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AN $L^p - L^q$ VERSION OF HARDY'S THEOREM FOR SPHERICAL FOURIER TRANSFORM ON SEMISIMPLE LIE GROUPS

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We consider a real semisimple Lie group G with finite center and K a maximal compact subgroup of G. We prove an $L^p - L^q$ version of Hardy's theorem for the spherical Fourier transform on G. More precisely, let a, b be positive real numbers, $1 \le p, q \le \infty$, and f a K-bi-invariant measurable function on G such that $h_a^{-1} f \in L^p(G)$ and $e^{b\|\lambda\|^2}\mathcal{F}(f) \in L^q(\mathfrak{a}_+^*)$ (h_a is the heat kernel on G). We establish that if $ab \ge 1/4$ and p or q is finite, then $f = 0$ almost everywhere. If $ab < 1/4$, we prove that for all p, q, there are infinitely many nonzero functions f and if $ab = 1/4$ with $p = q = \infty$, we have $f = \mathrm{const}\, h_a$.

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1. Introduction. In the usual Fourier analysis on $\mathbb{R}$, it is known, by a theorem of L. Schwartz, that a function f on $\mathbb{R}$ is rapidly decreasing if and only if its Fourier transform $\hat{f}$ is rapidly decreasing. A theorem of Hardy [8] measures this rapidity in the following meaning: for given positive real numbers a and b, suppose that f is a measurable function such that $|f(x)| \le \mathrm{const}\, e^{-ax^2}$ and $|\hat{f}(y)| \le \mathrm{const}\, e^{-by^2}$. Then $f = 0$ almost everywhere if $ab > 1/4$, $f(x) = \mathrm{const}\, e^{-ax^2}$ if $ab = 1/4$, and there are infinitely many nonzero functions f if $ab < 1/4$.

This theorem has been generalized in other situations as the Heisenberg group [3] and the Euclidean motion group [15]. In particular, for a semisimple Lie group G, Sitaram and Sundari proved in [14] a version of Hardy's theorem when G has one conjugacy class of Cartan subgroups. Their result has been extended to all semisimple Lie groups by Cowling et al. in [5], Ebata et al. in [6], and Sengupta in [12].

Another generalization is the $L^p - L^q$ version of Hardy's theorem, proved by Cowling and Price in [4]. More precisely, taking $1 \le p, q \le \infty$ (with p or q finite) and supposing that for a measurable function f, $e^{ax^2} f(x) \in L^p(\mathbb{R})$, $e^{by^2}\hat{f}(y) \in L^q(\mathbb{R})$, and $ab \ge 1/4$, then $f = 0$ almost everywhere. An analogue of this result has been also proved for the motion group in [7].

In this paper, we consider a real semisimple Lie group G with finite center, $G = K\exp(\overline{\mathfrak{a}_+})K$ a Cartan decomposition of G, and h_a the heat kernel on G. We establish the following $L^p - L^q$ version of Hardy's theorem for the spherical Fourier transform on G.

Let a, b be positive real numbers, $1 \le p, q \le \infty$, and f a K-bi-invariant measurable function such that $h_a^{-1} f \in L^p(G)$ and $e^{b\|\lambda\|^2}\mathcal{F}(f) \in L^q(\mathfrak{a}_+^*)$.

If p or q is finite and $ab \ge 1/4$, then $f = 0$ almost everywhere.

In case $p = q = \infty$,

(i) if $ab > 1/4$, then $f = 0$ almost everywhere,

(ii) if $ab = 1/4$, then $f = \text{const}\, h_a$.

If $ab < 1/4$, we prove that for all p, q, there are infinitely many nonzero functions satisfying the $L^p - L^q$ conditions.

This theorem is exactly the analogue of the result of Cowling and Price in [4]. We believe that comparing f to the heat kernel and $\mathcal{F}(f)$ to $e^{-b\|\lambda\|^2}$ is the natural way to generalize Hardy's theorem on G, because in the Euclidian case of $\mathbb{R}$, the function e^{-ax^2} is also the heat kernel.

In [11] and in order to give an L^p version of Hardy's theorem on G, Narayanan and Ray remark that if they compare f to $e^{-a\|x\|^2}$ and $\hat{f}$ to $e^{-b\|\lambda\|^2}$, they only obtain the result for $p \geq 2$. Then they take the function $e^{-a\|x\|^2}\varphi_0^{1-2/p}$ instead of $e^{-a\|x\|^2}$ to get the result for all $1 \leq p \leq \infty$, but this condition depends on p. (φ_0 is the spherical function defined in Section 2.2.)

To prove our result, we adapt the classical methods as in [4, 6, 14]. We use the fact that $h_a^{-1} f \in L^p(G)$ to give an estimation of $\mathcal{F}(f)$ (Lemma 4.2), then we add the hypothesis $e^{b\|\lambda\|^2}\mathcal{F}(f) \in L^q(\mathfrak{a}_+^*)$ to conclude with a Phragmén-Lindelöf-type theorem.

This paper is organized as follows. In Section 2, we introduce some notations and results for semisimple Lie groups and spherical Fourier transform. In Section 3, we prove different versions of the Phragmén-Lindelöf theorems, which are crucial for the proof of the principal results of the paper. We give the $L^p - L^q$ version of Hardy's theorem in Section 4.

