IJMMS 2004:32, 1715–1723
PII. S0161171204210389
http://ijmms.hindawi.com

RESERVICING SOME CUSTOMERS IN $M/G/1$ QUEUES UNDER THREE DISCIPLINES

M. R. SALEHI-RAD, K. MENGERSEN, and G. H. SHAHKAR

Received 23 October 2002

Consider an $M/G/1$ production line in which a production item is failed with some probability and is then repaired. We consider three repair disciplines depending on whether the failed item is repaired immediately or first stockpiled and repaired after all customers in the main queue are served or the stockpile reaches a specified threshold. For each discipline, we find the probability generating function (p.g.f.) of the steady-state size of the system at the moment of departure of the customer in the main queue, the mean busy period, and the probability of the idle period.

2000 Mathematics Subject Classification: 60K25, 68M20, 90B22.

1. Introduction. Consider an $M/G/1$ queueing system in which some customers must be reserviced with probability p. These models might be considered in a production line, in sending messages through the network, running computer programs, telephone network congestion, communication problems, and so on. In a production line, some items might be failed and require repair. When messages are sent through a network, some may be returned (postmaster) and must be sent again. An operator running computer programs may encounter some errors and need to run them again. In a telephone center, some calls may be not completed due to signal failure and must be reestablished. In these kinds of problems, we must reservice some items.

Optimal operating policies for single-server systems in which the server may be turned on and off have been studied over a period of more than three decades. Earlier authors include Yadin and Naor [12], Heyman [4], Sobel [9], and Bell [1]. Yadin and Naor [12] proposed shutdown control for an $M/M/1$ queue in order to increase the length of individual idle periods. Soon after, Heyman [4] proved that $(0,N)$ control (i.e., the server turns off if the system size is zero and turns on when the system size is N) is the optimal policy in an $M/G/1$ queue by considering the start-up and shutdown cost of a server, a cost per unit time when a server is running and a customer is waiting cost. This model was modified by Heyman and Marshal [5] by allowing for interarrival distributions that have increasing rates. These authors found an optimal policy for the undiscounted infinite horizon problem and bounds on the cost rate. Heyman's result was also improved by Bell [1] who proved that for an $M/G/1$ queue with a given operating cost structure, shutdown control is the optimal stationary operating policy. More generally, Sobel [9] showed that almost any type of stationary policy is equivalent to an (n,N) model, by considering the criterion of average cost rate over an infinite horizon.

This problem remains an active research area. For example, Kitaev and Serfozo [6] considered an $M/M/1$ queueing system with dynamically controlled arrival and service rates. Their results described natural conditions on the costs under which an optimal policy for either the discounted-cost or average-cost criterion is a hysteretic policy. Such a policy increases the service rate and decreases the arrival rate as the queue length increases. More recently, Deng and Tan [2] studied a single-server two-queue priority system with changeover times and switching threshold. They considered an $M/M/1$ queue, obtained the steady-state joint probability generating function of the length of the two queues, and then calculated the mean length of queue and mean delay.

In this note, we consider three alternatives for reservicing in a steady-state $M/G/1$ queue. These alternatives, denoted as disciplines I, II, and III, are described in Sections 3, 4, and 5. For each discipline, we find the probability generating function (p.g.f.) of the steady-state size of the system at the moment of departure of the customer in the main queue, the mean busy period, and the probability of the idle period (the proportion of time that the server is idle).

In Section 2, we describe the problem. In Sections 3, 4, and 5, for each discipline, explicit closed formula for the steady-state p.g.f. of the imbedded Markov chains will be derived. Also, we will find the proportion of time during which the server is idle in each discipline through finding the mean busy period and using Little's formula, that is, $E(\text{busy period})/E(\text{idle period}) = (1-\pi^*)/\pi^*$, where π^* is the probability of the idle period. The key mathematical tool that we use is Laplace-Stieltjes transform (LST) of a function, which for function $F(\cdot)$ we denote by $F^*(\cdot)$.

2. Description of the problem. As mentioned, we consider an $M/G/1$ queueing model in a steady state in which some items are failed with probability p, and require reservice. Three disciplines may be considered.

In discipline I, the server reservices a failed item immediately after completion of the service of the customer in the main queue (MQ), if the item served has failed.

