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ON THE CLASSIFICATION OF THE LIE ALGEBRAS $L_{r,t}^{s}$

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The Lie algebras $L_{r,t}^{s}$ introduced by the author (2003) are classified from an algebraic point of view. A matrix representation of least degree is given for each isomorphism class.

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1. Introduction. The aim of this note is to classify a family of Lie algebras, $L_{r,t}^{s}$, which were introduced in [4] as a generalization of the Tavis-Cummings model, $L_{2,1}^{1}$. The Lie algebras $L_{r,t}^{s}$ were presented by generators K_1, K_2, K_3, K_4 and relations

$$\begin{gathered} [K_1,K_2]=sK_3, \quad [K_3,K_1]=rK_1, \quad [K_3,K_2]=-rK_2, \\ [K_3,K_4]=0, \quad [K_4,K_1]=-tK_1, \quad [K_4,K_2]=tK_2, \quad \text{for } r,s,t\in\mathbb{R}. \end{gathered} \tag{1.1}$$

From [1], $\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right]$, and $\left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]$ are representation matrices of a faithful representation of $L_{2,1}^{1}$, for K_1, K_2, K_3, and K_4, respectively. Thus, the Lie algebras $L_{2,1}^{1}$ and $\mathfrak{gl}(2,\mathbb{R})$ are isomorphic.

Note that the Lie subalgebra L_r^s, of $L_{r,t}^{s}$, generated by K_1, K_2, K_3 and relations

$$[K_1,K_2]=sK_3, \qquad [K_3,K_1]=rK_1, \qquad [K_3,K_2]=-rK_2 \tag{1.2}$$

was introduced in [2, 3, 6] as a generalization of the coupled quantized harmonic oscillators [7], namely, the model of light amplifier L_1^{-2}, and the model of two-level optical atom L_1^2, whose Hamiltonian model $H=K_0+\lambda(K_+ + K_-)$, λ is the coupling parameter. The matrix representations of L_r^s of least degree satisfying the physical properties $K_2=K_1^{\dagger}$ ($\dagger$ stands for Hermitian conjugation and K_0 is a real diagonal operator representing energy) were discussed in [2, 3, 6].

Faithful matrix representations of least degree of $L_{r,t}^{s}$ for appropriate values of r, s, and t were given in [4], subject to the physical conditions, namely, $K_2=K_1^{\dagger}$, and K_3, K_4 are real diagonal operators representing energy. It was found that

(1) for $rs>0$, $t\in\mathbb{R}$, $L_{r,t}^{s}$ has faithful representations of degree 2 as the least degree, where the matrices $\left[\begin{smallmatrix}0 & a\pm i\sqrt{rs/2-a^2}\\ 0 & 0\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}0 & 0\\ a\mp i\sqrt{rs/2-a^2} & 0\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}r/2 & 0\\ 0 & -r/2\end{smallmatrix}\right]$, and $\left[\begin{smallmatrix}b & 0\\ 0 & b+t\end{smallmatrix}\right]$ are representation matrices for K_1, K_2, K_3, and K_4, respectively, with $a,b\in\mathbb{R}$, $b\neq -t/2$, and $|a|\leq\sqrt{rs/2}$, $i=\sqrt{-1}$,

(2) for $r=s=t=0$, $L_{0,0}^{0}$ has faithful representation of degree 4 as the least degree, where the representation matrices are linearly independent diagonal matrices, while the representation matrices of K_3 and K_4 are real matrices.

These are the only cases where $L^s_{r,t}$ has faithful representations satisfying the mentioned physical conditions.

The Lie algebras $L^s_{r,t}$, $r,s,t \in \mathbb{R}$, are classified from an algebraic point of view. A matrix representation of least degree is given for each isomorphism class. The classification is given by the following theorem.

Theorem 1.1. *Let r, s, t be any nonzero real numbers; then*

(1) $L^s_{r,t} \simeq L^s_{r,0} \simeq \mathfrak{gl}(2,\mathbb{R})$,
(2) $L^s_{0,t} \simeq L^1_{0,1}$,
(3) $L^0_{r,t} \simeq L^0_{1,1}$,
(4) $L^0_{r,0} \simeq L^0_{0,t}$,
(5) $L^s_{0,0} \simeq L^1_{0,0}$,
(6) *the Lie algebras $\mathfrak{gl}(2,\mathbb{R})$, $L^1_{0,1}$, $L^0_{1,1}$, $L^0_{1,0}$, $L^1_{0,0}$, and $L^0_{0,0}$ are nonisomorphic Lie algebras.*

Corollary 1.2. *A system of representatives for the isomorphism classes of the Lie algebras of the form $L^s_{r,t}$ consists of $\mathfrak{gl}(2,\mathbb{R})$, $L^1_{0,1}$, $L^0_{1,1}$, $L^0_{1,0}$, $L^1_{0,0}$, and $L^0_{0,0}$.*

Unless otherwise stated, whenever X and Y are Lie algebras and f is a mapping $f: X \to Y$, then X is the Lie algebra of type $L^s_{r,t}$ for the assigned values of r, s, t and is generated by K'_1, K'_2, K'_3, and K'_4 satisfying (1.1), respectively, and Y is the Lie algebra of type $L^s_{r,t}$ for the assigned values of r, s, t and is generated by K_1, K_2, K_3, and K_4 satisfying (1.1), respectively.