2. Preliminaries. In this section, we introduce some classical notations and results about semisimple Lie groups and spherical functions. For details, we refer to [10, 16].

2.1. Notations. Let G be a connected, noncompact real semisimple Lie group with finite center, and K a fixed maximal compact subgroup of G. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, respectively. Take $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ a Cartan decomposition of $\mathfrak{g}$ and $\mathfrak{a}$ a maximal commutative subspace of $\mathfrak{p}$. Denote by $\mathfrak{a}^*$ the real dual of $\mathfrak{a}$ and $\mathfrak{a}_{\mathbb{C}}^*$ its complexification. The associated Killing form defines a scalar product $\langle\cdot,\cdot\rangle$ on $\mathfrak{a}$.

For $\lambda \in \mathfrak{a}^*$, let H_λ be the unique element in $\mathfrak{a}$ such that $\lambda(H) = \langle H_\lambda, H\rangle$ for all $H \in \mathfrak{a}$. If $\lambda, \mu \in \mathfrak{a}^*$, then $\langle\lambda,\mu\rangle = \langle H_\lambda, H_\mu\rangle$ defines a scalar product on $\mathfrak{a}^*$ which can be extended to $\mathfrak{a}_{\mathbb{C}}^*$ as a Hermitian product, denoted also by $\langle\cdot,\cdot\rangle$. Let $\|\cdot\|$ be the associated norm.

For $\lambda \in \mathfrak{a}^*$, put $\mathfrak{g}_\lambda = \{X \in \mathfrak{g} \mid [H,X] = \lambda(H)X;\ \text{for all } H \in \mathfrak{a}\}$. If $\lambda \neq 0$ and $\mathfrak{g}_\lambda \neq \{0\}$, then λ is called a (restricted) root. Let $m_\lambda = \dim \mathfrak{g}_\lambda$.

As usual, denote by Σ the set of all roots. Let Σ^+ be a fixed set of positive roots, Σ_0^+ the set of positive indivisible roots, and $\mathfrak{a}_+$, $\mathfrak{a}_+^*$ the corresponding Weyl chambers, respectively, in $\mathfrak{a}$ and $\mathfrak{a}^*$. Denote by $\overline{\mathfrak{a}_+}$, $\overline{\mathfrak{a}_+^*}$ their usual closures. Let W be the Weyl group for Σ.

Choose $\beta_1,\dots,\beta_l$ a basis of $\mathfrak{a}^*$ constituted by simple roots in Σ^+. Take $\mu_1,\dots,\mu_l$ in $\mathfrak{a}^*$ such that for $1 \leq i, j \leq l$, $\langle\mu_i,\beta_j\rangle = \delta_{ij}$. Then $\mu_1,\dots,\mu_l$ is a basis of $\mathfrak{a}^*$ and $\mathfrak{a}_+^* = \sum_{i=1}^{l} \mathbb{R}_+^* \cdot \mu_i$.

We have the Cartan decomposition $G = K\exp(\overline{\mathfrak{a}_+})K$. For all $x \in G$, denote $|x| = \|x^+\|$, where x^+ is the $\mathfrak{a}_+$ component of x in the above decomposition.

One has also the usual Iwasawa decomposition $G = K\exp(\mathfrak{a})N$. For all $x \in G$, let $H(x)$ be the unique element in $\mathfrak{a}$ such that $x \in K\exp H(x)N$.

We normalize the Lebesgue measures dH and $d\lambda$, respectively, on $\mathfrak{a}$ and $\mathfrak{a}^*$ such that the Fourier transform,

$$\mathcal{F}_0(f)(\lambda) = \int_{\mathfrak{a}} f(H)e^{-i\lambda(H)}dH, \tag{2.1}$$

has the inversion formula

$$\mathcal{F}_0^{-1}(g)(H) = \int_{\mathfrak{a}^*} g(\lambda)e^{i\lambda(H)}d\lambda \tag{2.2}$$

defined for $g \in \mathcal{S}(\mathfrak{a}^*)$ (the Schwartz space on $\mathfrak{a}^*$).

2.2. Spherical functions. The spherical functions on G are defined by

$$\varphi_\lambda(x) = \int_K e^{(i\lambda-\rho)(H(xk))}dk, \quad x \in G,\ \lambda \in \mathfrak{a}_{\mathbb{C}}^*, \tag{2.3}$$

where dk is the Haar measure on K of total measure 1 and $\rho = (1/2)\sum_{\alpha\in\Sigma^+} m_\alpha \alpha$.