In discipline II, failed items are stockpiled in a failed queue (FQ) and reserviced only after all customers in MQ are serviced. After completion of reservice of all items in FQ, the server returns to MQ if there are customers waiting; otherwise the system is idle.

Discipline III is the same as discipline II, except that the server also switches to FQ if there are N failed items in FQ (threshold N). Again, all items in FQ are reserviced before returning to MQ.

Let $t_1, t_2, \ldots$ be times at which a service in MQ is completed. We suppose that service (s) and reservice $(\tilde{s})$ times are independent and have general distributions, denoted by $B_1(\cdot)$ and $B_2(\cdot)$ with means $1/\mu_1$ and $1/\mu_2$, respectively. $A_n(s)$ and $\tilde{A}_n(\tilde{s})$ are the numbers of arrivals in MQ, during servicing and reservicing in MQ and FQ, respectively, and are distributed as Poisson with parameters λs and $\lambda\tilde{s}$ at the moment of the nth departure in MQ, respectively. Of course, since these are independent of n, we show them by $A(s)$ and $\tilde{A}(\tilde{s})$.

In discipline I, the imbedded Markov chain is $X(t_n)$ (or, for convenience, X_n), the number of customers remaining in MQ at the completion of the nth customer's service time. In contrast, disciplines II and III are described by the bivariate Markov chain $(X(t_n),Y(t_n))$ (or, for convenience (X_n,Y_n)), where Y_n is the number of customers remaining in FQ at the completion of the nth customer's service time. Thus, for discipline I,

$$X_{n+1} = (X_n-1)_+ + A_{n+1}(s) + [\tilde{A}_{n+1}(\tilde{s})-1]_+ I_{\{X_n=0\}}, \tag{2.1}$$

for discipline II, (X_{n+1},Y_{n+1}) becomes

$$((X_n-1)_+ + A_{n+1}(s) + [\Sigma_{i=1}^{Y_n}\tilde{A}_{n+1}(\tilde{s}_i)-1]_+ I_{\{X_n=0\}}, Y_n I_{\{X_n>0\}} + U_{n+1}), \tag{2.2}$$

and, for discipline III, (X_{n+1},Y_{n+1}) is given by

$$\Big((X_n-1)_+ + A(s) + \Big[\Sigma_{i=1}^{Y_n}\tilde{A}(\tilde{s}_i)I_{C_2\cup C_3} - I_{C_3}\Big]_+, Y_n I_{C_1} + U_{n+1}\Big). \tag{2.3}$$

In these expressions, $(x-1)_+$ is $\max\{x-1,0\}$, $U_{n+1}=1$ if the departure has failed and is otherwise zero, $C_1=\{X_n>0,\ 0\le Y_n<N\}$, $C_2=\{X_n>0,\ Y_n=N\}$, $C_3=\{X_n=0,\ 0<Y_n\le N\}$, and $C_4=\{X_n=0,\ Y_n=0\}$.

3. The queueing discipline I. In this section, we consider discipline I and find the p.g.f. of the steady-state size of the system and $\pi_I^*=\Pr(\text{idle period})$ through finding the mean busy period. Three lemmas that are useful for finding the required p.g.f. stated are below. For proofs, see Salehi-Rad and Mengersen [8].

Lemma 3.1. *If $A(s)$ and $\tilde{A}(\tilde{s})$ are the numbers of arrivals during service and reservice times, respectively, then their p.g.f.'s are $Q_i(u)=B_i^*[\lambda(1-u)]$, $i=1,2$, where $B_1^*(\cdot)$ and $B_2^*(\cdot)$ are the LSTs of the distribution functions of the service and reservice times, respectively.*

Lemma 3.2. *By Lemma 3.1, $E[\Pr\{\tilde{A}(\tilde{s})=0\}]=B_2^*(\lambda)$.*

Lemma 3.3. *If X is a nonnegative integer-valued random variable with p.g.f. $P(u)$, then*

$$E[u^{(X-1)_+}] = u^{-1}[P(u)-(1-u)\Pr(X=0)]. \tag{3.1}$$

Now, by developing a proposition, we find the p.g.f. of X_n, denoted by $P(u)=E(u^{X_n})$.

