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SOME THEOREMS ON THE EXPLICIT EVALUATION OF RAMANUJAN'S THETA-FUNCTIONS

NAYANDEEP DEKA BARUAH and P. BHATTACHARYYA

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Bruce C. Berndt et al. and Soon-Yi Kang have proved many of Ramanujan's formulas for the explicit evaluation of the Rogers-Ramanujan continued fraction and theta-functions in terms of Weber-Ramanujan class invariants. In this note, we give alternative proofs of some of these identities of theta-functions recorded by Ramanujan in his notebooks and deduce some formulas for the explicit evaluation of his theta-functions in terms of Weber-Ramanujan class invariants.

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1. Introduction. Ramanujan's general theta-function $f(a,b)$ is given by

$$f(a,b)=\sum_{k=-\infty}^{\infty}a^{k(k+1)/2}b^{k(k-1)/2}, \tag{1.1}$$

where $|ab|<1$. If we set $a=q^{2iz}$, $b=q^{-2iz}$, and $q=e^{\pi i\tau}$, where z is complex and $\operatorname{Im}(\tau)>0$, then $f(a,b)=\vartheta_3(z,\tau)$, where $\vartheta_3(z,\tau)$ denotes one of the classical theta-functions in its standard notation [9, page 464]. After Ramanujan, we define the following special types of his theta-function.

If $|q|<1$, then

$$\phi(q):=f(q,q)=1+2\sum_{k=1}^{\infty}q^{k^2}, \tag{1.2}$$

$$\psi(q):=f(q,q^3)=\sum_{k=0}^{\infty}q^{k(k+1)/2}, \tag{1.3}$$

$$f(-q):=f(-q,-q^2)=\sum_{k=0}^{\infty}(-1)^kq^{k(3k-1)/2}+\sum_{k=1}^{\infty}(-1)^kq^{k(3k+1)/2}, \tag{1.4}$$

$$\chi(q):=(-q;q^2)_\infty, \tag{1.5}$$

where $(a;q)_\infty:=\Pi_{k=0}^{\infty}(1-aq^k)$. The function $\chi(q)$ is only for notational purposes. Also, note that $f(-q)=q^{-1/24}\eta(z)$, where $q=e^{2\pi iz}$ and η denotes the Dedekind eta-function. Much of Ramanujan's discoveries about theta-functions can be found in Chapters 16–21 of the organized pages of his second notebook [8]. Proofs and other references of all the identities can be found in [1]. However, in the unorganized pages of his notebooks [8], Ramanujan recorded many other beautiful identities. Proofs of these identities can be found in [2, 3]. In Section 2, we prove some of these identities by using some other identities of theta-functions. Berndt [2, 3] proved these identities via parameterization.

At scattered places in his notebooks [8], Ramanujan recorded several values of his theta-function $\phi(q)$. Proofs of all the values claimed by Ramanujan can be found in [3, Chapter 35]. Berndt and Chan [4] also verified all of Ramanujan's nonelementary values of $\phi(e^{-n\pi})$ and found three new values for $n = 13$, 27, and 63. Kang [6] also calculated some quotients of theta-functions ϕ and ψ. In Section 3, we give some theorems for the explicit evaluation of the quotients of theta-functions ϕ, ψ, and f, by combining Weber-Ramanujan class invariants with the identities proved in Section 2 and some other identities of theta-functions. Some of these evaluations can be used to find explicit values of the famous Rogers-Ramanujan continued fraction $R(q)$ defined by

$$R(q) := \frac{q^{1/5}}{1}\,\underset{+}{}\,\frac{q}{1}\,\underset{+}{}\,\frac{q^2}{1}\,\underset{+}{}\,\frac{q^3}{1}\,\underset{+}{}\cdots, \tag{1.6}$$

where $|q| < 1$.

