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MIRROR SYMMETRY AND CONFORMAL FLATNESS IN GENERAL RELATIVITY

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Using symmetry arguments only, we show that every spacetime with mirror-symmetric spatial sections is necessarily conformally flat. The general form of the Ricci tensor of such spacetimes is also determined.

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1. Introduction. It is well known that the curvature tensor of any four-dimensional differentiable manifold has only 20 algebraically independent components. Ten out of these 20 components can be associated with its Weyl tensor, the remaining ten making up its Ricci tensor. When the four-dimensional manifold corresponds to an empty spacetime, its Ricci tensor becomes identically zero. The Weyl tensor can thus be seen as describing that part of the curvature of the spacetime which is not due to the presence of matter. The spacetime is said to be conformally flat when its Weyl tensor is identically zero (see, e.g., [4, Chapter 8]).

In this note, we are interested in the conditions on the curvature tensor $\mathbf{R}$ of a spacetime $\mathcal{M}_4$ which follow from the assumption that $\mathcal{M}_4$ has mirror-symmetric spatial sections. We will show that any such $\mathcal{M}_4$ is conformally flat. We will also obtain the general form of the corresponding Ricci tensor.

2. Mirror symmetry. To mathematically translate the assumption concerning the existence of a mirror symmetry for the spatial sections of $\mathcal{M}_4$, we now introduce a system of coordinates on $\mathcal{M}_4$. Let x^i, $i = 0,1,2,3$, be a coordinate system such that the spatial sections of $\mathcal{M}_4$ are described by x^0 = constant. We also consider a change of coordinate system for $\mathcal{M}_4$ and designate by $\overline{x}^i$, $i = 0,1,2,3$, the new coordinate system. Since we are only interested in the application of a mirror symmetry to the spatial sections of $\mathcal{M}_4$, we can assume that this change of coordinate system leaves invariant the time coordinate. Without loss of generality, we can also assume that the spatial mirror symmetry is defined with respect to the symmetry hyperplane $x^1 = 0$. This implies that the space coordinates transform according to the matrix

$$A = \operatorname{diag}(-1,1,1). \tag{2.1}$$

The coordinates of $\mathcal{M}_4$ then transform as

$$(\overline{x}^0,\overline{x}^1,\overline{x}^2,\overline{x}^3)^T = \mathcal{A}(x^0,x^1,x^2,x^3)^T, \tag{2.2}$$

where

$$\mathcal{A} = \operatorname{diag}(1, A). \tag{2.3}$$

For the spatial sections of $\mathcal{M}_4$ to be invariant under the transformation A, the curvature tensor $\mathbf{R}$ of the whole spacetime $\mathcal{M}_4$ must be invariant under the change of coordinate (2.2). It follows that the algebraically independent components of $\mathbf{R}$, at any given point of $\mathcal{M}_4$, will also be invariant under the same transformation. This property will then hold for the ten independent components of the Weyl tensor and the ten independent components of the Ricci tensor, which form the 20 independent components of the most general form of $\mathbf{R}$.

3. Weyl tensor. A condition the Weyl tensor $\mathbf{C}$ must satisfy for the corresponding spacetime to be invariant under the transformation (2.2) results from the Petrov matrix expression of the independent components of $\mathbf{C}$. To obtain this condition, we use the following correspondence between pairs of tensor indices of $\mathbf{C}$ and single Petrov indices:

$$\begin{array}{lcccccccccc} \text{Tensor indices}: & ijkl & = & 23, & 31, & 12, & 10, & 20, & 30; & C_{ijkl} \\ & \updownarrow & & & & & & & & \updownarrow \\ \text{Petrov index}: & A,B & = & 1, & 2, & 3, & 4, & 5, & 6; & C_{AB} \end{array} \tag{3.1}$$

The matrix of the independent components of $\mathbf{C}$ can be simplified yet further if, instead of the fully covariant components C_{ijkl}, one considers the mixed components $C^{ij}{}_{kl} \leftrightarrow C^A{}_B$. Here, we have $C^A{}_B = G^{AC}C_{CB}$, where the matrix $(G^{AC}) = \operatorname{diag}(I_{3\times3}, -I_{3\times3})$, and $I_{3\times3}$ is the 3×3 identity matrix. The ten independent components of $\mathbf{C}$ are then given by

$$(C^A{}_B) = \begin{pmatrix} M & N \\ -N & M \end{pmatrix}, \tag{3.2}$$

where $M = (m_{ij})$ and $N = (n_{ij})$ are symmetric traceless 3×3 matrices.