Proposition 3.4. *The p.g.f. of X_n in the steady state is*

$$P(u) = \frac{(1-u)B_1^\star[\lambda(1-u)][1-p[1-B_2^\star(\lambda)]]}{(1-p)B_1^\star[\lambda(1-u)]+pC^\star[\lambda(1-u)]-u}\pi_0, \tag{3.2}$$

where $C^*(\cdot)$ is the LST of the convolution of the distribution functions of the service and *reservice times, and* π_0 *is the probability that the* MQ *is empty and is equal to*

$$(1-\rho)\{1-p[1-B_2^*(\lambda)]\}^{-1}. \tag{3.3}$$

Here, $\rho=\rho_1+p\rho_2$, $\rho_1=\lambda/\mu_1$, *and* $\rho_2=\lambda/\mu_2$ *and the* ρ_i *are traffic intensity in* MQ *and* FQ, *respectively.*

Proof. Using the definition of the p.g.f. of a random variable for (2.1) and after a long computation, (3.2) yields. For finding (3.3), we use Hôpital's rule and the fact that $P(1)=1$. □

Remark 3.5. We can think of the service time of an individual as being the service time in MQ plus any reservice time (conditional on requiring such reservice), times the probability p of the item being failed. Then $E[\text{service time}]=1/\mu_1+p/\mu_2=1/\mu$. Now, we have an $M/G/1$ queue with mean service time $1/\mu$. On the other hand, the mean busy period in $M/G/1$ is found by taking the derivative of the functional equation $\Gamma(u)=B^*[u+\lambda(1-\Gamma(u))]$ (see Takács [11]), with $u=0$, in which $\Gamma(u)$ and $B^*(\cdot)$ are the LSTs of the distribution functions of the busy period in MQ and service time, respectively, and is equal to $1/(\mu-\lambda)$ (see Gross and Harris [3]). Therefore, $E(\text{busy period})=\rho/\lambda(1-\rho)$. By using Little's law (see Stidham [10]) and the fact that the mean idle period $(1/\lambda)$ is exponentially distributed, we have $E(\text{busy period})/E(\text{idle period})=(1-\pi_I^*)/\pi_I^*$, which yields $\pi_I^*=1-\rho$.

4. The queueing discipline II. We now consider discipline II. In this case, since we store the failed items and then repair them when MQ is empty, we require two variables. One of these, X_n, is the number of the customers in MQ at the epochs $\{t_n\}$, and the other, Y_n, is the number of failed items in the store (FQ), again at the epochs $\{t_n\}$. We now have a bivariate imbedded Markov chain (X_n,Y_n). (X_{n+1},Y_{n+1}) has been given by (2.2). To evaluate the joint p.g.f. (X_n,Y_n) in the steady state, denoted by $P(u,v)=E(u^{X_n}v^{Y_n})$, we develop the proposition below.

Proposition 4.1. *The joint p.g.f. of* (X_n,Y_n) *in the steady state is*

$$\begin{aligned}P(u,v)&=\{(1-p+pv)B_1^*[\lambda(1-u)]-u\}^{-1}\\&\quad\times\{(1-p+pv)B_1^*[\lambda(1-u)][R(v)-G^*(u,p,\lambda)+(1-u)G^*(0,p,\lambda)]\}\pi_{0\bullet}\end{aligned} \tag{4.1}$$

in which

$$\begin{aligned}R(v)&=\Sigma\pi_{j|0}v^j,\quad j\ge 0,\\\pi_{j|0}&=\Pr\{Y_n=j\mid X_n=0\},\\\pi_{0\bullet}&=\Pr(X_n=0),\\G^*(u,p,\lambda)&=\psi(1-p+pB_2^*[\lambda(1-u)]),\\\psi(u)&=uB_1^*[\lambda(1-\psi(u))],\end{aligned} \tag{4.2}$$

where $\psi(u)$ *is the p.g.f. of the number of served customers (departures) in a busy period (here in* MQ*) (see (Takács [11] and Saaty [7])).* $\pi_{0\bullet}$ *is the probability that* MQ *is empty* *and is equal to*

$$(1-\rho_1)^2\left[p\rho_2+(1-\rho_1)G^\star(0,p,\lambda)\right]^{-1}. \tag{4.3}$$

It is clear that the number of failed items in FQ is distributed as a binomial distribution with parameters p and K, where K is the number of served customers in a busy period in MQ and has p.g.f. $\psi(u)$. Expression (4.2) is a functional equation. Takács [11] has proved the existence and uniqueness of an analytic solution of $\psi(u)$ for $|u| \le 1$ subject to $\psi(0) = 0$. In addition, he has shown that $\lim\psi(u)$, where $u \to 1$, equals the smallest positive real root of the equation $B_1^*[\lambda(1-x)] = x$. By solving expression (4.2) and using the p.g.f. of the binomial distribution, we can find the p.g.f. of Y_n, given $X_n = 0$, denoted by $R(\cdot)$, in terms of $\psi(u)$.

