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QUASI β-POWER INCREASING SEQUENCES

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We prove a theorem of Mazhar (1999) on $|\bar{N}, p_n|_k$ summability factors under weaker conditions by using a quasi β-power increasing sequence instead of an almost increasing sequence.

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1. Introduction. A positive sequence (b_n) is said to be almost increasing if there exist a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \le b_n \le Bc_n$ (see [1]). Obviously, every increasing sequence is almost increasing. However, the converse need not be true as can be seen by taking the example, say $b_n = ne^{(-1)^n}$. Let $\sum a_n$ be a given infinite series with partial sums (s_n). Let (t_n) denote the nth $(C,1)$ mean of the sequence (na_n). A series $\sum a_n$ is said to be summable $|C,1|_k$, $k \ge 1$, if (see [6, 8])

$$\sum_{n=1}^{\infty} \frac{|t_n|^k}{n} < \infty. \tag{1.1}$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \longrightarrow \infty \quad \text{as } n \longrightarrow \infty,\ P_{-i} = p_{-i} = 0,\ i \ge 1. \tag{1.2}$$

The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{1.3}$$

defines the sequence (σ_n) of the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [7]). The series $\sum a_n$ is said to be summable $|\bar{N}, p_n|_k$, $k \ge 1$, if (see [3])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |\Delta\sigma_{n-1}|^k < \infty, \tag{1.4}$$

where

$$\Delta\sigma_{n-1} = -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v, \quad n \ge 1. \tag{1.5}$$

In the special case when $p_n = 1$ for all values of n, $|\bar{N}, p_n|_k$ summability is the same as $|C,1|_k$ summability. Also if we take $p_n = 1/(n+1)$, then $|\bar{N}, p_n|_k$ summability reduces to $|\bar{N}, 1/(n+1)|_k$ summability.

Mazhar [9] has proved the following theorem on $|C,1|_k$ summability factors of an infinite series.

Theorem 1.1. *If (X_n) is a positive nondecreasing sequence such that*

$$\lambda_m X_m = O(1) \quad \text{as } m \longrightarrow \infty, \tag{1.6}$$

$$\sum_{n=1}^{m} n X_n \left|\Delta^2 \lambda_n\right| = O(1) \quad \text{as } m \longrightarrow \infty, \tag{1.7}$$

$$\sum_{n=1}^{m} \frac{\left|t_n\right|^k}{n} = O(X_m) \quad \text{as } m \longrightarrow \infty, \tag{1.8}$$

then the series $\sum a_n \lambda_n$ is summable $|C,1|_k$, $k \geq 1$.

Bor [5] has extended Theorem 1.1 for $|\bar{N}, p_n|_k$ summability method in the following form.

Theorem 1.2. *Under the conditions (1.6), (1.7),*

$$P_n = O(n p_n), \tag{1.9}$$

$$\sum_{n=1}^{m} \frac{p_n}{P_n} \left|t_n\right|^k = O(X_m) \quad \text{as } m \longrightarrow \infty, \tag{1.10}$$

the series $\sum a_n \lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

For $p_n = 1$, (1.10) is the same as (1.8), and (1.9) holds. In this case, Theorem 1.2 reduces to Theorem 1.1. Also if we assume that $(n p_n) = O(P_n)$, then (1.10) is equivalent to (1.8) and $|\bar{N}, p_n|_k$ is equivalent to the $|C,1|_k$ summability (see [2, 4]). Hence, under the additional assumption $(n p_n) = O(P_n)$, Theorem 1.1 is equivalent to Theorem 1.2.

Quite recently, Mazhar [10] obtained a further generalization of Theorem 1.2 under weaker conditions by using an almost increasing sequence instead of positive nondecreasing sequence. Also it is clear that (1.9) and (1.10) imply (1.8). On the other hand, (1.9) implies that

$$\sum_{n=1}^{m} \frac{P_n}{n} = O(P_m) \quad \text{as } m \longrightarrow \infty. \tag{1.11}$$

It may be remarked that (1.9) implies (1.11), but the converse need not be true. His theorem is as follows.

Theorem 1.3. *If (X_n) is an almost increasing sequence and the conditions (1.6), (1.7), (1.8), (1.10), and (1.11) hold, then the series $\sum a_n \lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.*

2. The main result.

The aim of this note is to prove Theorem 1.3 under weaker conditions. For this we need the concept of quasi β-power increasing sequence. A positive sequence (γ_n) is said to be quasi β-power increasing sequence if there exists a constant $K = K(\beta,\gamma) \geq 1$ such that

$$Kn^{\beta}\gamma_n \geq m^{\beta}\gamma_m \tag{2.1}$$

holds for all $n \geq m \geq 1$. It should be noted that every almost increasing sequence is a quasi β-power increasing sequence for any nonnegative β, but the converse need not be true as can be seen by taking the example, say $\gamma_n = n^{-\beta}$ for $\beta > 0$. So we are weakening the hypotheses of Theorem 1.3, replacing an almost increasing sequence by a quasi β-power increasing sequence. Now, we will prove the following theorem.

