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I-LINDELOF SPACES

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We define a space (X,T) to be I-Lindelof if every cover $\mathcal{A}$ of X by regular closed subsets of the space (X,T) contains a countable subfamily $\mathcal{A}'$ such that $X = \bigcup\{\text{int}(A) : A \in \mathcal{A}'\}$. We provide several characterizations of I-Lindelof spaces and relate them to some other previously known classes of spaces, for example, rc-Lindelof, nearly Lindelof, and so forth. Our study here of I-Lindelof spaces also deals with operations on I-Lindelof spaces and, in its last part, investigates images and inverse images of I-Lindelof spaces under some considered types of functions.

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1. Definitions and characterizations. In [2], a topological space (X,T) is called I-compact if every cover $\mathcal{A}$ of the space by regular closed subsets contains a finite subfamily $\{A_1,A_2,\dots,A_n\}$ such that $X = \bigcup_{k=1}^{n}\text{int}(A_k)$. Recall that a subset A of (X,T) is regular closed (regular open, resp.) if $A = \text{cl}(\text{int}(A))$ $(\text{int}(\text{cl}(A_k))$, resp.). We let $\text{RC}(X,T)$ $(\text{RO}(X,T)$, resp.) denote the family of all regular closed (all regular open, resp.) subsets of a space (X,T). A study that contains some properties of I-compact spaces appeared in [10]. In the present work, we study the class of I-Lindelof spaces.

Definition 1.1. A space (X,T) is called I-Lindelof if every cover $\mathcal{A}$ of the space (X,T) by regular closed subsets contains a countable subfamily $\{A_n : n \in N\}$ such that $X = \bigcup_{n\in N}\text{int}(A_n)$.

To obtain characterizations of I-Lindelof spaces, we need the definitions of some classes of generalized open sets.

Definition 1.2. A subset G of a space (X,T) is called semiopen (preopen, semi-preopen, resp.) if $G \subseteq \text{cl}(\text{int}(G))$ $(G \subseteq \text{int}(\text{cl}(G))$, $G \subseteq \text{cl}(\text{int}(\text{cl}(G)))$, resp.). $\text{SO}(X,T)$ $(\text{SPO}(X,T)$, resp.) is used to denote the family of all semiopen (all semi-preopen, resp.) subsets of a space (X,T). The complement of a semiopen subset (semi-preopen subset, resp.) is called semiclosed (semi-preclosed, resp.). It is clear that a subset G is semiopen if and only if $U \subseteq G \subseteq \text{cl}(U)$, for some open set U. A subset G is called regular semiopen if there exists a regular open set W such that $W \subseteq G \subseteq \text{cl}(W)$.

The following diagram relates some of these classes of sets:

$$\text{regular closed} \Longrightarrow \text{regular semiopen} \Longrightarrow \text{semiopen} \Longrightarrow \text{semi-preopen}. \tag{1.1}$$

It is well known that if G is a semi-preopen set, then $\text{cl}(G)$ is regular closed (see [6]). The next result gives several characterizations of I-Lindelof spaces and its proof is now clear.

Theorem 1.3. *The following statements are equivalent for a space (X,T).*

(a) (X,T) *is I-Lindelof.*

(b) *Every cover $\mathscr{A}$ of the space (X,T) by semi-preopen subsets contains a countable subfamily $\mathscr{A}'$ such that $X = \bigcup\{\text{int}(\text{cl}(A)) : A \in \mathscr{A}'\}$.*

(c) *Every cover $\mathscr{A}$ of the space (X,T) by semiopen subsets contains a countable subfamily $\mathscr{A}'$ such that $X = \bigcup\{\text{int}(\text{cl}(A)) : A \in \mathscr{A}'\}$.*

(d) *Every cover $\mathscr{A}$ of the space (X,T) by regular semiopen subsets contains a countable subfamily $\mathscr{A}'$ such that $X = \bigcup\{\text{int}(\text{cl}(A)) : A \in \mathscr{A}'\}$.*

Next we give another characterization of I-Lindelof spaces using the fact that a subset G is regular closed if and only if its complement is regular open.

