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ISHIKAWA ITERATIVE SEQUENCE WITH ERRORS FOR STRONGLY PSEUDOCONTRACTIVE OPERATORS IN ARBITRARY BANACH SPACES

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The Ishikawa iterative sequences with errors are studied for Lipschitzian strongly pseudocontractive operators in arbitrary real Banach spaces; some well-known results of Chidume (1998) and Zeng (2001) are generalized.

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1. Introduction. Let E be an arbitrary real Banach space with norm $\|\cdot\|$ and let E^* be the dual space of E. The duality mapping $J : E \to 2^{E^*}$ is defined by

$$Jx = \{f \in E^* : \langle x, f\rangle = \|x\| \cdot \|f\|,\ \|f\| = \|x\|\}, \tag{1.1}$$

where $\langle x, f\rangle$ denotes the value of the continuous linear function $f \in E^*$ at $x \in E$. It is well known that if E^* is strictly convex, then J is single valued.

An operator $T : D(T) \subset E \to E$ is said to be *accretive* if the inequality

$$\|x-y\| \le \big\|x-y+s(Tx-Ty)\big\| \tag{1.2}$$

holds for every $x, y \in D(T)$ and for all $s > 0$.

An operator T with domain $D(T)$ and range $R(T)$ in E is said to be a *strong pseudocontraction* if there exists $t > 1$ such that for all $x, y \in D(T)$ and $r > 0$, the following inequality holds:

$$\|x-y\| \le \big\|(1+r)(x-y)-rt(Tx-Ty)\big\|. \tag{1.3}$$

If $t = 1$ in inequality (1.3), then T is called *pseudocontractive*.

As a consequence of the result of Kato [3], T is pseudocontractive if and only if for each $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ such that

$$\big\langle (I-T)x-(I-T)y, j(x-y)\big\rangle \ge 0. \tag{1.4}$$

Furthermore, T is strongly pseudocontractive if and only if there exists $k > 0$ such that

$$\big\langle (I-T)x-(I-T)y, j(x-y)\big\rangle \ge k\|x-y\|^2. \tag{1.5}$$

Chidume [2] proved that if E is a real uniformly smooth Banach space, K is a nonempty closed convex bounded subset of E, and $T : K \to K$ is a strongly pseudocontraction with a fixed point x^* in K, then both the Mann and Ishikawa iteration schemes converge strongly to x^* for an arbitrary initial point $x_0 \in K$. Zeng [6] and Li and Liu [4] consider an iterative process for Lipschitzian strongly pseudocontractive operator in arbitrary real Banach spaces. In [2], Chidume proved the following theorem.

Theorem 1.1 [2]. *Suppose E is a real uniformly smooth Banach space and K is a bounded closed convex and nonempty subset of E. Suppose $T : E \to E$ is a strongly pseudocontractive map such that $Tx^* = x^*$ for some $x^* \in K$. Let $\{\alpha_n\}, \{\beta_n\}$ be real satisfying the following conditions:*

(i) $0 \le \alpha_n, \beta_n \le 1$ *for all* $n \ge 0$;
(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \beta_n = 0$;
(iii) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then, for arbitrary $x_0 \in K$, the sequence $\{x_n\}$ defined iteratively by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n, \quad n \ge 0, \end{aligned} \tag{1.6}$$

converges strongly to x^*; moreover, x^* is unique.

Our objective in this note is to consider an iterative sequence with errors for Lipschitzian strongly pseudocontractive operators in arbitrary real Banach spaces. Our results improve and extend the results of Chidume [2] and Zeng [6].

The following lemmas play an important role in proving our main results.

Lemma 1.2 [5]. *Let $\{a_n\}, \{b_n\}, \{c_n\}$ be a nonnegative sequence satisfying*

$$a_{n+1} \le (1-t_n)a_n + b_n + c_n. \tag{1.7}$$

With $\{t_n : n = 0, 1, 2, \ldots\} \subset [0,1]$, $\sum_{n=1}^{\infty} t_n = \infty$, $b_n = o(t_n)$, and $\sum_{n=1}^{\infty} c_n < \infty$, then $\lim_{n\to\infty} a_n = 0$.