To the coordinate transformation (2.2) corresponds a similarity transformation of the matrix $\mathcal{C} = (C^A{}_B)$. Denoting with an overbar the components of the Weyl tensor in the barred coordinate system $\overline{x}^i$, $i = 0,1,2,3$, one indeed obtains (see [1, page 178])

$$\overline{C}^{ij}{}_{kl} = \sum_{(mn,pq)\leftrightarrow \text{Petrov}} \left(2\frac{\partial \overline{x}^i}{\partial x^m}\frac{\partial \overline{x}^j}{\partial x^n}\right) C^{mn}{}_{pq} \left(2\frac{\partial x^p}{\partial \overline{x}^k}\frac{\partial x^q}{\partial \overline{x}^l}\right), \tag{3.3}$$

where the sum is taken only over the pairs mn and pq corresponding to Petrov indices. If the Petrov indices A, B, C, D correspond, respectively, to the pairs of tensor indices ij, kl, mn, pq, then (3.3) is equivalent to

$$\overline{C}^A{}_B = S^A{}_C C^C{}_D \tilde{S}^D{}_B, \tag{3.4}$$

where

$$S^A{}_C = 2\frac{\partial \overline{x}^i}{\partial x^m}\frac{\partial \overline{x}^j}{\partial x^n}, \quad A, C = 1, 2, \dots, 6, \tag{3.5}$$

and the expressions $\widetilde{S}^D{}_B$ are the components of $\widetilde{S}$, the inverse of the matrix $S = (S^A{}_B)$, when this inverse exists. The tensor $\mathbf{C}$ will be invariant under the transformation (2.2) if $\overline{C}^A{}_B = C^A{}_B$ for $A, B = 1, 2, \dots, 6$. Equation (3.4) thus becomes

$$C^A{}_B = S^A{}_C C^C{}_D \widetilde{S}^D{}_B, \tag{3.6}$$

which amounts to

$$\mathscr{C} S = S \mathscr{C}. \tag{3.7}$$

We will now apply (3.7) to the case of a mirror symmetry with respect to the hyperplane $x^1 = 0$, that is, when the coordinate transformation is given by (2.3). The expression of the corresponding matrix S is

$$S = 2\,\mathrm{diag}(1, -1, -1, -1, 1, 1). \tag{3.8}$$

It is then straightforward to show that all components of the matrices M and N in (3.2) vanish identically. This implies that $\mathbf{C}$ also vanishes identically, that is, that $\mathcal{M}_4$ is four-dimensional conformally flat.

4. Ricci tensor. To obtain a condition that the Ricci tensor (R_{ij}) of $\mathcal{M}_4$ must satisfy in order for $\mathcal{M}_4$ to be invariant under the transformation (2.2), we first observe that (R_{ij}) can be considered as the matrix realization of a bilinear form on $\mathcal{M}_4$. It follows that the change of coordinate (2.2) transforms (R_{ij}) according to

$$(\overline{R}_{ij}) = \mathscr{A}(R_{ij})\mathscr{A}^T. \tag{4.1}$$

Since the invariance of (R_{ij}) under the transformation (2.2) implies that $(\overline{R}_{ij}) = (R_{ij})$, we obtain

$$(R_{ij}) = \mathscr{A}(R_{ij})\mathscr{A}^T. \tag{4.2}$$

Substituting (2.3) into (4.2) directly leads to

$$(R_{ij}) = \begin{pmatrix} R_{00} & 0 & R_{02} & R_{03} \\ 0 & R_{11} & 0 & 0 \\ R_{02} & 0 & R_{22} & R_{23} \\ R_{03} & 0 & R_{23} & R_{33} \end{pmatrix}. \tag{4.3}$$

5. Conclusion. We have shown that every spacetime having mirror-symmetric spatial sections is conformally flat. This result applies in particular to spherically and cylindrically symmetric spacetimes, in rotation or not. It also applies to many of the simply or multiconnected spacetimes considered in cosmic crystallography [2, 3].

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