2. Isomorphism classes for $rs \neq 0$

Theorem 2.1. *The Lie algebras $L^s_{r,t}$ and $L^s_{r,0}$ are isomorphic to the general linear Lie algebra $\mathfrak{gl}(2,\mathbb{R})$ for $r,s,t \in \mathbb{R}^*$.*

Proof. The mapping $\phi: L^s_{r,0} \to L^s_{r,t}$ defined by $\phi(K'_i) = K_i$, $i = 1,2,3$, and $\phi(K'_4) = (1/r)K_3 + (1/t)K_4$ is a Lie algebra isomorphism. It was found in [5] that when $rs \neq 0$, the Lie algebras L^s_r and L^1_{rs} are isomorphic, and the lie algebras L^1_d and L^1_c are isomorphic whenever $cd \neq 0$, where, in particular, an element $u \in L^1_c$ should satisfy that adu has eigenvalues 0, d, and $-d$. Using [5, Lemma 5 and Theorem 6], the isomorphism $\phi_1: L^s_{r,t} \to \mathfrak{gl}(2,\mathbb{R})$ defined by $\phi_1(K'_1) = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$, $\phi_1(K'_2) = \left[\begin{smallmatrix} 0 & 0 \\ rs & 0 \end{smallmatrix}\right]$, $\phi_1(K'_3) = \left[\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right]$, and $\phi_1(K'_4) = \left[\begin{smallmatrix} 0 & 0 \\ 0 & t \end{smallmatrix}\right]$, where $rst \neq 0$, can be suggested. □

3. Isomorphism classes for $rst = 0$.

The case when $t = 0$ and $rs \neq 0$ is discussed in the previous section.

Lemma 3.1. *For $st \neq 0$, the Lie algebras $L^s_{0,t}$ and $L^1_{0,1}$ are isomorphic. Moreover, $L^s_{0,t}$ is not isomorphic to $\mathfrak{gl}(2,\mathbb{R})$ and has faithful representation of degree 3 as the least degree.*

Proof. In $\mathfrak{gl}(2,\mathbb{R})$, a central element has trace zero if and only if it is the zero element. Since in $L^1_{0,1}$, $K_3 = [K_1,K_2]$ is a central element and of trace zero, thus $L^s_{0,t} \not\simeq \mathfrak{gl}(2,\mathbb{R})$. The mapping $\phi: L^s_{0,t} \to L^1_{0,1}$ defined by $\phi(K'_i) = K_i$, $i = 1,2$, $\phi(K'_3) = (1/s)K_3$, and $\phi(K'_4) = (1/t)K_4$ is a Lie algebra isomorphism. Clearly, $\left[\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 0 & 0 & 1/s \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]$, and $\begin{bmatrix} 0&0&0\\ 0&t&0\\ 0&0&0 \end{bmatrix}$ are representation matrices for K_1, K_2, K_3, and K_4, respectively, of a faithful representation of least degree of $L^s_{0,t}$. □

Lemma 3.2. *For $rt \neq 0$, the Lie algebras $L^0_{r,t}$ and $L^0_{1,1}$ are isomorphic. Moreover, $L^0_{r,t}$ is not isomorphic to $\mathfrak{gl}(2,\mathbb{R})$ and has faithful representation of degree 3 as the least degree.*

Proof. The mapping $\phi : L^0_{r,t} \to L^0_{1,1}$ defined by $\phi(K'_i) = K_i$, $i = 1,2$, $\phi(K'_3) = rK_3$, and $\phi(K'_4) = tK_4$ is a Lie algebra isomorphism. The elements K_1+K_2, K_1-K_2, K_3+K_4 are linearly independent generators of an abelian Lie subalgebra of $L^0_{r,t}$. Thus, $L^0_{r,t}$ has no faithful representation of degree 2. Thus, $L^0_{r,t} \ncong \mathfrak{gl}(2,\mathbb{R})$. Obviously, $\begin{bmatrix} 0&1&0\\ 0&0&0\\ 0&0&0 \end{bmatrix}$, $\begin{bmatrix} 0&0&1\\ 0&0&0\\ 0&0&0 \end{bmatrix}$, $\begin{bmatrix} 0&0&0\\ 0&-r&0\\ 0&0&r \end{bmatrix}$, and $\begin{bmatrix} -t&0&0\\ 0&0&0\\ 0&0&-2t \end{bmatrix}$ are representation matrices for K_1, K_2, K_3, and K_4, respectively, of a faithful representation of least degree of $L^0_{r,t}$. □