2. Main results.

Now, we state and prove the following theorems.

Theorem 2.1. *Suppose E is an arbitrary real Banach space and $T : E \to E$ is a Lipschitzian strongly pseudocontractive map such that $Tx^* = x^*$ for some $x^* \in E$. Suppose $\{u_n\}, \{v_n\}$ are sequences in E and $\{\alpha_n\}, \{\beta_n\}$ are sequences in $[0,1]$ such that*

(1) $\sum_{n=1}^{\infty} \|u_n\| < \infty$, $\sum_{n=1}^{\infty} \|v_n\| < \infty$;
(2) $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\alpha_n \to 0$ *as* $n \to \infty$;
(3) $\beta_n \to 0$ *as* $n \to \infty$.

Then for any $x_0 \in E$, the Ishikawa iteration sequence $\{x_n\}$ with errors defined by

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n T x_n + v_n, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n + u_n, \end{aligned} \tag{2.1}$$

converges strongly to x^*; moreover, x^* is unique.

Proof. Since $T : E \to E$ is strongly pseudocontractive, we have that $(I-T)$ is strongly accretive, so for any $x, y \in E$, (1.5) holds, where $k = (t-1)/t$ and $t \in (1, \infty)$.

Thus

$$\langle ((1-k)I-T)x - ((1-k)I-T)y, j(x-y) \rangle \geq 0 \tag{2.2}$$

and so it follows from [3, Lemma 1.1] that

$$\|x-y\| \leq \|x-y+r[((1-k)I-T)x-((1-k)I-T)y]\| \tag{2.3}$$

for all $x, y \in E$ and $r > 0$.

From $x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n + u_n$, we obtain

$$\begin{aligned} x_n &= x_{n+1} + \alpha_n x_n - \alpha_n T y_n - u_n \\ &= (1+\alpha_n)x_{n+1} + \alpha_n[(I-T)x_{n+1} - kx_{n+1}] - (1-k)\alpha_n x_n \\ &\quad + (2-k)\alpha_n^2(x_n - Ty_n) + \alpha_n(Tx_{n+1} - Ty_n) - [(2-k)\alpha_n + 1]u_n. \end{aligned} \tag{2.4}$$

It is easy to see that

$$x^* = (1+\alpha_n)x^* + \alpha_n[((1-k)I-T)x^*] - (1-k)\alpha_n x^* \tag{2.5}$$

so that

$$\begin{aligned} x_n - x^* = {} & (1+\alpha_n)(x_{n+1} - x^*) + \alpha_n[((1-k)I-T)x_{n+1} - ((1-k)I-T)x^*] \\ & - (1-k)\alpha_n(x_n - x^*) + (2-k)\alpha_n^2(x_n - Ty_n) + \alpha_n(Tx_{n+1} - Ty_n) \\ & - [(2-k)\alpha_n + 1]u_n. \end{aligned} \tag{2.6}$$

Hence

$$\begin{aligned} \|x_n - x^*\| \geq {} & (1+\alpha_n)\left\|(x_{n+1} - x^*) + \frac{\alpha_n}{1+\alpha_n}[((1-k)I-T)x_{n+1} - ((1-k)I-T)x^*]\right\| \\ & - (1-k)\alpha_n\|x_n - x^*\| - (2-k)\alpha_n^2\|x_n - Ty_n\| \\ & - \alpha_n\|Tx_{n+1} - Ty_n\| - [(2-k)\alpha_n + 1]\|u_n\| \\ \geq {} & (1+\alpha_n)\|(x_{n+1} - x^*)\| - (1-k)\alpha_n\|x_n - x^*\| - (2-k)\alpha_n^2\|x_n - Ty_n\| \\ & - \alpha_n\|Tx_{n+1} - Ty_n\| - [(2-k)\alpha_n + 1]\|u_n\| \end{aligned} \tag{2.7}$$