Theorem 1.4. *A space (X,T) is I-Lindelof if and only if every family $\mathscr{U}$ of regular open subsets of (X,T) with $\bigcap\{U : U \in \mathscr{U}\} = \varnothing$ contains a countable subfamily $\mathscr{U}'$ such that $\bigcap\{\text{cl}(U) : U \in \mathscr{U}'\} = \varnothing$.*

Proof. To prove necessity, let $\mathscr{U} = \{U_\alpha : \alpha \in A\}$ be a family of regular open subsets of (X,T) such that $\bigcap\{U_\alpha : \alpha \in A\} = \varnothing$. Then the family $\{X - U_\alpha : \alpha \in A\}$ forms a cover of the I-Lindelof space (X,T) by regular closed subsets and therefore A contains a countable subset A' such that $X = \bigcup\{\text{int}(X - U_\alpha) : \alpha \in A'\}$. Then

$$\begin{aligned}\varnothing &= X - \bigcup\{\text{int}(X - U_\alpha) : \alpha \in A'\} \\ &= \bigcap\{X - \text{int}(X - U_\alpha) : \alpha \in A'\} = \bigcap\{\text{cl}(U_\alpha) : \alpha \in A'\}.\end{aligned} \tag{1.2}$$

To prove sufficiency, let $\mathscr{G} = \{G_\alpha : \alpha \in A\}$ be a cover of the space (X,T) by regular closed subsets. Then $\{X - G_\alpha : \alpha \in A\}$ is a family of regular open subsets of (X,T) with $\bigcap\{X - G_\alpha : \alpha \in A'\} = \varnothing$. By assumption, there exists a countable subset $A'$of A such that $\bigcap\{\text{cl}(X - G_\alpha) : \alpha \in A\} = \varnothing$. So $X = X - \bigcap\{\text{cl}(X - G_\alpha) : \alpha \in A'\} = \bigcup\{X - \text{cl}(X - G_\alpha) : \alpha \in A'\} = \bigcup\{\text{int}(G_\alpha) : \alpha \in A'\}$. This proves that (X,T) is I-Lindelof. □

In [7], a space (X,T) is called rc-Lindelof if every cover $\mathscr{A}$ of the space (X,T) by regular closed subsets contains a countable subcover for X. It is clear, by definitions, that every I-Lindelof space is rc-Lindelof. However, the converse is not true as we show in Example 1.7.

Recall that a space (X,T) is extremally disconnected (e.d.) if $\text{cl}(U)$ is open for each open $U \in T$. It is easy to show that a space (X,T) is e.d. if and only if, given any two regular open subsets U and V with $U \bigcap V = \varnothing$, $\text{cl}(U) \bigcap \text{cl}(V) = \varnothing$.

Proposition 1.5. *Every I-Lindelof space (X,T) is e.d.*

Proof. Suppose that (X,T) is not e.d. Then we find $U,V \in \text{RO}(X,T)$ such that $U \bigcap V = \varnothing$ but $\text{cl}(U) \bigcap \text{cl}(V) \neq \varnothing$, say $t \in \text{cl}(U) \bigcap \text{cl}(V)$. Now, the family $\{X - U, X - V\}$ forms a cover of the I-Lindelof space (X,T) by regular closed subsets. Thus $X = \text{int}(X - U) \bigcup \text{int}(X - V)$. Assume $t \in \text{int}(X - U)$. But $t \in \text{cl}(U)$ and therefore $\varnothing \neq \text{int}(X - U) \bigcap U \subseteq (X - U) \bigcap U$, a contradiction. The proof is now complete. □

Theorem 1.6. *A space (X,T) is I-Lindelof if and only if it is an e.d. rc-Lindelof space.*