These functions satisfy the following properties:

(i) for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, the function $x \mapsto \varphi_\lambda(x)$ is $\mathcal{C}^\infty$ and K-bi-invariant,

(ii) for $x \in G$, the function $\lambda \mapsto \varphi_\lambda(x)$ is analytic and W-invariant on $\mathfrak{a}_{\mathbb{C}}^*$,

(iii)

$$|\varphi_\lambda(x)| \le \varphi_{i\Im\lambda}(x) \quad \lambda \in \mathfrak{a}_{\mathbb{C}}^*, \tag{2.4}$$

(iv)

$$|\varphi_{i\lambda}(\exp H)| \le e^{\lambda(H)}\varphi_0(\exp H) \quad \lambda \in \overline{\mathfrak{a}_+^*},\ H \in \overline{\mathfrak{a}_+}, \tag{2.5}$$

(v)

$$e^{-\rho(H)} \le \varphi_0(\exp H) \le \mathrm{const}\,(1+\|H\|^d)e^{-\rho(H)}, \quad H \in \overline{\mathfrak{a}_+}, \tag{2.6}$$

for some constant $d > 0$.

2.3. The Harish-Chandra c-function. The Harish-Chandra c-function is given by the formula

$$c(\lambda) = c_0 \prod_{\alpha\in\Sigma_0^+} \frac{\Gamma((1/2)\langle i\lambda,\alpha_0\rangle)\Gamma((1/2)\langle i\lambda,\alpha_0\rangle+1/2)}{2\sqrt{\pi}\Gamma((1/4)m_\alpha+1/2+(1/2)\langle i\lambda,\alpha_0\rangle)\Gamma((1/4)m_\alpha+(1/2)m_{2\alpha}+(1/2)\langle i\lambda,\alpha_0\rangle)}, \tag{2.7}$$

where $\alpha_0 = \alpha/\langle\alpha,\alpha\rangle$, the constant c_0 is defined by $c(i\rho) = 1$, and Γ is the usual gamma function.

The c-function satisfies the following result.

Lemma 2.1. *Let $\lambda \in \mathfrak{a}_+^*$ and $r > 0$. Then*

$$|c(r\lambda)|^{-2} \le \max(r^{\dim N}, 1)|c(\lambda)|^{-2}. \tag{2.8}$$

Moreover, there are positive constants D and σ_0 such that for all $x \geq \sigma_0$,

$$|c(x\lambda)|^{-2} \geq Dx^{\dim N}. \tag{2.9}$$

Proof. Inequality (2.8) is an immediate deduction from [16, Proposition IV, page 248].

The proof of (2.9) is based on the fact that $|c(x\lambda)|^{-2}$ is a product of terms of the form $\Gamma(a+iy)/\Gamma(b+iy)$ with $a > b \geq 0$ and $y \in \mathbb{R}$, and the fact that

$$\lim_{|y|\to\infty} \frac{\Gamma(a+iy)}{\Gamma(b+iy)}|y|^{b-a} = 1. \tag{2.10}$$

This implies that

$$\lim_{x\to\infty} |c(x\lambda)|^{-2} x^{-\dim N} = c_0^{-1}\left(2^{2\,\mathrm{card}\,\Sigma_0^+ - \dim N}\pi^{\mathrm{card}\,\Sigma_0^+}\right) \prod_{\alpha\in\Sigma_0^+} \langle \lambda, \alpha_0\rangle^{m_\alpha + m_{2\alpha}}. \tag{2.11}$$

Since this limit is positive, then there are $\sigma_0 > 0$ and $D > 0$ such that for all $x \geq \sigma_0$, we have inequality (2.9). □

2.4. Spherical Fourier transform. We present the following notions:

(i) $\mathcal{D}(G)$ is the space of $\mathcal{C}^\infty$ functions on G with compact support,

(ii) $\mathcal{D}^\natural(G)$ is the space of K-bi-invariant elements of $\mathcal{D}(G)$,

(iii) $L^p(G)$, $1 \leq p < \infty$, are the spaces of measurable functions f on G such that

$$\|f\|_p = \left(\int_G |f(x)|^p dx\right)^{1/p} < \infty, \tag{2.12}$$

(iv) $L^\infty(G)$ is the space of measurable functions f on G such that

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x\in G} |f(x)| < \infty, \tag{2.13}$$

(v) $L^q(\mathfrak{a}_+^*, |c(\lambda)|^{-2}d\lambda)$, $1 \leq q < \infty$, are the spaces of measurable functions g on $\mathfrak{a}_+^*$ such that

$$\|g\|_{L^q(\mathfrak{a}_+^*,|c(\lambda)|^{-2}d\lambda)} = \left(\int_{\mathfrak{a}_+^*} |g(\lambda)|^q |c(\lambda)|^{-2} d\lambda\right)^{1/q} < \infty, \tag{2.14}$$

(vi) $L^\infty(\mathfrak{a}_+^*, |c(\lambda)|^{-2}d\lambda)$ is the space of measurable functions g on $\mathfrak{a}_+^*$ such that

$$\|g\|_\infty = \operatorname*{ess\,sup}_{\lambda\in\mathfrak{a}_+^*} |g(\lambda)| < \infty. \tag{2.15}$$

In the Cartan decomposition, the Haar measure on G is given by the formula

$$\int_G f(x)dx = \int_K \int_{\mathfrak{a}^+} \int_K f(k_1(\exp H)k_2)\delta(H)dk_1\,dH\,dk_2, \tag{2.16}$$

where $\delta(H) = \prod_{\alpha\in\Sigma^+}[2\sinh\alpha(H)]^{m_\alpha}$ and $f \in \mathcal{D}(G)$. Note that

$$0 \leq \delta(H) \leq e^{2\rho(H)}. \tag{2.17}$$

Let $1 \le p < \infty$ and let $f \in L^p(G)$ be a K-bi-invariant function; then

$$\int_G |f(x)|^p dx = \int_{\mathfrak{a}^+} |f(\exp H)|^p \delta(H) dH. \tag{2.18}$$