Proof. Using the definition of the joint p.g.f. of the bivariate random variable for (2.2), that is, $E(u^{X_n}v^{Y_n})$ and a long computation, (4.1) yields. □

Remark 4.2. Using an ergodicity argument, Remark 3.5, Little's law, the mean busy periods in MQ and FQ, and the idle period, we can find π_{II}^* which is Pr(idle period, discipline II). Moreover, we have

$$E(\text{busy period}) = E(\text{busy period in } MQ) + pE(\text{busy period in } FQ). \tag{4.4}$$

The first expression is equal to $1/(\mu_1-\lambda)$. The second expression is $p[\mu_2(1-\rho_1)]^{-1}$. Then, by $E(\text{busy period})/E(\text{idle period}) = (1-\pi_{II}^*)/\pi_{II}^*$, the probability of the idle period is $\pi_{II}^* = (1-\rho_1)(1+p^2\rho_2)^{-1}$.

5. The queueing distribution III.

Finally, consider discipline III. In this case, the server has to switch from MQ to FQ if the store is full (threshold N) or if there are no more items in MQ, and returns to MQ after reservicing all the failed items in FQ, if there are any items in MQ. As with the other disciplines, we find the joint p.g.f. (X_n, Y_n) in the steady state and $\pi_{III}^* = \Pr(\text{idle period, discipline III})$. However, before this, we find the probability that the store reaches the threshold N through a remark.

Remark 5.1. When the store is full, the server switches to FQ from MQ. At this time, the number of departures in MQ is a random variable D distributed as a negative binomial as follows:

$$\Pr(D=d) = \binom{d-1}{N-1}p^N(1-p)^d \quad \text{s.t. } d = N, N+1, \dots. \tag{5.1}$$

In other words, $\Pr(Y_n = N) = \Pr(D = d)$, denoted by $\pi_{\bullet N}$. We use this note for finding the joint p.g.f. (X_n, Y_n), by developing a proposition given below.

Proposition 5.2. *The joint p.g.f. of* (X_n, Y_n) *in the steady state is*

$$\begin{aligned} P(u,v) = \frac{(1-p+pv)B_1^*[\lambda(1-u)]}{(1-p+pv)B_1^*[\lambda(1-u)]-u}\Big\{&\left[R(v)-G_N^*(u,p,\lambda)+(1-u)G_N^*(0,p,\lambda)\right]\pi_{0\bullet} \\ &+\left[v^N-\left[B_2^*[\lambda(1-u)]\right]^N\right]R_N(u)\pi_{\bullet N}\Big\} \end{aligned} \tag{5.2}$$

in which

$$\begin{aligned}
R(v) &= \Sigma\pi_{j|0}v^{j}, \quad j=0,1,\dots,N,\\
\pi_{j|0} &= \Pr\{Y_n=j \mid X_n=0\},\\
R_N(u) &= E\left(u^{X_n}I_{\{X_n>0\}} \mid Y_n=N\right) = \Sigma\phi_{i|N}u^{i}, \quad i>0,\\
\phi_{i|N} &= \Pr\{X_n=i \mid Y_n=N\},\\
G_N^{*}(u,p,\lambda) &= E\Big[u^{\Sigma_{i=1}^{Y_n}\tilde{A}(\tilde{s}_i)} \mid X_n=0\Big] = R\left(B_2^{*}[\lambda(1-u)]\right),
\end{aligned} \tag{5.3}$$

where

$$\pi_{0\bullet} = \frac{1-\rho_1+\pi_{\bullet N}\rho_2}{R[B_2^{*}(\lambda)]-\rho_2E(Y_n \mid X_n=0)}. \tag{5.4}$$