Theorem 2.1. *Let (X_n) be a quasi β-power increasing sequence for some $0 < \beta < 1$. If the conditions (1.6), (1.7), (1.8), (1.10), and (1.11) are satisfied, then the series $\sum a_n\lambda_n$ is summable $|\bar{N},p_n|_k$, $k \geq 1$.*

We need the following lemma for the proof of Theorem 2.1.

Lemma 2.2. *If (X_n) is a quasi β-power increasing sequence for some $0 < \beta < 1$, then under the conditions (1.6) and (1.7),*

$$nX_n|\Delta\lambda_n| = O(1), \tag{2.2}$$

$$\sum_{n=1}^{\infty} X_n|\Delta\lambda_n| < \infty. \tag{2.3}$$

Proof. The condition (1.6) implies that $\lambda_n = O(1)$ and it is easy to see that (1.7) implies that $n\Delta\lambda_n = O(1)$. Thus $\Delta\lambda_n \to 0$, $n \to \infty$. Since $0 < \beta < 1$, for any $v \geq n$ we have $nX_n \leq KvX_v$, by (2.1). Hence, by (1.7), we get that

$$nX_n|\Delta\lambda_n| \leq nX_n\sum_{v=n}^{\infty}|\Delta^2\lambda_v| \leq K\sum_{v=n}^{\infty}vX_v|\Delta^2\lambda_v| < \infty, \tag{2.4}$$

thus $nX_n|\Delta\lambda_n| = O(1)$ as $n \to \infty$. Also,

$$\begin{aligned}\sum_{n=1}^{\infty}X_n|\Delta\lambda_n| &= \sum_{n=1}^{\infty}X_n\left|\sum_{v=n}^{\infty}\Delta^2\lambda_v\right| \leq \sum_{v=1}^{\infty}|\Delta^2\lambda_v|\sum_{n=1}^{v}X_n \\ &= \sum_{v=1}^{\infty}|\Delta^2\lambda_v|\sum_{n=1}^{v}n^{\beta}X_n n^{-\beta} \leq \sum_{v=1}^{\infty}|\Delta^2\lambda_v|Kv^{\beta}X_v\sum_{n=1}^{v}n^{-\beta} \\ &\leq K\sum_{v=1}^{\infty}|\Delta^2\lambda_v|v^{\beta}X_v\int_1^v\frac{dx}{x^{\beta}} \leq K\sum_{v=1}^{\infty}|\Delta^2\lambda_v|K(\beta)vX_v < \infty,\end{aligned} \tag{2.5}$$

where $K(\beta)$ is a constant depending only on β. This completes the proof of the lemma. □

3. Proof of Theorem 2.1. Let (T_n) denote the $(\bar{N},p_n)$ mean of the series $\sum a_n\lambda_n$. Then, by definition, and changing the order of summation, we have

$$T_n = \frac{1}{P_n}\sum_{v=0}^{n}p_v\sum_{i=0}^{v}a_i\lambda_i = \frac{1}{P_n}\sum_{v=0}^{n}(P_n - P_{v-1})a_v\lambda_v. \tag{3.1}$$

Then, for $n \geq 1$, we have

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v \lambda_v = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} \frac{P_{v-1}\lambda_v}{v} v a_v. \tag{3.2}$$

By Abel's transformation, we have

$$\begin{aligned} T_n - T_{n-1} &= \frac{n+1}{nP_n} p_n t_n \lambda_n - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v t_v \lambda_v \frac{v+1}{v} \\ &\quad + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v \Delta\lambda_v t_v \frac{v+1}{v} + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v t_v \lambda_{v+1} \frac{1}{v} \\ &= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}. \end{aligned} \tag{3.3}$$

Since

$$|T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}|^k \leq 4^k \left(|T_{n,1}|^k + |T_{n,2}|^k + |T_{n,3}|^k + |T_{n,4}|^k \right), \tag{3.4}$$

to complete the proof of Theorem 2.1, it is enough to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} |T_{n,r}|^k < \infty, \quad \text{for } r = 1,2,3,4. \tag{3.5}$$