The spherical Fourier transform on G is defined by

$$\mathcal{F}(f)(\lambda) = \int_G f(x)\varphi_{-\lambda}(x)dx, \quad f \in \mathcal{D}^\natural(G). \tag{2.19}$$

The inversion formula is given by

$$\mathcal{F}^{-1}(h)(x) = \int_{\mathfrak{a}_+^*} h(\lambda)\varphi_\lambda(x)|c(\lambda)|^{-2}d\lambda, \quad h = \mathcal{F}(f). \tag{2.20}$$

We remark that the transform $\mathcal{F}$ is injective on the space of K-bi-invariant functions of $L^1(G)$.

2.5. The heat kernel. The heat kernel $h_a(x)$ is defined for $x \in G$ and $a > 0$ by

$$h_a(x) = \int_{\mathfrak{a}_+^*} e^{-(1/4a)(\|\lambda\|^2+\|\rho\|^2)}\varphi_\lambda(x)|c(\lambda)|^{-2}d\lambda, \tag{2.21}$$

where h_a is a positive K-bi-invariant $\mathscr{C}^\infty$ function on G. Its spherical Fourier transform is defined for $\lambda \in \mathfrak{a}^*$ by

$$\mathcal{F}(h_a)(\lambda) = e^{-(1/4a)(\|\lambda\|^2+\|\rho\|^2)}. \tag{2.22}$$

Moreover, $\mathcal{F}(h_a)$ is analytic on $\mathfrak{a}_{\mathbb{C}}^*$ and defined for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ by

$$\mathcal{F}(h_a)(\lambda) = e^{-(1/4a)(\langle\lambda,\overline{\lambda}\rangle+\|\rho\|^2)}. \tag{2.23}$$

Anker [1] has proved the following estimate of the heat kernel:

$$h_a(\exp(H)) \le C(1+\|H\|^2)^{d'} e^{-a\|H\|^2} e^{-\langle\rho,H\rangle}, \tag{2.24}$$

where $d' = (\dim(G/K) - \dim(\mathfrak{a}))/2$ and C is a positive constant depending on a.

In the particular cases when G/K is of rank one or $G = \mathrm{SL}(3,\mathbb{R})$, we also have

$$h_a(\exp(H)) \ge D(1+\|H\|^2)^{d'} e^{-a\|H\|^2} e^{-\langle\rho,H\rangle}, \tag{2.25}$$

where D is a positive constant depending on a.

Note that Anker has conjectured this inequality for all noncompact symmetric spaces. (See [2] for details and references.)

3. Phragmén-Lindelöf-type results. The proofs of the main results of this paper depend on the following complex analysis results.

Let Q_θ denote the sector in $\mathbb{C}$ defined by $Q_\theta = \{re^{i\psi} : r > 0,\ \psi \in]0,\theta[\}$ and $Q = Q_{\pi/2} = \mathbb{R}_+^* + i\mathbb{R}_+^*$. Denote by $\overline{Q_\theta}$ the usual closure of Q_θ.

Fix γ a positive measurable function on $]0,\infty[$ verifying the following:

(i) there are an integer $k>0$ and a real $\sigma_0>0$ such that

$$\forall r>0,\ \forall x\geq\sigma_0,\quad \gamma(rx)\leq \operatorname{const}\max\left(r^k,1\right)\gamma(x), \tag{3.1}$$

(ii) there is $\varepsilon_0>0$ such that

$$\forall\sigma>\sigma_0,\quad d_\gamma(\sigma)=\int_\sigma^{\sigma+1}\gamma(x)dx\geq\varepsilon_0. \tag{3.2}$$

Lemma 3.1. *Let g be an analytic function on Q continuous on $\overline{Q}$. Suppose that for $q\in[1,\infty[$ and a constant $M>0$,*

$$\begin{gathered}|g(z)|\leq Me^{\pi\Re(z^2)},\quad z\in\overline{Q},\\ \int_0^\infty |g(x)|^q\gamma(x)dx\leq M.\end{gathered} \tag{3.3}$$

Then

$$\int_\sigma^{\sigma+1}\left|g\left(xe^{i\psi}\right)\right|\gamma(x)dx\leq \operatorname{const} d_\gamma(\sigma)\sigma^k \tag{3.4}$$

for all $\psi\in[0,\pi/2]$ and sufficiently large $\sigma\in\mathbb{R}_+$.