Proof. By the definition of the joint p.g.f. of (X_n,Y_n) for (2.3), we have

$$\begin{aligned}
P(u,v) &= E\left(u^{X_{n+1}}v^{Y_{n+1}}\right)\\
&= E\Big[u^{(X_n-1)_+ +A(s)+[\Sigma_{i=1}^{Y_n}\tilde{A}(\tilde{s}_i)I_{C_2\cup C_3}-I_{C_3}]_+}v^{Y_nI_{C_1}+U_{n+1}}\Big]\\
&= (1-p)E\left[u^{X_n+A(s)-1}v^{Y_n}I_{C_1}\right]+pE\left[u^{X_n+A(s)-1}v^{Y_n+1}I_{C_1}\right]\\
&\quad +(1-p)E\left[u^{X_n+\Sigma_{i=1}^{N}\tilde{A}(\tilde{s}_i)+A(s)-1}v^{0}I_{C_2}\right]+pE\left[u^{X_n+\Sigma_{i=1}^{N}\tilde{A}(\tilde{s}_i)+A(s)-1}v^{1}I_{C_2}\right]\\
&\quad +(1-p)E\left[u^{[\Sigma_{i=1}^{Y_n}\tilde{A}(\tilde{s}_i)-1]_+ +A(s)}v^{0}I_{C_3}\right]+pE\left[u^{[\Sigma_{i=1}^{Y_n}\tilde{A}(\tilde{s}_i)-1]_+ +A(s)}v^{1}I_{C_3}\right]\\
&\quad +(1-p)E\left[u^{A(s)}v^{0}I_{C_4}\right]+pE\left[u^{A(s)}v^{1}I_{C_4}\right]\\
&= (1-p+pv)E\left[u^{A(s)}\right]\Big\{E\left[I_{C_4}\right]+\frac{1}{u}\left[E\left[u^{X_n}v^{Y_n}I_{C_1}\right]+E\left[u^{X_n+\Sigma_{i=1}^{N}\tilde{A}(\tilde{s}_i)}I_{C_2}\right]\right]\\
&\qquad\qquad +E\left[u^{[\Sigma_{i=1}^{Y_n}\tilde{A}(\tilde{s}_i)-1]_+}I_{C_3}\right]\Big\}.
\end{aligned} \tag{5.5}$$

Now, by using the relations between indicator functions, $P(u,v)$ is equal to

$$\begin{aligned}
&(1-p+pv)B_1^{*}[\lambda(1-u)]\Big\{\pi_{00}+\frac{1}{u}\Big\{E\left[u^{X_n}v^{Y_n}\left(1-I_{C_2\cup C_3\cup C_4}\right)\right]+E\Big[u^{\Sigma_{i=1}^{N}\tilde{A}(\tilde{s}_i)}\Big]E\left[u^{X_n}I_{C_2}\right]\Big\}\\
&\qquad\qquad +E\Big[u^{[\Sigma_{i=1}^{Y_n}\tilde{A}(\tilde{s}_i)-1]_+}\left(I_{C_3\cup C_4}-I_{C_4}\right)\Big]\Big\}\\
&\quad= (1-p+pv)B_1^{*}[\lambda(1-u)]\\
&\qquad\times\Big\{\pi_{00}+\frac{1}{u}\big\{P(u,v)-E\left[u^{X_n}v^{Y_n}I_{C_2\cup C_3\cup C_4}\right]\\
&\qquad\qquad +\left[B_2^{*}[\lambda(1-u)]\right]^{N}\left[E\left(u^{X_n}I_{C_2\cup\{X_n=0,Y_n=N\}}\right)-E\left(u^{X_n}I_{\{X_n=0,Y_n=N\}}\right)\right]\big\}\\
&\qquad +E\Big[u^{[\Sigma_{i=1}^{Y_n}\tilde{A}(\tilde{s}_i)-1]_+}I_{C_3\cup C_4}\Big]-E\Big[u^{[\Sigma_{i=1}^{Y_n}\tilde{A}(\tilde{s}_i)-1]_+}I_{C_4}\Big]\Big\}
\end{aligned}$$

$$
\begin{aligned}
&= (1-p+pv)B_1^*[\lambda(1-u)] \\
&\quad\times\Bigg\{\pi_{00}+\frac{1}{u}\{P(u,v)-E[u^{X_n}v^{Y_n}I_{C_3\cup C_4}]-E[u^{X_n}v^{Y_n}I_{C_2\cup\{X_n=0,Y_n=N\}}] \\
&\qquad\qquad +E[u^{X_n}v^{Y_n}I_{\{X_n=0,Y_n=N\}}]+B_2^*[\lambda(1-u)][R_N(u)-\pi_{0N}]\} \\
&\qquad +E\left[\frac{[B_2^*[\lambda(1-u)]]^{Y_n}-(1-u)[B_2^*(\lambda)]^{Y_n}}{u}\right]\pi_{0\bullet}-\pi_{00}\Bigg\}.
\end{aligned}
\tag{5.6}
$$