In view of (1.6), (λ_n) is bounded. Hence, we have that

$$\begin{aligned} \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{k-1} |T_{n,1}|^k &= \sum_{n=1}^{m} \frac{p_n}{P_n} |\lambda_n|^{k-1} |\lambda_n| |t_n|^k = O(1) \sum_{n=1}^{m} |\lambda_n| \frac{p_n}{P_n} |t_n|^k \\ &= O(1) \sum_{n=1}^{m-1} \Delta |\lambda_n| \sum_{v=1}^{n} \frac{p_v}{P_v} |t_v|^k + O(1) |\lambda_m| \sum_{n=1}^{m} \frac{p_n}{P_n} |t_n|^k \\ &= O(1) \sum_{n=1}^{m-1} |\Delta\lambda_n| X_n + O(1) |\lambda_m| X_m \\ &= O(1) \quad \text{as } m \longrightarrow \infty, \end{aligned} \tag{3.6}$$

by virtue of (1.6), (1.10), and (2.3). Now, when $k > 1$, applying Hölder's inequality with indices k and k', where $1/k + 1/k' = 1$, as in $T_{n,1}$, we have that

$$\begin{aligned} \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{k-1} |T_{n,2}|^k &= O(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \left\{ \sum_{v=1}^{n-1} p_v |\lambda_v|^k |t_v|^k \right\} \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1} \\ &= O(1) \sum_{v=1}^{m} p_v |\lambda_v|^{k-1} |\lambda_v| |t_v|^k \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}} \\ &= O(1) \sum_{v=1}^{m} |\lambda_v| \frac{p_v}{P_v} |t_v|^k = O(1) \quad \text{as } m \longrightarrow \infty. \end{aligned} \tag{3.7}$$

In view of (2.3), it is clear that

$$\sum_{n=1}^{\infty} |\Delta\lambda_n| < \infty, \tag{3.8}$$

hence

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,3}|^k &= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\left\{\sum_{v=1}^{n-1}P_v|\Delta\lambda_v||t_v|^k\right\}\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}P_v|\Delta\lambda_v|\right\}^{k-1}\\ &= O(1)\sum_{v=1}^{m}P_v|t_v|^k|\Delta\lambda_v|\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}} = O(1)\sum_{v=1}^{m}|\Delta\lambda_v||t_v|^k\\ &= O(1)\sum_{v=1}^{m-1}\Delta(v|\Delta\lambda_v|)\sum_{i=1}^{v}\frac{1}{i}|t_i|^k + O(1)m|\Delta\lambda_m|\sum_{v=1}^{m}\frac{1}{v}|t_v|^k\\ &= O(1)\sum_{v=1}^{m-1}v|\Delta^2\lambda_v|X_v + O(1)\sum_{v=1}^{m-1}|\Delta\lambda_{v+1}|X_{v+1}\\ &\quad + O(1)m|\Delta\lambda_m|X_m = O(1) \quad \text{as } m\longrightarrow\infty,\end{aligned} \tag{3.9}$$

by virtue of (1.7), (1.8), (2.2), and (2.3). Since (λ_n) is bounded, finally we have that

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,4}|^k &\le \sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}P_v|\lambda_{v+1}|^k|t_v|^k\frac{1}{v}\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}\frac{P_v}{v}\right\}^{k-1}\\ &= O(1)\sum_{v=1}^{m}P_v|\lambda_{v+1}||t_v|^k\frac{1}{v}\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}}\\ &= O(1)\sum_{v=1}^{m}|\lambda_{v+1}|\frac{|t_v|^k}{v}\\ &= O(1)\sum_{v=1}^{m-1}\Delta|\lambda_{v+1}|\sum_{r=1}^{v}\frac{1}{r}|t_r|^k + O(1)|\lambda_{m+1}|\sum_{v=1}^{m}\frac{1}{v}|t_v|^k\\ &= O(1)\sum_{v=1}^{m-1}|\Delta\lambda_{v+1}|X_{v+1} + O(1)|\lambda_{m+1}|X_{m+1}\\ &= O(1) \quad \text{as } m\longrightarrow\infty,\end{aligned} \tag{3.10}$$

by virtue of (1.6), (1.8), (1.11), and (2.3). Therefore, we get that

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,r}|^k = O(1) \quad \text{as } m\longrightarrow\infty, \text{ for } r=1,2,3,4. \tag{3.11}$$

This completes the proof of Theorem 2.1.

Finally, if we take $p_n = 1$ for all values of n in Theorem 2.1, then we get a new result concerning the $|C,1|_k$ summability factors. Furthermore, if we take $p_n = 1/(n+1)$, then we get another new result for $|\bar{N}, 1/(n+1)|_k$ summability factors.

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