We end this introduction by defining Weber-Ramanujan class invariants G_n and g_n. For $q = \exp(-\pi\sqrt{n})$, where n is a positive rational number, the Weber-Ramanujan class invariants G_n and g_n are defined by

$$G_n := 2^{-1/4}q^{-1/24}\chi(q), \tag{1.7}$$

$$g_n := 2^{-1/4}q^{-1/24}\chi(-q). \tag{1.8}$$

2. Theta-function identities. The following identity was recorded by Ramanujan on page 295 of his first notebook [8]. Berndt [3, page 366] proved this by using parameterization. Here we give an alternative proof.

Theorem 2.1. *If $\phi(q)$, $\psi(q)$, and $\chi(q)$ are defined by (1.2), (1.3), and (1.5), respectively, then*

$$\psi^2(-q) + 5q\psi^2(-q^5) = \frac{\phi^2(q)}{\chi(q)\chi(q^5)}. \tag{2.1}$$

Proof. From [1, Entry 9(vii), page 258, and Entry 10(v), page 262], we find that

$$\psi^2(q) - q\psi^2(q^5) = \frac{\phi(-q^5)f(-q^5)}{\chi(-q)}. \tag{2.2}$$

From [1, Entry 24(iii), page 39], we note that

$$f(q) = \frac{\phi(q)}{\chi(q)}. \tag{2.3}$$

From (2.2) and (2.3), we deduce that

$$\psi^2(q) - q\psi^2(q^5) = \frac{\phi^2(-q^5)}{\chi(-q)\chi(-q^5)}. \tag{2.4}$$

Now, we recall from [1, Entry 9(iii), page 258] that

$$\phi^2(q) - \phi^2(q^5) = 4q\chi(q)f(-q^5)f(-q^{20}). \tag{2.5}$$

Replacing q by $-q$ in (2.5), we deduce that

$$\phi^2(-q^5) = \phi^2(-q) + 4q\chi(-q)f(q^5)f(-q^{20}). \tag{2.6}$$

Employing (2.6) in (2.4), we find that

$$\psi^2(q) - q\psi^2(q^5) = \frac{\phi^2(-q)}{\chi(-q)\chi(-q^5)} + 4q\frac{f(q^5)f(-q^{20})}{\chi(-q^5)}. \tag{2.7}$$

Again, by [1, Entry 24(iii), page 39], we find that

$$f(-q^4) = \psi(q^2)\chi(-q^2). \tag{2.8}$$

Using (2.8) in (2.7), we obtain

$$\psi^2(q) - q\psi^2(q^5) = \frac{\phi^2(-q)}{\chi(-q)\chi(-q^5)} + 4q\frac{f(q^5)\psi(q^{10})\chi(-q^{10})}{\chi(-q^5)}. \tag{2.9}$$

Now, by [1, Entry 24(iv), page 39], we note that

$$\chi(q)\chi(-q) = \chi(-q^2). \tag{2.10}$$

Thus, from (2.9), we obtain

$$\psi^2(q) - q\psi^2(q^5) = \frac{\phi^2(-q)}{\chi(-q)\chi(-q^5)} + 4qf(q^5)\psi(q^{10})\chi(q^5). \tag{2.11}$$

From [1, Entry 25(iv), page 40], we note that

$$\phi(q)\psi(q^2) = \psi^2(q). \tag{2.12}$$

Employing (2.3) and (2.12), with q replaced by q^5, we conclude from (2.11) that

$$\psi^2(q) - q\psi^2(q^5) = \frac{\phi^2(-q)}{\chi(-q)\chi(-q^5)} + 4q\psi^2(q^5). \tag{2.13}$$

Replacing q by $-q$ in (2.13), we complete the theorem. □

The next theorem was recorded by Ramanujan on page 4 of his second notebook [8]. Berndt [2, page 202] proved this theorem by parameterization. Here we give an alternative proof by using some identities of theta-functions.