Proof. As necessity is clear, we prove only sufficiency. We let $\mathcal{A}$ be a cover of (X,T) by regular closed subsets. If $A \in \mathcal{A}$, then A is regular closed and can be written as $A = \mathrm{cl}(U)$ for some $U \in T$. Since (X,T) is e.d., the set $A = \mathrm{cl}(U)$ is open. Now, since (X,T) is rc-Lindelof, the cover $\mathcal{A}$ contains a countable subfamily $\mathcal{A}'$ such that $X = \bigcup\{A : A \in \mathcal{A}'\} = \bigcup\{\mathrm{int}(A) : A \in \mathcal{A}'\}$ because $A = \mathrm{int}(A)$ for each $A \in \mathcal{A}$. This proves that (X,T) is I-Lindelof as required. □

Example 1.7. We construct an rc-Lindelof space which is not I-Lindelof. We let X be a countable infinite set and we fix a point $t \in X$. We provide X with the topology $T = \{U \subseteq X : t \notin U\} \bigcup \{U \subseteq X : t \in U \text{ and } X - U \text{ is finite}\}$. It is immediate that (X,T) is rc-Lindelof. However, (X,T) is not e.d. and therefore, by Theorem 1.6, is not I-Lindelof. To see that (X,T) is not e.d., we write $X = A \bigcup B$, where A and B are disjoint infinite subsets. Assume that $t \in A$. Then B is an open subset of (X,T) and $\mathrm{cl}(B) = B \bigcup \{t\}$. But $\mathrm{cl}(B)$ is not open and hence (X,T) is not e.d.

Definition 1.8. A space (X,T) is called:

(a) nearly Lindelof if every open cover $\mathcal{U}$ of (X,T) contains a countable subfamily $\mathcal{U}'$ such that $X = \bigcup\{\mathrm{int}(\mathrm{cl}(U)) : U \in \mathcal{U}'\}$ (see [3]);

(b) countably nearly compact if every countable open cover $\mathcal{U}$ of (X,T) contains a finite subfamily $\mathcal{U}'$ such that $X = \bigcup\{\mathrm{int}(\mathrm{cl}(U)) : U \in \mathcal{U}'\}$.

It is clear that a space (X,T) is I-compact if and only if it is I-Lindelof and countably nearly compact.

Theorem 1.9. *A space (X,T) is I-Lindelof if and only if it is an e.d. nearly Lindelof space.*

Proof. To prove necessity, we see that (X,T) is, by Proposition 1.5, e.d. Now, let $\mathcal{U}$ be an open cover of (X,T). Then $\{\mathrm{cl}(U) : U \in \mathcal{U}\}$ is a cover of the I-Lindelof space(X,T) by regular closed subsets. So $\mathcal{U}$ contains a countable subfamily $\mathcal{U}'$ such that $X = \bigcup\{\mathrm{int}(\mathrm{cl}(U)) : U \in \mathcal{U}'\}$. This proves that (X,T) is nearly Lindelof. Next, to prove sufficiency, we let $\mathcal{A}$ be a cover of (X,T) by regular closed subsets. Since (X,T) is e.d., then each $A \in \mathcal{A}$ is open. So $\mathcal{A}$ is an open cover of the nearly Lindelof space (X,T) and therefore $\mathcal{A}$ contains a countable subfamily $\mathcal{A}'$ such that $X = \bigcup\{\mathrm{int}(\mathrm{cl}(A)) : A \in \mathcal{A}'\} = \bigcup\{\mathrm{int}(A) : A \in \mathcal{A}'\}$ and we conclude that (X,T) is I-Lindelof. □

Theorem 1.10. *Let (X,T) be e.d. Then the following statements are equivalent:*

(a) *(X,T) is I-Lindelof;*

(b) *(X,T) is rc-Lindelof;*

(c) *(X,T) is nearly Lindelof.*

Recall that the family of all regular open subsets of a space (X,T) is a base for a topology T_s on X, weaker than T. The space (X,T_s) is called the semiregularization of (X,T) (see [7]). A property P of topological spaces is called a semiregular property if a space (X,T) has property P if and only if (X,T_s) has property P.