By using (5.3) and summarizing, the proof is completed. □

In order to find $\pi_{0\bullet}$ in (5.4), we use Hôpital's rule and the fact that $\lim P(u,1)=1$, where $u\to 1$.

Special cases. Two special cases, denoted below by (S1) and (S2), are important.
(S1) If $N\to\infty$, then $v^N-\{B_2^*[\lambda(1-u)]\}^N\to 0$ and $G_N^*(u,p,\lambda)=G^*(u,p,\lambda)$. Thus

$$
\begin{aligned}
P(u,v) &= \{(1-p+pv)B_1^*[\lambda(1-u)]-u\}^{-1} \\
&\quad\times\{(1-p+pv)B_1^*[\lambda(1-u)] \\
&\qquad\times[R(v)-G^*(u,p,\lambda)+(1-u)G^*(0,p,\lambda)]\}\pi_{0\bullet},
\end{aligned}
\tag{5.7}
$$

that is, discipline II.

(S2) If $p=0$ and $v=1$, then

$$
P(u,1)=P(u)=\{(1-u)B_1^\star[\lambda(1-u)]\}\{B_1^\star[\lambda(1-u)]-u\}^{-1}\pi_0,
\tag{5.8}
$$

with $\pi_0=1-\lambda/\mu_1=1-\lambda E(\text{service time})=1-\rho_1$. This is similar to the $M/G/1$ queue without any conditions (see Gross and Harris [3]).

Remark 5.3. We now find the proportion of time that the server is idle, that is, π_{III}^*. For this, first we find the mean busy period (denoted by T), and then, by using $(1-\pi_{III}^*)/\pi_{III}^*=E(T)/E(\text{idle period})$, we can find π_{III}^*. We can divide the busy period into four subperiods. The first (denoted by T_1) is when the server starts in MQ with one customer. In the second (T_2), the server switches from MQ to FQ for reservicing the waiting failed items, when the level of FQ is the threshold N. In the third (T_3), the server returns to MQ from FQ for servicing the waiting customers, after reservicing all of the failed items in FQ. At this time, there are $V_n=X_n+\Sigma_{i=1}^N\tilde{A}(\tilde{s}_i)$ waiting customers in MQ, where the X_n are the remaining customers from before, that is, $\Sigma_{i=1}^D A(s_i)+1-D$ and $\Sigma_{i=1}^N\tilde{A}(\tilde{s}_i)$ are new arrivals during reservicing in FQ. In the fourth period (T_4), the server again switches to FQ after servicing all customers in MQ and the level of the store (FQ) is Y_n, where $Y_n=1,2,\ldots,\min\{N,K^*\}$, where K^* is the number of departures in MQ when the server starts with V_n customers. Now, by this discussion, we have

$$
E(T)=E(T_1)+\Pr(D=d)\times E(T_2)+E(T_3)+p\times E(T_4).
\tag{5.9}
$$

The means of the subbusy periods are found as below.

(a) T_1 comprises D service times that are i.i.d $B_1(\cdot)$ with mean $1/\mu_1$. Thus $E(T_1) = E(D)E(s) = N/p\mu_1$.

(b) T_2 comprises N reservice times that are i.i.d $B_2(\cdot)$ with mean $1/\mu_2$. Thus $E(T_2) = NE(\tilde{s}) = N/\mu_2$.