Theorem 2.2. *With $\psi(q)$ and $\chi(q)$ defined in (1.3) and (1.5), respectively,*

$$\frac{\chi^3(q)}{\chi(q^3)} = 1 + 3q\frac{\psi(-q^9)}{\psi(-q)}, \tag{2.14}$$

$$\frac{\chi^5(q)}{\chi(q^5)} = 1 + 5q\frac{\psi^2(-q^5)}{\psi^2(-q)}. \tag{2.15}$$

Proof of (2.14). From [1, Chapter 16, Corollary (ii) of Entry 31, page 49], we find that

$$\psi(q) - q\psi(q^9) = f(q^3, q^6). \tag{2.16}$$

Using the Jacobi triple product identity, Berndt [1, page 350] proved that

$$f(q, q^2) = \frac{\phi(-q^3)}{\chi(-q)}. \tag{2.17}$$

Replacing q by q^3 in (2.17) and then using the resultant identity in (2.16), we find that

$$\psi(q) - q\psi(q^9) = \frac{\phi(-q^9)}{\chi(-q^3)}. \tag{2.18}$$

Now, from [1, Corollary (i) of Entry 31, page 49 and Example (v), page 51], we find that

$$\phi(-q^9) = \phi(-q) + 2q\psi(q^9)\chi(-q^3). \tag{2.19}$$

Invoking (2.19) in (2.18), we deduce that

$$\psi(q) - 3q\psi(q^9) = \frac{\phi(-q)}{\chi(-q^3)}. \tag{2.20}$$

Thus,

$$1 - 3q\frac{\psi(q^9)}{\psi(q)} = \frac{\phi(-q)}{\chi(-q^3)\psi(q)}. \tag{2.21}$$

Now, from [1, Entry 24(iii), page 39], we note that

$$\chi(q) = \sqrt[3]{\frac{\phi(q)}{\psi(-q)}}. \tag{2.22}$$

Replacing q by $-q$ in (2.21) and then using (2.22), we complete the proof of (2.14). □

Proof of (2.15). From Theorem 2.1, we find that

$$1 + 5q\frac{\psi^2(-q^5)}{\psi^2(-q)} = \frac{\phi^2(q)}{\chi(q)\chi(q^5)\psi^2(-q)}. \tag{2.23}$$

Employing (2.22) in (2.23), we arrive at (2.15), which completes the proof. □

3. Explicit evaluations of theta-functions

Theorem 3.1. *If $\psi(q)$, G_n, and g_n are defined by (1.3), (1.7), and (1.8), respectively, then*

$$e^{-\pi\sqrt{n}}\frac{\psi(-e^{-9\pi\sqrt{n}})}{\psi(-e^{-\pi\sqrt{n}})} = \frac{1}{3}\left(\sqrt{2}\frac{G_n^3}{G_{9n}}-1\right), \tag{3.1}$$

$$e^{-\pi\sqrt{n}}\frac{\psi(e^{-9\pi\sqrt{n}})}{\psi(e^{-\pi\sqrt{n}})} = \frac{1}{3}\left(1-\sqrt{2}\frac{g_n^3}{g_{9n}}\right). \tag{3.2}$$

Proof. From (2.14) and the definition of G_n from (1.7), we easily arrive at (3.1). To prove (3.2), we replace q by $-q$ in (2.14) and then use the definition of g_n from (1.8). □

Since G_{9n} and g_{9n} can be calculated from the respective values of G_n and g_n [5], from the theorem above, we see that the quotients of theta-functions on the left-hand sides can be evaluated if the corresponding values of G_n and g_n are known. We give a few examples below.

Corollary 3.2.

$$e^{-\pi}\frac{\psi(-e^{-9\pi})}{\psi(-e^{-\pi})} = \frac{\sqrt[3]{2(\sqrt{3}-1)}-1}{3}. \tag{3.3}$$

Proof. Putting $n = 1$ in (3.1), we find that

$$e^{-\pi}\frac{\psi(-e^{-9\pi})}{\psi(-e^{-\pi})} = \frac{1}{3}\left(\sqrt{2}\frac{G_1^3}{G_9}-1\right). \tag{3.4}$$

From [3, page 189],

$$G_1 = 1, \qquad G_9 = \left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)^{1/3}. \tag{3.5}$$

Employing (3.5) in (3.4) and then simplifying, we complete the proof. □

From [1, Entry 11(ii), page 123], we find that

$$\psi(-e^{-\pi}) = \phi(e^{-\pi})2^{-3/4}e^{\pi/8}. \tag{3.6}$$

Since

$$\phi(e^{-\pi}) = \frac{\pi^{1/4}}{\Gamma(3/4)} \tag{3.7}$$

is classical [9], (3.3) and (3.6) provide an explicit evaluation for $\psi(-e^{-9\pi})$.