We will prove that I-Lindelofness is a semiregular property. First, we need the following result.

Proposition 1.11 [8, Proposition 2.2]. *Given a space* (X,T)*, let* $G \in \mathrm{SO}(X,T)$*. Then* $\mathrm{cl}_T(G) = \mathrm{cl}_{T_s}(G)$.

Theorem 1.12. *The property of being an I-Lindelof space is a semiregular property.*

Proof. First, the property of being an e.d. space is a semiregular property (see [7, page 99]). Now let (X,T) be an I-Lindelof space. Then (X,T) is, by Proposition 1.5, e.d. and hence (X,T_s) is also e.d. So $\mathrm{RC}(X,T) = \mathrm{RO}(X,T)$ and $\mathrm{RC}(X,T_s) = \mathrm{RO}(X,T_s)$. To show that (X,T_s) is rc-Lindelof, let $\mathcal{A}$ be a cover of (X,T) by regular closed subsets. Then each $A \in \mathcal{A}$ is T_s-open and $\mathcal{A} \subseteq T_s \subseteq T$. Thus $\mathcal{A}$ contains a countable subfamily $\mathcal{A}'$ such that $X = \bigcup\{\mathrm{cl}_T(A) : A \in \mathcal{A}'\}$ = (Proposition 1.11)$\bigcup\{\mathrm{cl}_{T_s}(A) : A \in \mathcal{A}'\} = \bigcup\{A : A \in \mathcal{A}'\}$ and therefore (X,T_s) is rc-Lindelof and hence I-Lindelof. Conversely, let (X,T_s) be I-Lindelof. Then both (X,T) and (X,T_s) are e.d. We show that (X,T) is rc-Lindelof. We let $\mathcal{A}$ be a cover of (X,T) by regular closed subsets, that is, $\mathcal{A} \subseteq \mathrm{RC}(X,T) = \mathrm{RO}(X,T) \subseteq T_s$. Since (X,T_s) is rc-Lindelof, there exists a countable subfamily $\mathcal{A}'$ of $\mathcal{A}$ such that $X = \bigcup\{\mathrm{cl}_{T_s}(A) : A \in \mathcal{A}'\}$ = (Proposition 1.11)$\bigcup\{\mathrm{cl}_T(A) : A \in \mathcal{A}'\} = \bigcup\{A : A \in \mathcal{A}'\}$. This shows that (X,T) is rc-Lindelof and the proof is complete. □

2. Operations on I-Lindelof spaces.

We note that the property of being an I-Lindelof space is not hereditary. Consider the discrete space N of all natural numbers and let βN be its Stone-Čech compactification. Then βN is an rc-compact Hausdorff space (see [7, page 102]) and therefore βN is e.d. (see [11]). So βN is an I-Lindelof space. However, the subspace $\beta N - N$ is not I-Lindelof as it is not e.d. (see [7, page 102]). Here, (X,T) is called rc-compact or S-closed if every cover of X by regular closed subsets contains a finite subcover (see [7]).

Recall that a subset A of a space (X,T) is called preopen if $A \subseteq \mathrm{int}(\mathrm{cl}(A))$. We let $\mathrm{PO}(X,T)$ denote the family of all preopen subsets of (X,T).

Proposition 2.1 [4, Corollary 2.12]. *Let* (X,T) *be rc-Lindelof and let* $U \in \mathrm{RO}(X,T)$*. Then the subspace* $(U, T|_U)$ *is rc-Lindelof.*

Proposition 2.2 [8, Proposition 4.2]. *The property of being an e.d. space is hereditary with respect to preopen subspaces.*

Remark 2.3. It is well known that a space (X,T) is e.d. if and only if $\mathrm{RC}(X,T) = \mathrm{RO}(X,T)$ if and only if $\mathrm{SO}(X,T) \subseteq \mathrm{PO}(X,T)$. Thus if (X,T) is e.d., then

$$\mathrm{RO}(X,T) = \mathrm{RC}(X,T) \subseteq \mathrm{SO}(X,T) \subseteq \mathrm{PO}(X,T). \tag{2.1}$$

In view of Propositions 2.1, 2.2, and Remark 2.3, the proof of the following result is now clear.