Lemma 3.3. *For $rt \neq 0$, the Lie algebras $L^0_{r,0}$ and $L^0_{0,t}$ are isomorphic. Moreover, $L^0_{0,t}$ is not isomorphic to $\mathfrak{gl}(2,\mathbb{R})$ and has faithful representation of degree 3 as the least degree.*

Proof. The mapping $\phi : L^0_{r,0} \to L^0_{0,t}$ defined by $\phi(K'_i) = K_i$, $i = 1,2$, $\phi(K'_3) = -(r/t)K_4$, and $\phi(K'_4) = K_3$ is a Lie algebra isomorphism. The elements K_1, K_2, K_3 are linearly independent generators of an abelian Lie subalgebra of $L^0_{0,t}$. Thus, $L^0_{0,t} \ncong \mathfrak{gl}(2,\mathbb{R})$. Clearly, $\begin{bmatrix} 0&0&0\\ 1&0&0\\ 0&0&0 \end{bmatrix}$, $\begin{bmatrix} 0&0&0\\ 0&0&1\\ 0&0&0 \end{bmatrix}$, $\begin{bmatrix} 1&0&0\\ 0&1&0\\ 0&0&1 \end{bmatrix}$, and $\begin{bmatrix} t&0&0\\ 0&0&0\\ 0&0&-t \end{bmatrix}$ are representation matrices for K_1, K_2, K_3, and K_4, respectively, of a faithful representation of least degree of $L^0_{0,t}$. □

Lemma 3.4. *For $s \neq 0$, the Lie algebras $L^s_{0,0}$ and $L^1_{0,0}$ are isomorphic. Moreover, $L^s_{0,0}$ is not isomorphic to $\mathfrak{gl}(2,\mathbb{R})$ and has faithful representation of degree 3 as the least degree.*

Proof. The mapping $\phi : L^s_{0,0} \to L^1_{0,0}$ defined by $\phi(K'_i) = K_i, i = 1,3,4$, and $\phi(K'_2) = sK_2$ is a Lie algebra isomorphism.

The elements K_1, K_3, K_4 are linearly independent generators of an abelian Lie subalgebra of $L^s_{0,0}$. Thus, $L^s_{0,0} \ncong \mathfrak{gl}(2,\mathbb{R})$. Obviously, $\begin{bmatrix} 0&1&0\\ 0&0&0\\ 0&0&0 \end{bmatrix}$, $\begin{bmatrix} 0&0&0\\ 0&0&s\\ 0&0&0 \end{bmatrix}$, $\begin{bmatrix} 0&0&1\\ 0&0&0\\ 0&0&0 \end{bmatrix}$, and $\begin{bmatrix} 1&0&0\\ 0&1&0\\ 0&0&1 \end{bmatrix}$ are representation matrices for K_1, K_2, K_3, and K_4, respectively, of a faithful representation of least degree of $L^s_{0,0}$. □

Theorem 3.5. *The Lie algebras $L^1_{0,1}$, $L^0_{1,1}$, $L^0_{0,1}$, $L^1_{0,0}$, and $L^0_{0,0}$ are not isomorphic.*

Proof. The Lie algebra $L^0_{0,0}$ is an abelian Lie algebra, while $L^1_{0,1}$, $L^0_{1,1}$, $L^0_{0,1}$, and $L^1_{0,0}$ are nonabelian Lie algebras. From (1.1), the dimension of the center of $L^1_{0,0}$ is 2. Let $Z = a_1K_1 + a_2K_2 + a_3K_3 + a_4K_4$ be a central element of $L^0_{0,1}$. Since $[Z,K_1] = 0$, then $a_4 = 0$, and since $[Z,K_4] = 0$, then $a_1K_1 - a_2K_2 = 0$. For the linear independence of K_1 and K_2, we must have $a_1 = a_2 = 0$. Thus, the center of $L^0_{0,1}$ can be generated by K_3. Thus, $L^1_{0,0} \ncong L^0_{0,1}$. Similarly, it can be proved that the center of $L^0_{1,1}$ is trivial. Thus, $L^0_{1,1}$ is not isomorphic to either $L^1_{0,0}$ or $L^0_{0,1}$. Thus, the Lie algebras $L^0_{1,1}$, $L^0_{0,1}$, and $L^1_{0,0}$ are not isomorphic.

The dimensions of $[L^0_{1,1},L^0_{1,1}]$, $[L^1_{0,0},L^1_{0,0}]$, and $[L^0_{0,1},L^0_{0,1}]$ are 2, 1, and 2, respectively, while the dimension of $[L^1_{0,1},L^1_{0,1}]$ is 3. Thus, $L^1_{0,1}$ is not isomorphic to any of the Lie algebras $L^0_{1,1}$, $L^0_{0,1}$, and $L^1_{0,0}$. □

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