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COEFFICIENT ESTIMATES FOR RUSCHEWEYH DERIVATIVES

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We consider functions f, analytic in the unit disc and of the normalized form $f(z) = z+\sum_{n=2}^{\infty}a_nz^n$. For functions $f \in \bar{R}_\delta(\beta)$, the class of functions involving the Ruscheweyh derivatives operator, we give sharp upper bounds for the Fekete-Szegö functional $|a_3-\mu a_2^2|$.

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1. Introduction. Let S denote the class of normalized analytic univalent functions f defined by

$$f(z) = z+\sum_{n=2}^{\infty}a_nz^n \tag{1.1}$$

in the unit disc $D = \{z : |z| < 1\}$. Suppose that

$$\begin{aligned} S^* &= \left\{f \in S : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0,\ z \in D\right\}, \\ S^*(\beta) &= \left\{f \in S : \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\beta\pi}{2},\ z \in D\right\} \end{aligned} \tag{1.2}$$

are classes of starlike and strongly starlike functions of order β $(0 < \beta \le 1)$, respectively. Note that $S^*(\beta) \subset S^*$ for $0 < \beta < 1$ and $S^*(1) = S^*$ [5]. Kanas [2] introduced the subclass $\bar{R}_\delta(\beta)$ of function $f \in S$ as the following.

Definition 1.1. For $\delta \ge 0$, $\beta \in (0,1]$, a function f normalized by (1.1) belongs to $\bar{R}_\delta(\beta)$ if, for $z \in D-\{0\}$ and $D^\delta f(z) \ne 0$, the following holds:

$$\left|\arg\frac{z(D^\delta f(z))'}{D^\delta f(z)}\right| \le \frac{\beta\pi}{2}, \tag{1.3}$$

where $D^\delta f$ denotes the generalized Ruscheweyh derivative which was originally defined as the following.

Definition 1.2 [6]**.** Let $D^n f$ and f be defined by (1.1). Then for $n \in \mathbb{N}\cup\{0\}$,

$$D^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z), \tag{1.4}$$

where $*$ denotes the Hadamard product of two analytic functions and $\mathbb{N}$ is a set of natural numbers.

Later in [1], Al-Amiri generalized the Ruscheweyh derivative D^δ for real numbers $\delta \geq -1$ as a Hadamard product of the functions f and $z/(1-z)^{\delta+1}$.

Note that $\bar{R}_0(\beta) = S^*(\beta)$ for each $\beta \in (0,1]$ and $\bar{R}_0(1) = S^*$. In this note, we obtain sharp estimates for $|a_2|$, $|a_3|$ and the Fekete-Szegö functional for the class $\bar{R}_\delta(\beta)$. For the subclass S^*, sharp upper bounds for the functional $|a_3 - \mu a_2^2|$ have been obtained for all real μ [3, 4].

2. Preliminary results.

In proving our results, we will need the following lemmas. However, we first denote P to be the class of analytic functions with positive real part in D.

Lemma 2.1. *Let $p \in P$ and let it be of the form $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ with* $\operatorname{Re} p(z) > 0$. *Then*

(i) $|c_n| \leq 2$ *for* $n \geq 1$,

(ii) $|c_2 - c_1^2/2| \leq 2 - |c_1|^2/2$.

Lemma 2.2. *Let $\delta \geq 0$ and $\beta \in (0,1]$. If $f \in \bar{R}_\delta(\beta)$ and is given by (1.1), then*

$$|a_2| \leq \frac{2\beta}{\delta+1},$$
$$|a_3| \leq \begin{cases} \dfrac{2\beta}{(\delta+2)(\delta+1)} & \text{if } \beta \leq \dfrac{1}{3}, \\ \dfrac{6\beta^2}{(\delta+2)(\delta+1)} & \text{if } \beta \geq \dfrac{1}{3}. \end{cases} \tag{2.1}$$

