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AN EXTENSION OF q-ZETA FUNCTION

T. KIM, L. C. JANG, and S. H. RIM

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We will define the extension of q-Hurwitz zeta function due to Kim and Rim (2000) and study its properties. Finally, we lead to a useful new integral representation for the q-zeta function.

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1. Introduction. Let $0 < q < 1$ and for any positive integer k, define its q-analogue $[k]_q = (1-q^k)/(1-q)$. Let $\mathbb{C}$ be the field of complex numbers. The q-zeta function due to T. Kim was defined as

$$\zeta_q^{(h)}(s) = \sum_{n=1}^{\infty} \frac{q^{nh}}{[n]_q^s} + (q-1)\frac{1-s+h}{1-s}\sum_{n=1}^{\infty}\frac{q^{nh}}{[n]_q^{s-1}} \tag{1.1}$$

for any $s, h \in \mathbb{C}$ (cf. [3, 4]). This function can be considered on the spectral zeta function of the quantum group $\mathrm{SU}_q(2)$ (cf. [2, 4]). Also, the q-zeta function $\zeta_q^{(h)}(s)$ was studied at negative integers (see [4]). In this note, we lead to a useful new integral representation for the q-zeta function $\zeta_q^{(h)}(s)$. Finally, we define the extension of q-Hurwitz zeta function, and study its properties.

2. q-zeta functions. For $q \in \mathbb{C}$ with $|q| < 1$, we define q-Bernoulli polynomials as follows:

$$\begin{aligned} F_q^{(h)}(t,x) &= \sum_{n=0}^{\infty} \frac{\beta_{n,q}^{(h)}(x)}{n!}t^n \\ &= e^{(1/(1-q))t}\sum_{j=0}^{\infty}\frac{j+h}{[j+h]_q}(-1)^j q^{jx}\left(\frac{1}{1-q}\right)^j\frac{t^j}{j!} \\ &= -t\sum_{l=0}^{\infty} q^{l(h+1)+x}e^{[l+x]_q t} + (1-q)h\sum_{l=0}^{\infty} q^{lh}e^{[l+x]_q t} \end{aligned} \tag{2.1}$$

for $h \in \mathbb{Z}$, $x \in \mathbb{C}$ (cf. [2, 4]). In the case $x = 0$, $\beta_{n,q}^{(h)} (= \beta_{n,q}^{(h)}(0))$ will be called the q-Bernoulli numbers (cf. [4]). By (2.1), we easily see that

$$\begin{aligned}\beta_{n,q}^{(h)}(x) &= \sum_{j=0}^{m}\binom{m}{j}[x]_q^{n-j}q^{jx}\beta_{j,q}^{(h)} \\ &= \left(\frac{1}{1-q}\right)^n\sum_{j=0}^{n}\binom{n}{j}(-1)^j\frac{j+h}{[j+h]_q}q^{jx} \quad (\text{cf. [2]}),\end{aligned} \tag{2.2}$$

where $\binom{n}{j}$ is a binomial coefficient.

Thus we note that

$$q^h(q\beta^{(h)}+1)^n-\beta_{n,q}^{(h)}=\delta_{1,n}, \tag{2.3}$$

where we use the usual convention about replacing $(\beta^{(h)})^n$ by $\beta_{n,q}^{(h)}$ and $\delta_{1,n}$ is the Kronecker symbol.

Example 2.1.

$$\beta_0^{(2)}=\frac{2}{[2]},\quad \beta_1^{(2)}=-\frac{2q+1}{[2][3]},\quad \beta_2^{(2)}=\frac{2q^2}{[3][4]},\quad \beta_3^{(2)}=-\frac{q^2(q-1)(2[3]_q+q)}{[3][4][5]},\quad \ldots. \tag{2.4}$$

Let $F_q^{(h)}(t)=\sum_{n=0}^{\infty}(\beta_{n,q}^{(h)}/n!)t^n$. Then we easily see that

$$\begin{aligned}F_q^{(h)}(x,t) &= e^{[x]_qt}F_q^{(h)}(q^xt) \\ &= -t\sum_{l=0}^{\infty}q^{l(h+1)+x}e^{[l+x]_qt}+(1-q)h\sum_{l=0}^{\infty}q^{lh}e^{[l+x]_qt}.\end{aligned} \tag{2.5}$$

By (2.1) and (2.5), we note that

$$e^{-t}F_q^{(h)}(-qt)=qt\sum_{l=0}^{\infty}q^{l(h+1)}e^{-[l+1]_qt}+(1-q)h\sum_{l=0}^{\infty}q^{lh}e^{-[l+1]_qt}. \tag{2.6}$$

Thus we have

$$\frac{1}{\Gamma(s)}\int_0^{\infty}q^ht^{s-2}e^{-t}F_q^{(h)}(-qt)dt=\sum_{n=1}^{\infty}\frac{q^{nh}}{[n]_q^s}+(q-1)\frac{h+1-s}{1-s}\sum_{n=1}^{\infty}\frac{q^{nh}}{[n]_q^{s-1}}. \tag{2.7}$$

For $h,s\in\mathbb{C}$, we define the q-zeta function as follows:

$$\zeta_q^{(h)}(s)=\sum_{n=1}^{\infty}\frac{q^{nh}}{[n]_q^s}+(q-1)\frac{1-s+h}{1-s}\sum_{n=1}^{\infty}\frac{q^{nh}}{[n]^{s-1}} \quad (\text{cf. [1, 4]}). \tag{2.8}$$

Note that $\zeta_q^{(h)}(s)$ is a meromorphic function for $\operatorname{Re}(s)>1$.

Let $\Gamma(s)$ be the gamma function and let $\mathbb{Z}$ be the set of integers. By (2.3), (2.7), and (2.8), we obtain the following.

For $h,n(>1)\in\mathbb{Z}$, we have

$$\zeta_q^{(h)}(1-n)=-\frac{q^h(q\beta^{(h)}+1)^n}{n}=-\frac{\beta_{n,q}^{(h)}}{n}. \tag{2.9}$$

Let x be any nonzero positive real number. Then we define the q-analogue of Hurwitz zeta function as follows:

$$\zeta_q^{(h)}(s,x) = \sum_{n=0}^{\infty} \frac{q^{nh}}{[n+x]_q^s} + \frac{h+1-s}{1-s}(q-1)\sum_{n=0}^{\infty} \frac{q^{nh}}{[n+x]_q^{s-1}} \tag{2.10}$$

for $s,h \in \mathbb{C}$. By (2.5) and (2.10), we easily see that

$$\zeta_q^{(h)}(s,x) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-2} F_q^{(h)}(x,-t)\,dt. \tag{2.11}$$

Thus we obtain the following: for $n \in \mathbb{N}$, $h \in \mathbb{Z}$, we have

$$\zeta_q^{(h)}(1-n) = -\frac{\beta_{n,q}^{(h)}(x)}{n} \tag{2.12}$$

because

$$\zeta_q^{(h)}(s,x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \beta_{n,q}^{(h)}(x) \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s+n-2}\,dt. \tag{2.13}$$

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T. Kim: Institute of Science Education, Kongju National University, Kongju 314-701, Korea
E-mail address: tkim@kongju.ac.kr

L. C. Jang: Department of Mathematics and Computer Science, Konkuk University, Choongju 380-701, Korea
E-mail address: leechae.jang@kku.ac.kr

S. H. Rim: Department of Mathematics Education, Kyungpook National University, Daegu 702-701, Korea
E-mail address: shrim@knu.ac.kr