(c) T_3 is the same as the busy period for an $M/G/1$ queue when the server starts with V_n customers. This means that we repeat an $M/G/1$ queue that starts with one customer, V_n times. By Remark 4.2, the mean busy period for such a queue is $1/(\mu_1-\lambda)$. Thus the mean busy period in MQ for the queue that starts with V_n customers is $E(V_n)/(\mu_1-\lambda)$. Now, by using the mean of the negative binomial distribution, we can compute $E(V_n)$ as follows:

$$\begin{aligned} E(V_n) &= E\left[X_n + \sum_{i=1}^{N} \tilde{A}(\tilde{s}_i)\right] \\ &= E\left[\sum_{i=1}^{D} A(s_i) + 1 - D\right] + NE[\tilde{A}(\tilde{s})] \\ &= E(D)E[A(s)] + 1 - E(D) + \frac{N\lambda}{\mu_2} \\ &= N\rho_2 - \frac{N(1-\rho_1)}{p+1}. \end{aligned} \tag{5.10}$$

Therefore, $E(T_3) = [N\rho_2 - N(1-\rho_1)/p+1]/\mu_1(1-\rho_1)$.

(d) T_4 comprises Y_n reservice times that are i.i.d $B_2(\cdot)$ with mean $1/\mu_2$. Then $E(T_4) = E(Y_n)E(\tilde{s})$, where $Y_n = 1,\dots,\min\{N,K^*\}$. For an $M/G/1$ queue in which the server starts with one customer, we know that the mean of the number of the departures in MQ is $1/(1-\rho_1)$ (see Salehi-Rad and Mengersen [8]). However, at the third subbusy period, the server starts with V_n customers, thus

$$E(K^*) = \frac{E(V_n)}{1-\rho_1} = \frac{N\rho_2 - N(1-\rho_1)/p+1}{1-\rho_1} = \mu_1 E(T_3). \tag{5.11}$$

Now, we can find $E(T_4)$ as follows:

$$E(T_4) = E(\tilde{s})E(Y_n) = {\mu_2}^{-1} E[E(Y_n \mid \min\{N,K^*\})], \tag{5.12}$$

that is, Np/μ_2 if $\min\{N,K^*\} = N$, otherwise $E(K^*)p/\mu_2$. Finally, by (5.9) and using (a), (b), (c), and (d), we can find π^*_{III}.

References

[1] C. E. Bell, *Characterization and computation of optimal policies for operating an $M/G/1$ queueing system with removable server*, Oper. Res. **19** (1971), 208–218.

[2] Y. Deng and J. Tan, *Priority queueing model with changeover times and switching threshold*, J. Appl. Probab. **38A** (2001), 263–273.

[3] D. Gross and C. M. Harris, *Fundamentals of Queueing Theory*, Wiley Series in Probability and Mathematical Statistics: Applied Probability and Statistics, John Wiley & Sons, New York, 1985.

[4] D. P. Heyman, *Optimal operating policies for $M/G/1$ queuing systems*, Oper. Res. **16** (1968), 362–382.

[5] D. P. Heyman and K. T. Marshall, *Bounds on the optimal operating policy for a class of single-server queues*, Oper. Res. **16** (1968), 1138–1146.

[6] M. Y. Kitaev and R. F. Serfozo, *$M/M/1$ queues with switching costs and hysteretic optimal control*, Oper. Res. **47** (1999), no. 2, 310–312.

[7] T. L. Saaty, *Elements of Queueing Theory, with Applications*, McGraw-Hill, New York, 1961.

[8] M. R. Salehi-Rad and K. Mengersen, *Reservicing some customers in $M/G/1$ queues, under two disciplines*, Advances in Statistics, Combinatorics and Related Areas, World Scientific Publishing, New Jersey, 2002, pp. 267–274.

[9] M. J. Sobel, *Optimal average-cost policy for a queue with start-up and shut-down costs*, Oper. Res. **17** (1969), 145–162.

[10] S. Stidham, *A last word on $L = \lambda W$*, Oper. Res. **22** (1974), no. 2, 417–421.

[11] L. Takács, *Investigation of waiting time problems by reduction to Markov processes*, Acta Math. Acad. Sci. Hungar. **6** (1955), 101–129.

[12] M. Yadin and P. Naor, *On queueing system with variable service capacities*, Nav. Res. Logist. Q. **14** (1967), 43–53.

M. R. Salehi-Rad: Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad 9177948953, Iran

E-mail address: salehi_rad@hotmail.com

K. Mengersen: School of Mathematical and Physical Sciences, The University of Newcastle, NSW 2308, Australia

E-mail address: kerrie.mengersen@newcastle.edu.au

G. H. Shahkar: Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad 9177948953, Iran

E-mail address: shahkar@math.um.ac.ir