & + \end{pmatrix}_{n\times n}. \tag{3.2}$$

In [2], Drew et al. have considered the matrix $A_n \in Q(T_n)$:

$$A_n = \begin{pmatrix} -a_0 & 1 & & & \\ -a_1 & 0 & 1 & & \\ & -a_2 & \ddots & \ddots & \\ & & \ddots & 0 & 1 \\ & & & -a_{n-1} & a_n \end{pmatrix}. \tag{3.3}$$

They proved that if $\det(zI_n - A_n) = z^n$, then A_n is symmetric about the reverse diagonal.

Theorem 3.1 [2]. *If there exists $A_n \in Q(T_n)$ such that $\det(\lambda I_n - A_n) = z^n$, then $a_i = a_{n-i}$ for $i = 0, 1, \ldots, \lfloor (n-1)/2 \rfloor$.*

This can be generalized in the following way.

Theorem 3.2. *Let $A_n \in Q(T_n)$ and $\phi_n(\lambda) = \det(\lambda I_n - A_n)$. Then $\phi_n(-\lambda) = (-1)^n \phi_n(\lambda)$ if and only if A_n is symmetric about the reverse diagonal.*

Recently, Elsner et al. [4] gave a proof of this theorem. We present a different proof by using anti-associated polynomials as defined in the first section. Let $P_{n-1}^{(-1)}$ be the anti-associated polynomial, with $\gamma_0 = a_1$ and $\beta_{-1} = 0$, for the characteristic polynomial P_{n-1} of the matrix

$$\begin{pmatrix} 0 & 1 & & \\ a_2 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & a_{n-1} & 0 \end{pmatrix}. \tag{3.4}$$

Let $\tilde{\phi}_n(\lambda) = \det(\lambda I_n - iA_n)$. Then

$$\tilde{\phi}_n(\lambda) = P_n^{(-1)}(\lambda) - i a_n P_{n-1}^{(-1)}(\lambda) + i a_0 P_{n-1}(\lambda) + a_0 a_n P_{n-2}(\lambda). \tag{3.5}$$

If $\tilde{\phi}_n(-\lambda) = (-1)^n \tilde{\phi}_n(\lambda)$, then

$$a_n P_{n-1}^{(-1)}(\lambda) = a_0 P_{n-1}(\lambda). \tag{3.6}$$

Since the polynomials are monic, $a_n = a_0$. On the other hand, $P_{n-1}(\lambda) = \lambda P_{n-2}(\lambda) - a_{n-1} P_{n-3}(\lambda)$ and, by (1.9), $P_{n-1}^{(-1)}(\lambda) = \lambda P_{n-2}(\lambda) - a_1 P_{n-3}^{(1)}(\lambda)$, which implies $a_{n-1} = a_1$. Furthermore, $P_{n-3}(\lambda) = P_{n-3}^{(1)}(\lambda)$, and now we only have to apply the procedure to the matrix

$$\begin{pmatrix} 0 & 1 & & \\ a_3 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & a_{n-2} & 0 \end{pmatrix}. \tag{3.7}$$

Notice that the condition $\phi_n(-\lambda) = (-1)^n\phi_n(\lambda)$ is equivalent to $\tilde{\phi}_n(-\lambda) = (-1)^n\tilde{\phi}_n(\lambda)$. This completes the proof.

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C. M. da Fonseca: Departamento de Matemática, Universidade de Coimbra, 3001-454 Coimbra, Portugal

E-mail address: cmf@mat.uc.pt

J. Petronilho: Departamento de Matemática, Universidade de Coimbra, 3001-454 Coimbra, Portugal

E-mail address: josep@mat.uc.pt