Proof. Let $F(z) = D^\delta f(z) = z + A_2 z^2 + A_3 z^3 + \cdots$. Since $f \in \bar{R}_\delta(\beta)$ and $D^\delta f(z) \in S^*(\beta)$, then

$$\frac{zF'(z)}{F(z)} = p^\beta(z) \tag{2.2}$$

and so

$$\frac{z(1 + 2A_2 z + 3A_3 z^2 + \cdots)}{z + A_2 z^2 + A_3 z^3 + \cdots} = (1 + c_1 z + c_2 z^2 + \cdots)^\beta, \tag{2.3}$$

which implies that

$$z + 2A_2 z^2 + 3A_3 z^3 + \cdots = z + (\beta c_1 + A_2) z^2 + \left(\beta c_2 + \frac{\beta(\beta-1)}{2} c_1^2 + \beta A_2 c_1 + A_3\right) z^3 + \cdots. \tag{2.4}$$

Equating the coefficients, we have

$$A_2 = \beta c_1, \tag{2.5}$$
$$A_3 = \frac{\beta}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{3}{4}\beta^2 c_1^2. \tag{2.6}$$

Now, for $\delta \geq -1$, $D^\delta f$ has the Taylor expansion

$$D^\delta f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+\delta)}{(n-1)!\Gamma(1+\delta)} a_n z^n, \quad z \in D, \tag{2.7}$$

where $\Gamma(n+\delta)$ denotes Euler's functions with

$$\Gamma(n+\delta) = \delta(\delta+1)\cdots(\delta+n-1)\Gamma(\delta). \tag{2.8}$$

Then

$$z + A_2 z^2 + A_3 z^3 + \cdots = z + \frac{\Gamma(2+\delta)}{\Gamma(1+\delta)} a_2 z^2 + \frac{\Gamma(3+\delta)}{2\Gamma(1+\delta)} a_3 z^3 + \cdots. \tag{2.9}$$

Equating the coefficients in (2.9), we have

$$a_2 \frac{\Gamma(2+\delta)}{\Gamma(1+\delta)} = a_2(\delta+1) = A_2. \tag{2.10}$$

Then, from (2.5), we obtain

$$a_2 = \frac{\beta c_1}{\delta+1}. \tag{2.11}$$

It follows that from Lemma 2.1(i)

$$|a_2| \leq \frac{2\beta}{\delta+1}, \tag{2.12}$$

whereas the coefficient of z^3 in (2.9) is

$$a_3 \frac{\Gamma(3+\delta)}{2\Gamma(1+\delta)} = a_3 \frac{(\delta+1)(\delta+2)}{2} = A_3. \tag{2.13}$$

From (2.6), we obtain

$$a_3 = \frac{2}{(\delta+1)(\delta+2)} \left[\frac{\beta}{2} \left(c_2 - \frac{c_1^2}{2} \right) + \frac{3}{4} \beta^2 c_1^2 \right]. \tag{2.14}$$

It follows from Lemma 2.1(ii) that

$$|a_3| \leq \frac{2}{(\delta+1)(\delta+2)} \left[\frac{\beta}{2} \left(2 - \frac{|c_1|^2}{2} \right) + \frac{3}{4} \beta^2 |c_1|^2 \right], \tag{2.15}$$

that is,

$$|a_3| \leq \begin{cases} \dfrac{2\beta}{(\delta+2)(\delta+1)} & \text{if } \beta \leq \dfrac{1}{3}, \\[2ex] \dfrac{6\beta^2}{(\delta+2)(\delta+1)} & \text{if } \beta \geq \dfrac{1}{3}. \end{cases} \tag{2.16}$$

□

3. Results. We first consider the functional $|a_3-\mu a_2^2|$ for complex μ.

Theorem 3.1. *Let $f\in\bar{R}_\delta(\beta)$ and $\beta\in(0,1]$. Then for μ complex,*

$$|a_3-\mu a_2^2|\le\frac{2\beta}{(\delta+1)(\delta+2)}\max\left[1,\frac{|\beta(3(\delta+1)-2\mu(\delta+2))|}{(\delta+1)}\right]. \tag{3.1}$$

For each μ there is a function in $\bar{R}_\delta(\beta)$ such that equality holds.