so that

$$\begin{aligned}
\|x_{n+1}-x^*\| &\le \left[\frac{1+(1-k)\alpha_n}{1+\alpha_n}\right]\|x_n-x^*\|+\alpha_n^2\|x_n-Ty_n\| \\
&\quad+\alpha_n\|Tx_{n+1}-Ty_n\|+(2\alpha_n+1)\|u_n\| \\
&\le \left(1-\frac{k\alpha_n}{1+\alpha_n}\right)\|x_n-x^*\|+\alpha_n^2\|x_n-Ty_n\| \\
&\quad+\alpha_n\|Tx_{n+1}-Ty_n\|+(2\alpha_n+1)\|u_n\| \\
&\le \left(1-\frac{k\alpha_n}{2}\right)\|x_n-x^*\|+\alpha_n^2\|x_n-Ty_n\| \\
&\quad+\alpha_n\|Tx_{n+1}-Ty_n\|+3\|u_n\|.
\end{aligned} \tag{2.8}$$

Since T is a Lipschitzian operator and L is Lipschitz bound, we have

$$\begin{aligned}
\|y_n-x^*\| &= \|(1-\beta_n)(x_n-x^*)+\beta_n(Tx_n-x^*)+v_n\| \\
&\le [1+\beta_n(L-1)]\|x_n-x^*\|+\|v_n\| \le L\|x_n-x^*\|+\|v_n\|, \\
\|x_n-Ty_n\| &\le \|x_n-x^*\|+L\|y_n-x^*\| \le (1+L^2)\|x_n-x^*\|+L\|v_n\|, \\
\|Tx_{n+1}-Ty_n\| &\le L\|(1-\alpha_n)(x_n-y_n)+\alpha_n(Ty_n-y_n)+u_n\| \\
&\le L(1-\alpha_n)[\beta_n(1+L)\|x_n-x^*\|+\|v_n\|] \\
&\quad+L\alpha_n(1+L)[L\|x_n-x^*\|+\|v_n\|]+L\|u_n\| \\
&\le [L(1+L)\beta_n+(1+L)L^2\alpha_n]\|x_n-x^*\|+L(1+L)\|v_n\|+L\|u_n\|.
\end{aligned} \tag{2.9}$$

So there exist $M_1>0$ and $M_2>0$ such that

$$\begin{aligned}
\|x_{n+1}-x^*\| &\le \left(1-\frac{k\alpha_n}{2}\right)\|x_n-x^*\| \\
&\quad+[L(1+L)\beta_n+(L^3+3L^2+2)\alpha_n]\alpha_n\|x_n-x^*\|+M_1\|u_n\|+M_2\|v_n\|.
\end{aligned} \tag{2.10}$$

Since $\alpha_n\to 0$ and $\beta_n\to 0$, there exists $N>0$ such that for all $n>N$, we have

$$L(1+L)\beta_n+(L^3+3L^2+2)\alpha_n<\frac{k}{4}. \tag{2.11}$$

Thus

$$\|x_{n+1}-x^*\| \le \left(1-\frac{k\alpha_n}{4}\right)\|x_n-x^*\|+M_1\|u_n\|+M_2\|v_n\|. \tag{2.12}$$

Set

$$t_n=\frac{k\alpha_n}{4}, \qquad b_n=0, \qquad c_n=M_1\|u_n\|+M_2\|v_n\|. \tag{2.13}$$

Then we have

$$a_{n+1} \leq (1-t_n)a_n + b_n + c_n. \tag{2.14}$$

According to the above argument, it is easily seen that

$$\sum_{k=0}^{\infty} t_n = \infty, \qquad b_n = o(t_n), \qquad \sum_{k=0}^{\infty} c_n < \infty \tag{2.15}$$

and so, by Lemma 1.2, we have $\lim a_n = \lim \|x_n - x^*\| = 0$. Uniqueness follows as in [1]. The proof of the theorem is complete. □

Remark 2.2. Our Theorem 2.1 generalized the theorem of Chidume [2] from uniformly smooth Banach space to arbitrary Banach space and from Ishikawa iteration to Ishikawa iteration with errors. In addition, our results extend, generalize, and improve the corresponding results obtained by Zeng [6] and Li and Liu [4].

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