Corollary 3.3.

$$e^{-\pi\sqrt{5}/3}\frac{\psi(-e^{-3\pi\sqrt{5}})}{\psi(-e^{-\pi\sqrt{5}/3})} = \frac{(3+\sqrt{5})(\sqrt{5}-\sqrt{3})-2}{6}. \tag{3.8}$$

Proof. Putting $n=5/9$ in (3.1), we obtain

$$e^{-\pi\sqrt{5}/3}\frac{\psi(-e^{-3\pi\sqrt{5}})}{\psi(-e^{-\pi\sqrt{5}/3})}=\frac{1}{3}\left(\sqrt{2}\frac{G_{5/9}^3}{G_5}-1\right). \tag{3.9}$$

Now, from [3, pages 189 and 345], we note that

$$G_5=\left(\frac{1+\sqrt{5}}{2}\right)^{1/4},\qquad G_{5/9}=(\sqrt{5}+2)^{1/4}\left(\frac{\sqrt{5}-\sqrt{3}}{\sqrt{2}}\right)^{1/3}. \tag{3.10}$$

Employing (3.10) in (3.9) and then simplifying, we arrive at (3.8). □

Corollary 3.4.

$$e^{-\pi\sqrt{2}}\frac{\psi(e^{-9\pi\sqrt{2}})}{\psi(e^{-\pi\sqrt{2}})}=\frac{1-\sqrt{2}\sqrt[3]{\sqrt{3}-\sqrt{2}}}{3}. \tag{3.11}$$

Proof. Putting $n=2$ in (3.2), we find that

$$e^{-\pi\sqrt{2}}\frac{\psi(e^{-9\pi\sqrt{2}})}{\psi(e^{-\pi\sqrt{2}})}=\frac{1}{3}\left(1-\sqrt{2}\frac{g_2^3}{g_{18}}\right). \tag{3.12}$$

From [3, page 200], we note that

$$g_2=1,\qquad g_{18}=(\sqrt{2}+\sqrt{2})^{1/3}. \tag{3.13}$$

Using (3.13) in (3.12), we easily arrive at (3.11). □

Theorem 3.5. *With $\psi(q)$, G_n, and g_n defined in (1.3), (1.7), and (1.8), respectively,*

$$e^{-\pi\sqrt{n}}\frac{\psi^2(-e^{-5\pi\sqrt{n}})}{\psi^2(-e^{-\pi\sqrt{n}})}=\frac{1}{5}\left(2\frac{G_n^5}{G_{25n}}-1\right), \tag{3.14}$$

$$e^{-\pi\sqrt{n}}\frac{\psi^2(e^{-5\pi\sqrt{n}})}{\psi^2(e^{-\pi\sqrt{n}})}=\frac{1}{5}\left(1-2\frac{g_n^5}{g_{25n}}\right). \tag{3.15}$$

Proof. From (2.15) and the definition of G_n from (1.7), we easily arrive at (3.14). Replacing q by $-q$ in (2.15) and then using the definition of g_n from (1.8), we arrive at (3.15). □

If the class invariants are known, then we can explicitly find the values of the quotients of the left-hand-side expressions of the theorem. Next we give some examples.