Proof. From (2.11) and (2.14), we write

$$\begin{aligned}a_3-\mu a_2^2&=\frac{2}{(\delta+1)(\delta+2)}\left[\frac{\beta}{2}\left(c_2-\frac{c_1^2}{2}\right)+\frac{3}{4}\beta^2c_1^2\right]-\mu\left(\frac{\beta c_1}{\delta+1}\right)^2,\\&=\frac{1}{(\delta+1)(\delta+2)}\left[\beta\left(c_2-\frac{c_1^2}{2}\right)\right]+\frac{\beta^2(3(\delta+1)-2\mu(\delta+2))}{2(\delta+1)^2(\delta+2)}c_1^2.\end{aligned} \tag{3.2}$$

It follows from (3.2) and Lemma 2.1(ii) that

$$\begin{aligned}|a_3-\mu a_2^2|&\le\frac{\beta}{(\delta+1)(\delta+2)}\left(2-\frac{|c_1|^2}{2}\right)+\left|\frac{\beta^2(3(\delta+1)-2\mu(\delta+2))}{2(\delta+1)^2(\delta+2)}\right||c_1|^2,\\&=\frac{2\beta}{(\delta+1)(\delta+2)}+\frac{|\beta^2(3(\delta+1)-2\mu(\delta+2))|-\beta(\delta+1)}{2(\delta+1)^2(\delta+2)}|c_1|^2,\end{aligned} \tag{3.3}$$

which on using Lemma 2.1(i), that is, $|c_1|\le 2$, gives

$$|a_3-\mu a_2^2|\le\begin{cases}\dfrac{2\beta}{(\delta+1)(\delta+2)} & \text{if } \kappa(\delta)\le\beta(\delta+1),\\[2ex]\dfrac{|\beta^2(6(\delta+1)-4\mu(\delta+2))|}{(\delta+1)^2(\delta+2)} & \text{if } \kappa(\delta)\ge\beta(\delta+1),\end{cases} \tag{3.4}$$

where $\kappa(\delta)=|\beta^2(3(\delta+1)-2\mu(\delta+2))|$.

Equality is attained for functions in $\bar{R}_\delta(\beta)$ given by

$$\frac{z(D^\delta f(z))'}{D^\delta f(z)}=\left(\frac{1+z^2}{1-z^2}\right)^\beta,\qquad\frac{z(D^\delta f(z))'}{D^\delta f(z)}=\left(\frac{1+z}{1-z}\right)^\beta, \tag{3.5}$$

respectively. □

We next consider the cases where μ is real and prove the following.

Theorem 3.2. *Let $f \in \bar{R}_\delta(\beta)$ and $\beta \in (0,1]$. Then for μ real,*

$$|a_3 - \mu a_2^2| \le \begin{cases} \dfrac{\beta^2(6(\delta+1)-4\mu(\delta+2))}{(\delta+1)^2(\delta+2)} & \textit{if } \mu \le \dfrac{(6\beta-2)(\delta+1)}{4\beta(\delta+2)}, \\ \dfrac{2\beta}{(\delta+1)(\delta+2)} & \textit{if } \dfrac{(6\beta-2)(\delta+1)}{4\beta(\delta+2)} \le \mu \le \dfrac{(2+6\beta)(\delta+1)}{4\beta(\delta+2)}, \\ \dfrac{\beta^2(4\mu(\delta+2)-6(\delta+1))}{(\delta+1)^2(\delta+2)} & \textit{if } \mu \ge \dfrac{(2+6\beta)(\delta+1)}{4\beta(\delta+2)}. \end{cases} \tag{3.6}$$

For each μ, there is a function in $\bar{R}_\delta(\beta)$ such that equality holds.