Proof. The proof uses the same arguments of Cowling and Price in [4]. Let $\theta\in\,]0,\pi/2[$ and $\varepsilon\in]0,\pi/2-\theta[$. Define the function g on $\overline{Q_\theta}$ by

$$g_\varepsilon(z)=g(z)\exp\left[i\varepsilon e^{i\varepsilon}z^{(\pi-2\varepsilon)/\theta}+\frac{i\pi}{2}\cdot\cot(\theta)\cdot z^2\right]. \tag{3.5}$$

The function $g_\varepsilon(re^{i\psi})$ is bounded on $\overline{Q_\theta}$ and tends to 0 as $r\to+\infty$ uniformly in $\psi\in[0,\theta]$. Fix $\sigma\geq\sigma_0$ and take S a measurable function on $[\sigma,\sigma+1]$ of L^∞-norm 1. Define F on $\overline{Q_\theta}$ by

$$F\left(re^{i\psi}\right)=\int_\sigma^{\sigma+1}S(x)g_\varepsilon\left(rxe^{i\psi}\right)\gamma(x)dx, \tag{3.6}$$

then F is analytic on Q_θ, continuous on $\overline{Q_\theta}$, and tends to 0 as $r\to+\infty$ uniformly in $\psi\in[0,\theta]$. By the maximum principle, we have

$$\sup_{\substack{r>0\\0\leq\psi\leq\theta}}\left|F\left(re^{i\psi}\right)\right|\leq\max\left(\sup_{r>0}\left|F\left(re^{i\theta}\right)\right|,\sup_{r>0}|F(r)|\right). \tag{3.7}$$

Remark that for $r>0$, we have $|g_\varepsilon(re^{i\theta})|\leq M$; so

$$\left|F\left(re^{i\theta}\right)\right|\leq Md_\gamma(\sigma), \tag{3.8}$$

and then

$$\sup_{r>0}\left|F\left(re^{i\theta}\right)\right|\leq Md_\gamma(\sigma). \tag{3.9}$$

We now estimate $\sup_{r>0}|F(r)|$.

(i) For $r\in[0,1/(\sigma+1)]$, we have

$$|F(r)| \le \int_\sigma^{\sigma+1} |g_\varepsilon(rx)|\gamma(x)dx \le Me^{\pi}d_\gamma(\sigma). \tag{3.10}$$

(ii) For $r\in]1/(\sigma+1),+\infty[$, we have

$$|F(r)| \le \int_\sigma^{\sigma+1} |g_\varepsilon(rx)|\gamma(x)dx, \tag{3.11}$$

so, by Hölder's inequality, we obtain

$$|F(r)| \le (d_\gamma(\sigma))^{1-1/q}\left(\int_\sigma^{\sigma+1} |g_\varepsilon(rx)|^q\gamma(x)dx\right)^{1/q}, \tag{3.12}$$

and, by the change of variable $t=rx$, we obtain

$$|F(r)| \le (d_\gamma(\sigma))^{1-1/q}\sigma^{-1/q}\left(\int_0^{\infty} |g_\varepsilon(t)|^q\gamma\left(\frac{t}{r}\right)dt\right)^{1/q}. \tag{3.13}$$

Using the estimates (3.1) and (3.2) of γ, we get

$$|F(r)| \le \mathrm{const}(\sigma+1)^k\sigma^{-k-1/q}d_\gamma(\sigma)\sigma^k. \tag{3.14}$$

Finally, we deduce that for sufficiently large σ and for all $r>0$, $\theta\in]0,\pi/2[$,

$$|F(r)| \le \mathrm{const}\, d_\gamma(\sigma)\sigma^k, \qquad |F(re^{i\theta})| \le \mathrm{const}\, d_\gamma(\sigma)\sigma^k. \tag{3.15}$$

Letting first ε tend to 0, θ tend to $\pi/2$, and taking the supremum over all S, we obtain (3.4) for all $\psi\in[0,\pi/2]$. □

Lemma 3.2. *Let f be an analytic even function on $\mathbb{C}$. Suppose that for $1\le q<+\infty$ and a constant $\nu>0$,*

$$\begin{aligned} |f(z)| &\le \mathrm{const}\, e^{\nu\Re(z^2)}, \quad z\in\mathbb{C}, \\ \int_0^\infty &|f(x)|^q\gamma(x)dx<\infty. \end{aligned} \tag{3.16}$$

Then $f=0$ on $\mathbb{C}$.

Proof. By the change of variable $z\mapsto\sqrt{\pi/\nu}\cdot z$, we may assume that $\nu=\pi$. If we apply Lemma 3.1 to the functions $z\mapsto f(z)$ and $z\mapsto\overline{f(\overline{z})}$ and use the fact that f is even, then, for large $\sigma>0$ and for all $\psi\in\mathbb{R}$, we have

$$\int_\sigma^{\sigma+1} |f(xe^{i\psi})|\gamma(x)dx \le \mathrm{const}\, d_\gamma(\sigma)\sigma^k. \tag{3.17}$$

On the other hand, Cauchy's integral formula gives

$$|f^{(n)}(0)| \leq \frac{n!}{2\pi}\int_0^{2\pi} |f(xe^{i\psi})|x^{-n}d\psi \tag{3.18}$$

for all $x > 0$ and $n \in \mathbb{N}$. Integrating this inequality between a large σ and $\sigma+1$, we get

$$d_\gamma(\sigma)|f^{(n)}(0)| \leq \frac{n!}{2\pi}\int_\sigma^{\sigma+1}\int_0^{2\pi} |f(xe^{i\psi})|x^{-n}\gamma(x)d\psi\, dx, \tag{3.19}$$

and by Lemma 3.1, we obtain

$$|f^{(n)}(0)| \leq \operatorname{const}\sigma^{k-n}, \tag{3.20}$$

which implies that f is a polynomial function, the integrability condition of the hypothesis gives that $f = 0$ on $\mathbb{C}$. □

We also need the following Phragmén-Lindelöf theorem [9].