Corollary 3.6 [6].

$$e^{-\pi}\frac{\psi^2(-e^{-5\pi})}{\psi^2(-e^{-\pi})}=\frac{1}{5\sqrt{5}+10}. \tag{3.16}$$

Proof. Putting $n=1$ in (3.14), we find that

$$e^{-\pi}\frac{\psi^2(-e^{-5\pi})}{\psi^2(-e^{-\pi})}=\frac{1}{5}\left(2\frac{G_1^5}{G_{25}}-1\right). \tag{3.17}$$

From [3, page 189],

$$G_1 = 1, \qquad G_{25} = \frac{1+\sqrt{5}}{2}. \tag{3.18}$$

Employing (3.18) in (3.17) and then simplifying, we complete the proof. □

Corollary 3.7.

$$e^{-\pi/\sqrt{5}} \frac{\psi^2(-e^{-\sqrt{5}\pi})}{\psi^2(-e^{-\pi/\sqrt{5}})} = \frac{1}{\sqrt{5}}. \tag{3.19}$$

Proof. We put $n = 1/5$ in (3.14) to obtain

$$e^{-\pi/\sqrt{5}} \frac{\psi^2(-e^{-\sqrt{5}\pi})}{\psi^2(-e^{-\pi/\sqrt{5}})} = \frac{1}{5}(2G_5^4 - 1). \tag{3.20}$$

Since, from [3, page 189],

$$G_5 = \left(\frac{1+\sqrt{5}}{2}\right)^{1/4}, \tag{3.21}$$

we can easily complete the proof by (3.20). □

Corollary 3.8.

$$e^{-\pi\sqrt{3/5}} \frac{\psi^2(-e^{-\pi\sqrt{15}})}{\psi^2(-e^{-\pi\sqrt{3/5}})} = \frac{3-\sqrt{5}}{5+\sqrt{5}}. \tag{3.22}$$

Proof. Putting $n = 3/5$ in (3.14), we obtain

$$e^{-\pi\sqrt{3/5}} \frac{\psi^2(-e^{-\pi\sqrt{15}})}{\psi^2(-e^{-\pi\sqrt{3/5}})} = \frac{1}{5}\left(2\frac{G_{3/5}^5}{G_{15}} - 1\right). \tag{3.23}$$

Now, from [3, page 341], we note that

$$G_{15} = 2^{-1/12}(1+\sqrt{5})^{1/3}, \qquad G_{3/5} = 2^{-1/12}(\sqrt{5}-1)^{1/3}. \tag{3.24}$$

Employing (3.24) in (3.23) and then simplifying, we arrive at (3.22). □

Corollary 3.9.

$$e^{-\pi\sqrt{2}} \frac{\psi^2(e^{-5\pi\sqrt{2}})}{\psi^2(e^{-\pi\sqrt{2}})} = \frac{1}{5}\left(1 - \frac{2}{a}\right), \tag{3.25}$$

where

$$a = g_{50} = \frac{1}{3}\left(1 + \left(\frac{5+\sqrt{5}}{4}\right)^{1/3}\left(\sqrt[3]{1+7\sqrt{5}+6\sqrt{6}} + \sqrt[3]{1+7\sqrt{5}-6\sqrt{6}}\right)\right). \tag{3.26}$$

Proof. We put $n = 2$ in (3.15) to obtain

$$e^{-\pi\sqrt{2}}\frac{\psi^2(e^{-5\pi\sqrt{2}})}{\psi^2(e^{-\pi\sqrt{2}})} = \frac{1}{5}\left(1-2\frac{g_2^5}{g_{50}}\right). \tag{3.27}$$

From [3, page 201],

$$g_{50} = \frac{1}{3}\left(1+\left(\frac{5+\sqrt{5}}{4}\right)^{1/3}\left(\sqrt[3]{1+7\sqrt{5}+6\sqrt{6}}+\sqrt[3]{1+7\sqrt{5}-6\sqrt{6}}\right)\right). \tag{3.28}$$