Proof. Here we consider two cases.
Case (i): $\mu \le 3(\delta+1)/2(\delta+2)$.
In this case, (3.2) and Lemma 2.1(ii) give

$$\begin{aligned} |a_3 - \mu a_2^2| &\le \frac{\beta}{(\delta+1)(\delta+2)}\left(2 - \frac{|c_1|^2}{2}\right) + \frac{\beta^2(6(\delta+1)-4\mu(\delta+2))}{4(\delta+1)^2(\delta+2)}|c_1|^2, \\ &= \frac{2\beta}{(\delta+1)(\delta+2)} + \frac{\beta^2(6(\delta+1)-4\mu(\delta+2))-2\beta(\delta+1)}{4(\delta+1)^2(\delta+2)}|c_1|^2, \end{aligned} \tag{3.7}$$

and so, using the fact that $|c_1| \le 2$, we obtain

$$|a_3 - \mu a_2^2| \le \begin{cases} \dfrac{\beta^2(6(\delta+1)-4\mu(\delta+2))}{(\delta+1)^2(\delta+2)} & \text{if } \mu \le \dfrac{(6\beta-2)(\delta+1)}{4\beta(\delta+2)}, \\ \dfrac{2\beta}{(\delta+1)(\delta+2)} & \text{if } \dfrac{(6\beta-2)(\delta+1)}{4\beta(\delta+2)} \le \mu \le \dfrac{3(\delta+1)}{2(\delta+2)}. \end{cases} \tag{3.8}$$

Equality is attained on choosing $c_1 = c_2 = 2$ and $c_1 = 0$, $c_2 = 2$, respectively, in (3.2).
Case (ii): $\mu \ge 3(\delta+1)/2(\delta+2)$.
It follows from (3.2) and Lemma 2.1(ii) that

$$\begin{aligned} |a_3 - \mu a_2^2| &\le \frac{\beta}{(\delta+1)(\delta+2)}\left(2 - \frac{|c_1|^2}{2}\right) + \frac{\beta^2(4\mu(\delta+2)-6(\delta+1))}{4(\delta+1)^2(\delta+2)}|c_1|^2, \\ &= \frac{2\beta}{(\delta+1)(\delta+2)} + \frac{\beta^2(4\mu(\delta+2)-6(\delta+1))-2\beta(\delta+1)}{4(\delta+1)^2(\delta+2)}|c_1|^2, \end{aligned} \tag{3.9}$$

and so, using the fact that $|c_1| \le 2$, we obtain

$$|a_3 - \mu a_2^2| \le \begin{cases} \dfrac{2\beta}{(\delta+1)(\delta+2)} & \text{if } \dfrac{3(\delta+1)}{2(\delta+2)} \le \mu \le \dfrac{(6\beta+2)(\delta+1)}{4\beta(\delta+2)}, \\ \dfrac{\beta^2(4\mu(\delta+2)-6(\delta+1))}{(\delta+1)^2(\delta+2)} & \text{if } \mu \le \dfrac{(6\beta+2)(\delta+1)}{4\beta(\delta+2)}. \end{cases} \tag{3.10}$$

Equality is attained on choosing $c_1 = 0$, $c_2 = 2$ and $c_1 = 2i$, $c_2 = -2$, respectively, in (3.2). Thus the proof is complete. □

Theorem 3.3. *Let $f \in \bar{R}_\delta(\beta)$ and let it be given by (1.1). Then*

$$|a_3| - |a_2| \le \frac{2\beta}{(\delta+1)(\delta+2)} \quad \text{if } \beta \le \frac{3(\delta+1)}{5\delta+1}. \tag{3.11}$$

Proof. Write

$$|a_3| - |a_2| \le \left| a_3 - \frac{2}{3}a_2^2 \right| + \frac{2}{3}|a_2|^2 - |a_2|. \tag{3.12}$$

Then since $(6\beta-2)(\delta+1)/4\beta(\delta+2) \le 2/3$ for $\beta \le 3(\delta+1)/(5\delta+1)$, it follows from Theorem 3.2 that

$$|a_3| - |a_2| \le \frac{2\beta}{(\delta+1)(\delta+2)} + \frac{2}{3}|a_2|^2 - |a_2| = \lambda(x), \tag{3.13}$$

where $x = |a_2| \in [0, 2\beta/(\delta+1)]$. Since $\lambda(x)$ attains its maximum value at $x = 0$, the theorem follows. This is sharp. □

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