Theorem 3.3. *Suppose that F is an entire function on $\mathbb{C}$, where a, C are positive numbers. If*

$$|F(\zeta)| \leq Ce^{a|\zeta|} \quad (\zeta \in \mathbb{C}), \qquad |F(x)| \leq Ce^{-a|x|} \quad (x > 0), \tag{3.21}$$

then there is a number C' such that $|F(\zeta)| = C'e^{-a\zeta}$.

As a corollary, we have the following lemma.

Lemma 3.4. *Let f be an analytic even function on $\mathbb{C}$. Suppose that for a constant $\nu > 0$,*

$$\begin{aligned} |f(z)| &\leq \operatorname{const} e^{\nu|z|^2}, \quad z \in \mathbb{C}, \\ |f(x)| &\leq \operatorname{const} e^{-\nu x^2}, \quad x \in \mathbb{R}. \end{aligned} \tag{3.22}$$

Then $f = \operatorname{const} e^{-\nu z^2}$ on $\mathbb{C}$.

Proof. The function f is even and analytic on $\mathbb{C}$; then there is an entire function F on $\mathbb{C}$ such that $f(z) = F(z^2)$. Applying Theorem 3.3 to F, we obtain the result. □

We conclude this section by the following lemma.

Lemma 3.5. *Let f be an analytic function on $\mathbb{C}^l$. Suppose that for all $t_2 > 0,\dots,t_l > 0$, the function $z \longmapsto f(z,t_2z,\dots,t_lz)$ is null on $\mathbb{C}$. Then $f = 0$ on $\mathbb{C}^l$.*

Proof. Fix $z \in \mathbb{C}^*$ and $t_3 > 0,\dots,t_l > 0$. Note that the analytic function on $\mathbb{C}$ defined by $\zeta \longmapsto f(z,\zeta,t_3z,\dots,t_lz)$ vanishes on the half-axis $\mathbb{R}_+ \cdot z$, and then it is null on $\mathbb{C}$.

So, for all $t_3 > 0,\dots,t_l > 0$, the function $(z_1,z_2) \longmapsto f(z_1,z_2,t_3z_1,\dots,t_lz_1)$ is identically null on $\mathbb{C}^2$. By induction, we can conclude that $f = 0$ on $\mathbb{C}^l$. □

4. The $L^p - L^q$ version of Hardy's theorem. We begin this section by the following remark.

Remark 4.1. Let $1 \le p \le \infty$ and let f be a K-bi-invariant measurable function on G such that $h_a^{-1} f$ is in $L^p(G)$. Then the property (2.5) of the spherical functions and the estimate (2.24) of the heat kernel imply that

$$\mathcal{F}(f)(\lambda) = \int_{\mathfrak{a}^+} f(\exp H)\varphi_{-\lambda}(\exp H)\delta(H)\,dH \tag{4.1}$$

is well defined, W-invariant, and analytic on $\mathfrak{a}^*_{\mathbb{C}}$.

Moreover, we have the following lemma.

Lemma 4.2. *Let $1 \le p \le \infty$ and let f be a K-bi-invariant measurable function on G such that $h_a^{-1} f$ is in $L^p(G)$. Then*

$$|\mathcal{F}(f)(\lambda)| \le \text{const}\, e^{(1/4a)\|\eta\|^2} \tag{4.2}$$

for all $\xi, \eta \in \mathfrak{a}^*$ and $\lambda = \xi + i\eta$.

Proof. Since inequality (4.2) is invariant under the Weyl group which acts transitively on the set of Weyl chambers, it is sufficient to establish it for η in $\mathfrak{a}^*_+$.

Let $\xi, \eta \in \mathfrak{a}^*_+$ and $\lambda = \xi + i\eta$.

By the property (2.4) such that $|\varphi_{-\lambda}(x)| \le \varphi_{-i\eta}(x)$, we get

$$|\mathcal{F}(f)(\lambda)| \le \int_{\mathfrak{a}^+} |f(\exp H)|\varphi_{-i\eta}(\exp H)\delta(H)\,dH. \tag{4.3}$$

Now, to prove (4.2), we distinguish three cases.