Employing (3.13) and (3.28) in (3.27), we complete the proof. □

Since for $q = e^{-\pi\sqrt{n}}$, n positive rational, the explicit formulas for $\phi^2(q^5)/\phi^2(q)$, $\phi(q^9)/\phi(q)$, and $\phi^4(q^3)/\phi^4(q)$ are known [3, page 339, (8.11); page 334, (5.7); page 330, (4.5), respectively], namely,

$$\frac{\phi^2(e^{-5\pi\sqrt{n}})}{\phi^2(e^{-\pi\sqrt{n}})} = \frac{1}{5}\left(1+2\frac{G_{25n}}{G_n^5}\right), \tag{3.29}$$

$$\frac{\phi(e^{-9\pi\sqrt{n}})}{\phi(e^{-\pi\sqrt{n}})} = \frac{1}{3}\left(1+\sqrt{2}\frac{G_{9n}}{G_n^3}\right), \tag{3.30}$$

$$\frac{\phi^4(e^{-3\pi\sqrt{n}})}{\phi^4(e^{-\pi\sqrt{n}})} = \frac{1}{9}\left(1+2\sqrt{2}\frac{G_{9n}^3}{G_n^9}\right), \tag{3.31}$$

we now derive some identities by which the corresponding values of the quotients $\psi^2(-q^5)/\psi^2(-q)$, $\psi(-q^9)/\psi(-q)$, and $\psi^4(-q^3)/\psi^4(-q)$ can be found.

Theorem 3.10 [7]. *If $\phi(q)$ and $\psi(q)$ are defined by (1.2) and (1.3), respectively, then*

$$q\frac{\psi^2(-q^5)}{\psi^2(-q)} = \frac{1-\phi^2(q^5)/\phi^2(q)}{(5\phi^2(q^5)/\phi^2(q))-1}. \tag{3.32}$$

Proof. We replace q by $-q$ in (2.4) and then divide the resulting identity by (2.1) to obtain

$$\frac{\phi^2(q^5)}{\phi^2(q)} = \frac{\psi^2(-q)+q\psi^2(-q^5)}{\psi^2(-q)+5q\psi^2(-q^5)}. \tag{3.33}$$

This is indeed equivalent to (3.32). □

Theorem 3.11. *With $\phi(q)$ and $\psi(q)$ defined in (1.2) and (1.3), respectively,*

$$q\frac{\psi(-q^9)}{\psi(-q)} = \frac{1-\phi(q^9)/\phi(q)}{(3\phi(q^9)/\phi(q))-1}. \tag{3.34}$$

Proof. Replace q by $-q$ in (2.18) and (2.20) and then, dividing the first resulting identity by the second, we find that

$$\frac{\phi(q)}{\phi(q^9)} = \frac{\psi(-q)+q\psi(-q^9)}{\psi(-q)+3q\psi(-q^9)}. \tag{3.35}$$

It is now easy to see that (3.35) and (3.34) are equivalent. □

Theorem 3.12. *With $\phi(q)$ and $\psi(q)$ defined in (1.2) and (1.3), respectively,*

$$1+9q\frac{\psi^4(-q^3)}{\psi^4(-q)}=\frac{8}{(9\phi^4(q^3)/\phi^4(q))-1}. \tag{3.36}$$

Proof. From Theorem 3.11, we note that

$$1+3q\frac{\psi(-q^9)}{\psi(-q)}=\frac{2}{(3\phi(q^9)/\phi(q))-1}. \tag{3.37}$$

From the third equality of [1, Entry 1(ii), page 345] and the second equality of [1, Entry 1(iii), page 345], we note that

$$\begin{aligned}1+3q\frac{\psi(-q^9)}{\psi(-q)}&=\left(1+9q\frac{\psi^4(-q^3)}{\psi^4(-q)}\right)^{1/3},\\ 3\frac{\phi(q^9)}{\phi(q)}-1&=\left(9\frac{\phi^4(q^3)}{\phi^4(q)}-1\right)^{1/3},\end{aligned} \tag{3.38}$$

respectively. Employing (3.38) in (3.37) and then cubing the resultant identity, we complete the proof. □

Corollary 3.13.

$$e^{-\pi}\frac{\psi^4(-e^{-3\pi})}{\psi^4(-e^{-\pi})}=\frac{2-\sqrt{3}}{3\sqrt{3}}. \tag{3.39}$$