Theorem 2.4. *Every regular open (and hence every regular closed) subspace of an I-Lindelof space is I-Lindelof.*

Theorem 2.5. *If a space* (X,T) *is a countable union of open I-Lindelof subspaces, then it is I-Lindelof.*

Proof. Assume that $X = \bigcup\{U_n : n \in N\}$, where $(U_n, T|_{U_n})$ is an I-Lindelof subspace for each $n \in N$. Let $\mathcal{A}$ be a cover of the space (X,T) by regular closed subsets. For each $n \in N$, the family $\{A \bigcap U_n : A \in \mathcal{A}\}$ is a cover of U_n by regular closed subsets of the I-Lindelof subspace $(U_n, T|_{U_n})$ (see [4, Lemma 2.5]). So we find a countable subfamily $\mathcal{A}_n$ of $\mathcal{A}$ such that $U_n = \bigcup\{\text{int}_{U_n}(A \bigcap U_n) : A \in \mathcal{A}_n\}$. Put $\mathcal{B} = \bigcup\{\mathcal{A}_n : n \in N\}$. Then $\mathcal{B}$ is a countable subfamily of $\mathcal{A}$ such that $X = \bigcup\{U_n : n \in N\} = \bigcup_{n\in N}\bigcup\{\text{int}_{U_n}(A\bigcap U_n) : A \in \mathcal{A}_n\} = \bigcup_{n\in N}\bigcup\{\text{int}_X(A\bigcap U_n) : A \in \mathcal{A}_n\} \subseteq \bigcup\{\text{int}_X(A) : A \in \mathcal{B}\} \subseteq X$, that is, $X = \bigcup\{\text{int}(A) : A \in \mathcal{B}\}$. Therefore (X,T) is I-Lindelof. □

If $\{(X_\alpha, T_\alpha) : \alpha \in A\}$ is a family of spaces, we let $\oplus_{\alpha\in A} X_\alpha$ denote their topological sum. Now we have, as a consequence of Theorem 2.5, the following result.

Theorem 2.6. *The topological sum $\oplus_{\alpha\in A} X_\alpha$ of a family $\{(X_\alpha, T_\alpha) : \alpha \in A\}$ is I-Lindelof if and only if (X_α, T_α) is I-Lindelof for each $\alpha \in A$ and that A is a countable set.*

Proof. It is clear that sufficiency is a direct consequence of Theorem 2.5. To prove necessity, we note that (X_α, T_α) is a clopen (and hence regular open) subspace of the I-Lindelof space $\oplus_{\alpha\in A} X_\alpha$ and therefore (X_α, T_α) is, by Theorem 2.5, I-Lindelof for each $\alpha \in A$. Moreover, the family $\{X_\alpha : \alpha \in A\}$ forms a cover of the rc-Lindelof space $\oplus_{\alpha\in A} X_\alpha$ by mutually disjoint regular closed subsets and therefore must contain a countable subfamily whose union is $\oplus_{\alpha\in A} X_\alpha$. Thus A must be a countable set. □

We now turn to products of I-Lindelof spaces. As noted earlier, the space βN is I-Lindelof while $\beta N \times \beta N$ is not even e.d. However, we have the next special case.

Theorem 2.7. *Let (X,T) be a compact space and (Y,M) an I-Lindelof space. If the product $X \times Y$ is e.d., then it is I-Lindelof.*

Proof. By [1, Theorem 2.4], the space $X \times Y$ is rc-Lindelof. Since it is, by assumption, e.d., then it is, by Theorem 1.6, I-Lindelof. □

3. Images and inverse images of I-Lindelof spaces.

Let $f : (X,T) \to (Y,M)$. Recall that f is semicontinuous (see [9]) if $f^{-1}(V) \in \text{SO}(X,T)$ whenever $V \in M$, and f is almost open (see [8, page 86]) if $f^{-1}(\text{cl}(V)) \subseteq \text{cl}(f^{-1}(V))$ for each $V \in M$. Finally, f is preopen (see [8, page 86]) if $f(U)$ is a preopen subset of (Y,M) for each $U \in T$. It is mentioned in [8] that preopenness and almost openness coincide. Accordingly, we have the following result.