First case ($1 < p < \infty$). Let p' be such that $1/p + 1/p' = 1$. Since $h_a^{-1} f$ is in $L^p(G)$, let $M = \|h_a^{-1} f\|_p$. We have, by Hölder's inequality,

$$|\mathcal{F}(f)(\lambda)| \le M\left(\int_{\mathfrak{a}^+} h_a^{p'}(\exp H)[\varphi_{-i\eta}]^{p'}(\exp H)\delta(H)\,dH\right)^{1/p'}. \tag{4.4}$$

By inequalities (2.5), (2.6), (2.17), and (2.24), we deduce that

$$\begin{aligned} &h_a^{p'}(\exp H)[\varphi_{-i\eta}]^{p'}(\exp H)\delta(H) \\ &\quad \le \text{const}\, P(\|H\|)^{p'} e^{2(1-p')\langle H_\rho, H\rangle} e^{-p'(a\|H\|^2 - \langle H_\eta, H\rangle)}, \end{aligned} \tag{4.5}$$

where $P(\|H\|) = (1 + \|H\|^d)(1 + \|H\|^2)^{d'}$. Note that $P(\|H\|)^{p'} e^{2(1-p')\langle H_\rho, H\rangle}$ is bounded on $\mathfrak{a}^+$; so

$$|\mathcal{F}(f)(\lambda)| \le \text{const}\left(\int_{\mathfrak{a}^+} e^{-p'(a\|H\|^2 - \langle H_\eta, H\rangle)}\,dH\right)^{1/p'}. \tag{4.6}$$

Since

$$a\|H\|^2 - \langle H_\eta, H\rangle = \frac{1}{a}\left\|aH - \frac{1}{2}H_\eta\right\|^2 - \frac{1}{4a}\|\eta\|^2, \tag{4.7}$$

we get

$$\left|\mathcal{F}(f)(\lambda)\right| \le \operatorname{const} e^{(1/4a)\|\eta\|^2}\left(\int_{\mathfrak{a}^+} e^{-(p'/a)\|aH-(1/2)H_\eta\|^2}dH\right)^{1/p'}. \tag{4.8}$$

If we use the fact that

$$\int_{\mathfrak{a}^+} e^{-(p'/a)\|aH-(1/2)H_\eta\|^2}dH \le \int_{\mathfrak{a}} e^{-p'a\|H\|^2}dH, \tag{4.9}$$

we obtain

$$\left|\mathcal{F}(f)(\lambda)\right| \le \operatorname{const} e^{(1/4a)\|\eta\|^2}. \tag{4.10}$$

Second case ($p = 1$). We have, for $H \in \mathfrak{a}^+$ and $P(\|H\|) = (1+\|H\|^d)(1+\|H\|^2)^{d'}$ as in the first case, that formula (4.7) gives

$$\left|h_a(\exp H)\varphi_{-i\eta}(\exp H)\right| \le \operatorname{const} P(\|H\|)e^{-2\langle H_\rho,H\rangle}e^{-a\|H\|^2}e^{\langle\eta,H\rangle}. \tag{4.11}$$

By the same arguments used in the first case, we get

$$\left|h_a(\exp H)\varphi_{-i\eta}(\exp H)\right| \le \operatorname{const} e^{(1/4a)\|\eta\|^2}, \tag{4.12}$$

and then

$$\left|\mathcal{F}(f)(\lambda)\right| \le \operatorname{const} e^{(1/4a)\|\eta\|^2}. \tag{4.13}$$

Third case ($p = \infty$). Since $h_a^{-1}f$ is in $L^\infty(G)$, let $M = \|h_a^{-1}f\|_\infty$. By the property (2.4), we have

$$\left|\mathcal{F}(f)(\lambda)\right| \le M\int_{\mathfrak{a}^+} h_a(\exp H)\varphi_{-i\eta}(\exp H)\delta(H)dH. \tag{4.14}$$

Using the spherical Fourier transform of the heat kernel (2.23), we obtain

$$\left|\mathcal{F}(f)(\lambda)\right| \le \operatorname{const} e^{(1/4a)\|\eta\|^2}. \tag{4.15}$$

□

Theorem 4.3. *Suppose $1 \le p$, $q \le \infty$ with at least one of them finite. Let f be a measurable K-bi-invariant function on G such that*

$$\|h_a^{-1}f\|_{L^p(G)} \le M, \qquad \|e^{b\|\lambda\|^2}\mathcal{F}(f)(\lambda)\|_{L^q(\mathfrak{a}_+^*,|c(\lambda)|^{-2}d\lambda)} \le M, \tag{4.16}$$

for $M > 0$, $a > 0$, and $b > 0$. If $ab \ge 1/4$, then $f = 0$ almost everywhere.

Proof. First, suppose that $1 \le q < \infty$ and $1 \le p \le \infty$.

In the basis $\mu_1,\dots,\mu_l$ of $\mathfrak{a}^*$, the second inequality of (4.16) can be written in the form

$$\begin{aligned}\int_{\mathbb{R}_+^l} &\left|e^{b\|x_1\mu_1+\cdots+x_l\mu_l\|^2}\mathcal{F}(f)(x_1\mu_1+\cdots+x_l\mu_l)\right|^q \\ &\times\left|c(x_1\mu_1+\cdots+x_l\mu_l)\right|^{-2}dx_1\cdots dx_l \le M.\end{aligned} \tag{4.17}$$