Proof. It is known from [3, page 327] (or can be found easily from (3.31)) that

$$\frac{\phi^4(e^{-3\pi})}{\phi^4(e^{-\pi})}=\frac{1}{6\sqrt{3}-9}. \tag{3.40}$$

The proof of the corollary now follows immediately by putting $q=e^{-\pi}$ in Theorem 3.12 and then using (3.40). □

Now, from [1, Entries 24(ii) and 24(iv), page 39], we note that

$$\begin{aligned}f^3(q)&=\phi^2(q)\psi(-q),\\ f^3(-q^2)&=\phi(q)\psi^2(-q).\end{aligned} \tag{3.41}$$

From (3.41), we find the following quotients of f in terms of ϕ and ψ:

$$\begin{aligned}F_1(q)&:=\frac{f^6(q)}{qf^6(q^5)}=\frac{\psi^2(-q)}{q\psi^2(-q^5)}\times\frac{\phi^4(q)}{\phi^4(q^5)},\\ F_2(q)&:=\frac{f^6(-q^2)}{q^2f^6(-q^{10})}=\frac{\phi^2(q)}{\phi^2(q^5)}\times\frac{\psi^4(-q)}{q^2\psi^4(-q^5)}.\end{aligned} \tag{3.42}$$

The values of $F_1(q)$ and $F_2(q)$ can be determined explicitly for $q = e^{-\pi\sqrt{n}}$ by employing Theorem 3.5 and (3.29). We give a couple of examples below.

Corollary 3.14.

$$F_1(e^{-\pi/\sqrt{5}}) = 5\sqrt{5},$$
$$F_2(e^{-\pi/\sqrt{5}}) = 5\sqrt{5}. \tag{3.43}$$

Proof. As in Corollary 3.7, by putting $n = 1/5$ in (3.29), it can be easily seen that

$$\frac{\phi^2(e^{-\sqrt{5}\pi})}{\phi^2(e^{-\pi/\sqrt{5}})} = \frac{1}{\sqrt{5}}. \tag{3.44}$$

Putting $q = e^{-\pi/\sqrt{5}}$ in (3.42) and then employing (3.44) and Corollary 3.7, we complete the proof. □

Corollary 3.15.

$$F_1(e^{-\pi\sqrt{3/5}}) = \frac{5(5+\sqrt{5})}{2},$$
$$F_2(e^{-\pi\sqrt{3/5}}) = \frac{5(25+11\sqrt{5})}{2}. \tag{3.45}$$

Proof. As in Corollary 3.8, by putting $n = 3/5$ in (3.29), it can be easily seen that

$$\frac{\phi^2(e^{-\sqrt{15}\pi})}{\phi^2(e^{-\pi\sqrt{3/5}})} = \frac{2}{5-\sqrt{5}}. \tag{3.46}$$

Putting $q = e^{-\pi\sqrt{3/5}}$ in (3.42) and then employing (3.46) and Corollary 3.8, we complete the proof. □

Now, for the explicit evaluation of $R(q)$ defined in (1.6), we note from [6] that

$$\frac{1}{R^5(q^2)} - 11 - R^5(q^2) = \frac{f^6(-q^2)}{q^2 f^6(-q^{10})},$$
$$\frac{1}{S^5(q)} + 11 - S^5(q) = \frac{f^6(q)}{q f^6(q^5)}, \tag{3.47}$$

where $S(q) = -R(-q)$.

From (3.47) and (3.42), we see that to find the explicit values of $R(q^2)$ and $S(q)$, for $q = e^{-\pi\sqrt{n}}$, it is enough to find $F_1(q)$ and $F_2(q)$. See [6].

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Nayandeep Deka Baruah: Department of Mathematical Sciences, Tezpur University, Napaam 784 028, Sonitpur, Assam, India

E-mail address: nayan@tezu.ernet.in

P. Bhattacharyya: Department of Mathematical Sciences, Tezpur University, Napaam 784 028, Sonitpur, Assam, India

E-mail address: pb@tezu.ernet.in