Theorem 3.1. *Let $f : (X,T) \to (Y,M)$ be semicontinuous almost open and let (X,T) be I-Lindelof. Then (Y,M) is I-Lindelof.*

Proof. First we have, by [8, Proposition 4.4], that (Y,M) is e.d. Next, by [1, Theorem 3.4], we have that (Y,M) is rc-Lindelof. Then, by Theorem 1.6, (Y,M) is I-Lindelof. □

Corollary 3.2. *Every open continuous image of an I-Lindelof space is I-Lindelof.*

Corollary 3.3. *If a product space $\Pi_{\alpha\in I} X_\alpha$ is I-Lindelof, then (X_α, T_α) is I-Lindelof, for each $\alpha \in I$.*

We recall that a function $f:(X,T)\to(Y,M)$ is irresolute if $f^{-1}(S)\in \mathrm{SO}(X,T)$ for each $S\in\mathrm{SO}(Y,M)$. Each irresolute is semicontinuous (see [1, Lemma 3.8]).

Corollary 3.4. *Every preopen irresolute image of an I-Lindelof space is I-Lindelof.*

We turn now to the inverse image of I-Lindelof spaces under certain class of functions. Recall that A is a semi-preclosed subset of a space (X,T) if its complement is semi-preopen.

Definition 3.5. A function $f:(X,T)\to(Y,M)$ is called (weakly) semi-preclosed if $f(A)$ is a semi-preclosed subset of (Y,M) for each (regular) closed subset A of (X,T).

The easy proof of the next result is omitted.

Lemma 3.6. *A function $f:(X,T)\to(Y,M)$ is (weakly) semi-preclosed if and only if, for every $y\in Y$ and for each $(U\in\mathrm{RO}(X,T))$ $U\in T$ with $f^{-1}(y)\subseteq U$, there exists $W\in\mathrm{SPO}(Y,M)$ such that $y\in W$ and $f^{-1}(W)\subseteq U$.*

Corollary 3.7. *Let $f:(X,T)\to(Y,M)$ be weakly semi-preclosed. If $B\subseteq Y$ and $f^{-1}(B)\subseteq U$, with $U\in\mathrm{RO}(X,T)$, then there exists $W\in\mathrm{SPO}(Y,M)$ such that $B\subseteq W$ and $f^{-1}(W)\subseteq U$.*

We recall that a space (X,T) is km-perfect (see [5]) if, for each $U\in\mathrm{RO}(X,T)$ and each point $x\in X-U$, there exists a sequence $\{U_n:n\in N\}$ of open subsets of (X,T) such that $\bigcup\{U_n:n\in N\}\subseteq U\subseteq\bigcup\{\mathrm{cl}(U_n):n\in N\}$ and $x\notin\bigcup\{\mathrm{cl}(U_n):n\in N\}$.

It is easy to see that every e.d. space is km-perfect.The converse, however, is not true as the space constructed in Example 1.7 is easily seen to be km-perfect but not e.d.

Lemma 3.8. *If (X,T) is a km-perfect P-space ($\equiv$ the countable union of closed subsets is closed), then (X,T) is e.d.*