By the change of variables $(x_1,\dots,x_l) = x(1,t_2,\dots,t_l)$, we have

$$\int_{\mathbb{R}_+^{l-1}}\left(\int_0^\infty \left|e^{b|x|^2\|\Lambda_t\|^2}\mathcal{F}(f)(x\Lambda_t)\right|^q \left|c(x\Lambda_t)\right|^{-2}x^{l-1}dx\right)dt_2\cdots dt_l \le M, \tag{4.18}$$

where $\Lambda_t = \mu_1 + t_2\mu_2 + \cdots + t_l\mu_l$. Fubini's theorem implies that for almost every $t_2 > 0,\dots,t_l > 0$,

$$\int_0^\infty \left|e^{b|x|^2\|\Lambda_t\|^2}\mathcal{F}(f)(x\Lambda_t)\right|^q \left|c(x\Lambda_t)\right|^{-2}x^{l-1}dx < +\infty. \tag{4.19}$$

For such $t_2,\dots,t_l$, let $g : \mathbb{C} \to \mathbb{C}$ be the function defined by

$$g(z) = e^{(1/4a)\|\Lambda_t\|^2 z^2}\mathcal{F}(f)(z\Lambda_t). \tag{4.20}$$

By Remark 4.1 and inequality (4.2), we deduce that g is an even analytic function on $\mathbb{C}$, which verifies

$$|g(z)| \le \operatorname{const} e^{(1/4a)\|\Lambda_t\|^2\Re(z^2)}, \quad z \in \mathbb{C}. \tag{4.21}$$

Moreover,

$$\int_0^\infty |g(x)|^q \gamma(x)dx < +\infty, \tag{4.22}$$

where $\gamma(x) = |c(x\Lambda_t)|^{-2}x^{l-1}$. Using (2.9) and (2.8), we can see that γ satisfies the hypotheses (3.1) and (3.2). So, by Lemma 3.2, we conclude that $g = 0$ on $\mathbb{C}$, and by Lemma 3.5, we conclude that $\mathcal{F}(f) = 0$ on $\mathfrak{a}_{\mathbb{C}}^*$. From the injectivity of the transform $\mathcal{F}$, we deduce that $f = 0$ almost everywhere.

Second, let $q = \infty$ and $1 \le p < \infty$.

We have, by the hypothesis (4.16),

$$|\mathcal{F}(f)(x\Lambda_t)| \le Me^{-b\|\Lambda_t\|^2x^2}, \quad x \in \mathbb{R}. \tag{4.23}$$

The fact that $ab \ge 1/4$ gives

$$|\mathcal{F}(f)(x\Lambda_t)| \le Me^{-(1/4a)\|\Lambda_t\|^2x^2}, \quad x \in \mathbb{R}. \tag{4.24}$$

By formula (4.2), we have

$$|\mathcal{F}(f)(z\Lambda_t)| \le \operatorname{const} e^{(1/4a)\|\Lambda_t\|^2|z|^2}, \quad z \in \mathbb{C}. \tag{4.25}$$

Hence, by Lemma 3.4, we conclude that

$$\mathcal{F}(f)(z\Lambda_t) = C \cdot e^{-(1/4a)\|\Lambda_t\|^2z^2} \tag{4.26}$$

for some constant C, and by Lemma 3.5, we obtain that

$$\mathcal{F}(f)(\lambda) = C \cdot e^{-(1/4a)\langle\lambda,\bar{\lambda}\rangle}, \quad \lambda \in \mathfrak{a}_{\mathbb{C}}^*. \tag{4.27}$$

Formula (2.23) implies that $f = \operatorname{const} h_a$.

Now, the hypothesis $\|h_a^{-1}f\|_{L^p(G)} \le M$ implies that $f = 0$ almost everywhere. $\square$

From the proof of the precedent theorem we deduce the following results.

Theorem 4.4. *Let $p = q = \infty$.*

(i) *If $ab > 1/4$, then every function f satisfying (4.16) is equal to zero almost everywhere.*

(ii) *If $ab = 1/4$, then the only functions verifying (4.16) are of the form $f = \text{const}\, h_a$.*

Remark 4.5. (i) This theorem is analogous to the classical theorem of Hardy.
(ii) The case $ab = 1/4$ of this theorem is also proved in [13].

We consider now the case $ab < 1/4$. We are going to prove that if $ab < 1/4$, $1 \le p$, and $q \le \infty$, then there are infinitely many functions satisfying conditions (4.16). For this, we need inequality (2.25) which is verified by h_t for $a < t < 1/4b$. This inequality is proved at least when G/K is of rank one or $G = \mathrm{SL}(3,\mathbb{R})$, and conjectured by Anker for any noncompact symmetric space.

Theorem 4.6. *If $ab < 1/4$, then for all $1 \le p, q \le \infty$,*

$$||h_a^{-1}h_t||_{L^p(G)} < \infty, \qquad ||e^{b\|\lambda\|^2}\mathcal{F}(h_t)(\lambda)||_{L^q(\mathfrak{a}_+^*,|c(\lambda)|^{-2}d\lambda)} < \infty. \tag{4.28}$$

Proof. Formulas (2.24) and (2.25) imply that $h_a^{-1}h_t \le \text{const}\, e^{(a-t)\|H\|^2}$. Since $a < t$, then $\|h_a^{-1}h_t\|_{L^p(G)} < \infty$.

The fact that $\mathcal{F}(h_t)(\lambda) = e^{-(1/4t)(\langle\lambda,\overline{\lambda}\rangle + \|\rho\|^2)}$ and $t < 1/4b$ gives the result. □

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