Proof. We show that $\mathrm{cl}(U)$ is open for each $U\in T$. Note that $\mathrm{int}(\mathrm{cl}(U))$ is regular open and if $x\notin\mathrm{int}(\mathrm{cl}(U))$, then, since (X,T) is km-perfect, there exists a sequence $\{U_n:n\in N\}$ of open subsets such that $\bigcup\{U_n:n\in N\}\subseteq\mathrm{int}(\mathrm{cl}(U))\subseteq\bigcup\{\mathrm{cl}(U_n):n\in N\}$ and $x\notin\bigcup\{\mathrm{cl}(U_n):n\in N\}$. Since (X,T) is a P-space, then $\bigcup\{\mathrm{cl}(U_n):n\in N\}$ is closed and contains $\mathrm{int}(\mathrm{cl}(U))$ and so it contains $\mathrm{cl}(\mathrm{int}(\mathrm{cl}(U)))$. Thus $x\notin\mathrm{cl}(\mathrm{int}(\mathrm{cl}(U)))$ and we obtain that $\mathrm{cl}(\mathrm{int}(\mathrm{cl}(U)))=\mathrm{int}(\mathrm{cl}(U))$. But $U\subseteq\mathrm{int}(\mathrm{cl}(U))$ and therefore $\mathrm{cl}(U)\subseteq\mathrm{int}(\mathrm{cl}(U))=\mathrm{cl}(\mathrm{int}(\mathrm{cl}(U)))\subseteq\mathrm{cl}(U)$, that is, $\mathrm{cl}(U)=\mathrm{int}(\mathrm{cl}(U))$, which shows that $\mathrm{cl}(U)$ is open. □

Definition 3.9. A subset A of a space (X,T) is called an rc-Lindelof set (see [4]) if each cover of A by regular closed subsets of (X,T) contains a countable subcover of A.

We now state our final result which deals with an inverse image of an I-Lindelof space.

Theorem 3.10. *Let (X,T) be a km-perfect P-space. Let $f:(X,T)\to(Y,M)$ be weakly semi-preclosed almost open with $f^{-1}(y)$ an rc-Lindelof set for each $y\in Y$. If (Y,M) is I-Lindelof, then so is (X,T).*

Proof. It is clear, by Lemma 3.8, that (X,T) is e.d. and therefore we only show that (X,T) is rc-Lindelof (Theorem 1.6). We let $\mathcal{A}$ be a cover of X by regular closed subsets of the space (X,T). For each $y \in Y$, $\mathcal{A}$ forms a cover of the rc-Lindelof subset $f^{-1}(y)$ so we find a countable subfamily $\mathcal{A}_y$ of $\mathcal{A}$ such that $f^{-1}(y) \subseteq \bigcup\{A : A \in \mathcal{A}_y\} = G_y$. Then G_y is open, because (X,T) is e.d. and therefore $\mathrm{RC}(X,T) = \mathrm{RO}(X,T)$. But $f^{-1}(y) \subseteq G_y$, then we find, by Lemma 3.6, a subset $V_y \in \mathrm{SPO}(X,T)$ such that $y \in V_y$ and $f^{-1}(V_y) \subseteq G_y$. Now, the family $\{V_y : y \in Y\}$ forms a cover of Y by semi-preopen subsets of the rc-Lindelof space (Y,M). By [1, Theorem 1.9], it contains a countable subfamily $\{V_{y_n} : n \in N\}$ such that $Y = \bigcup \{\mathrm{cl}(V_{y_n}) : n \in N\}$. We put $\mathcal{A}' = \bigcup \{\mathcal{A}_{y_n} : n \in N\}$. Then $\mathcal{A}'$ is countable and $\mathcal{A}'$ is a cover of X. To see this, let $x \in X$ and let $y = f(x)$. Choose $k \in N$ such that $y \in \mathrm{cl}(V_{y_k})$. Then $x \in f^{-1}(\mathrm{cl}(V_{y_k})) \subseteq (f$ is almost open$)\, \mathrm{cl}(f^{-1}(V_{y_k})) \subseteq \mathrm{cl}(G_{y_k}) = G_{y_k}$ (because (X,T) is a P-space and G_{y_k} is a countable union of closed subsets). We have $x \in G_{y_k} = \bigcup\{A : A \in \mathcal{A}_{y_k}\} \subseteq \bigcup\{A : A \in \mathcal{A}'\}$. The proof is now complete. □

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