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ON THE SUBLINEAR OPERATORS FACTORING THROUGH L_q

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Let $0 < p \leq q \leq +\infty$. Let T be a bounded sublinear operator from a Banach space X into an $L_p(\Omega,\mu)$ and let ∇T be the set of all linear operators $\leq T$. In the present paper, we will show the following. Let C be a positive constant. For all u in ∇T, $C_{pq}(u) \leq C$ (i.e., u admits a factorization of the form $X \xrightarrow{\tilde{u}} L_q(\Omega,\mu) \xrightarrow{M_{g_u}} L_p(\Omega,\mu)$, where $\tilde{u}$ is a bounded linear operator with $\|\tilde{u}\| \leq C$, M_{g_u} is the bounded operator of multiplication by g_u which is in $B_{L_r^+(\Omega,\mu)}$ $(1/p = 1/q + 1/r)$, $u = M_{g_u} \circ \tilde{u}$ and $C_{pq}(u)$ is the constant of q-convexity of u) if and only if T admits the same factorization; this is under the supposition that $\{g_u\}_{u\in\nabla T}$ is latticially bounded. Without this condition this equivalence is not true in general.

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1. Introduction. The origin of this kind of factorization comes first from the work of Grothendieck [4] where he established that for each linear operator from an $L_\infty(S,\lambda)$ into an $L_1(\Omega,\mu)$ admits a factorization of the form

$$L_\infty(S,\lambda) \xrightarrow{\tilde{u}} L_2(\Omega,\mu) \xrightarrow{M_{g_u}} L_1(\Omega,\mu), \tag{1.1}$$

where $\tilde{u}$ is a bounded linear operator and M_{g_u} is the bounded linear operator of multiplication by a function g_u which is in $L_2(\Omega,\mu)$.

In the same circle of ideas Nikishin has proved in [12] that, any bounded linear operator from a Banach space X into $L_0(\Omega,\mu)$ ((Ω,μ) probability space) admits a factorization of the form

$$X \xrightarrow{\tilde{u}} L_p(\Omega,\mu) \xrightarrow{M_{g_u}} L_0(\Omega,\mu), \tag{1.2}$$

where $\tilde{u}$ is a bounded linear operator, M_{g_u} is the bounded linear operator of multiplication by a measurable function g_u and $0 < p < 1$.

This type of factorization through L_p was amplified and generalized by Kwapien [6]. For more informations, we refer to Pisier's book [14, Chapter 2].

In his thesis [8], Maurey gave some necessary and sufficient conditions so that every linear operator $u : X \to L_p(\Omega,\mu)$ ((Ω,μ) any measure space) factors through $L_q(\Omega,\mu)$ of the form

$$X \xrightarrow{\tilde{u}} L_q(\Omega,\mu) \xrightarrow{M_{g_u}} L_p(\Omega,\mu), \tag{1.3}$$

where $\tilde{u}$ is a bounded linear operator, M_{g_u} is the bounded linear operator of multiplication by a function g_u in $L_r(\Omega,\mu)$ and p, q, r are such that $0 < p \leq q \leq \infty$ with $1/p = 1/r + 1/q$.

Mezrag has suppressed in [10] the metric approximation hypothesis in Maurey's theorems and improved them in [11].

In [15], Pisier proved a more general result and gave some necessary and sufficient conditions so that every bounded linear operator $u : X \to L_r(\Omega,\mu)$ ((Ω,μ) an arbitrary measure space) factors through $L_{p\infty}(\Omega,\nu)$, $0 < r < p < \infty$, of the form

$$X \xrightarrow{\tilde{u}} L_{p\infty}(\Omega,\nu) \xrightarrow{M} L_r(\Omega,\mu), \tag{1.4}$$

where $\tilde{u}$ is a bounded linear operator, M is the bounded linear operator of multiplication by a function $g^{1/r}$ in $L_r(\Omega,\mu)$ and $\nu = g \cdot \mu$.

He also treated the dual problem (i.e., factorization of linear operators from $L_s(S,\lambda)$ into a Banach space Y by $L_{q1}(S,\nu)$, for $1 \le q < s < \infty$). This work was generalized in [1] to sublinear operators.

In this paper, we give the generalization of Maurey's theorem of factorization to sublinear operators which is a simple application of [8, Theorem 2] (also, by another method Defant in [3] has generalized this type of factorization to homogeneous operators) and we study the following equivalence which is our main result.

Let $0 < p \le q \le \infty$. Let $T : X \to L_p(\Omega,\mu)$ be a bounded sublinear operator. $\forall u \in \nabla T$, u factors through $L_q(\Omega,\mu) \Leftrightarrow T$ factors through $L_q(\Omega,\mu)$ where $\nabla T = \{$linear operators $u : X \to L_p(\Omega,\mu)$ such that $u \le T\}$.

In Section 2, we give some preliminaries on the (quasi-)Banach lattices and sublinear operators. We also give some relations between linear and sublinear operators.

In Section 3, we give our main result. We establish necessary and sufficient condition under which the above equivalence has a positive answer. In other words, we show that, if u is a q-convex linear operator for all u in ∇T and $C_{pq}(u)$ (the constant of q-convexity of u) are uniformly bounded then the equivalence as mentioned in the abstract holds under the assumption that the $\{g_u\}$ for u in ∇T is latticially bounded. Without this condition the equivalence fails.

2. Sublinear operators. In this section, we give some elementary definitions and fundamental properties on the (quasi-)Banach lattice. The reader is referred to the monographs [7, 9, 16] as general references for (quasi-)Banach lattices. We also study the class of sublinear operators by giving briefly some relation between linear and sublinear operators. We refer the reader to Pallu de la Barrière [13] for more information on sublinear operators as well as [1, 2].

A real vector space X that is partially ordered by a partial order denoted by $\le$ is called an order vector space if

$$\begin{aligned} &x \le y \Longrightarrow x + z \le y + z, \quad \text{for every } z \in X; \\ &x \ge 0 \Longrightarrow \alpha x \ge 0, \quad \text{for every } \alpha \ge 0 \text{ in } \mathbb{R}. \end{aligned} \tag{2.1}$$

We denote by $X^+ = \{x \in X : x \ge 0\}$. An element x of X is positive if $x \in X^+$. A subset A of X is called order bounded (or simply bounded) if there exists an element y in X such that $x \le y$ for all $x \in A$ and y is then called an upper bound for A or the supremum of A. If A is order bounded then z is called the least upper bound of A if z

is an upper bound for A and $z \le y$ for every upper bound of A. An order vector space in which each pair of elements has least upper bound is called a vector lattice. The least upper bound of a set with two elements x, y is denoted by

$$x \vee y \qquad \text{or} \qquad \sup\{x,y\}. \tag{2.2}$$

By a complete vector lattice we mean an order space for which every non-empty order bounded subset has a least upper bound.

Recall that a quasi-norm on a real vector space X is a function $x \to \|x\|$ from X to $\mathbb{R}^+$ which satisfies

(i) $\|x\| > 0$ *for all* $x \neq 0$;

(ii) $\|tx\| = |t|\|x\|$ *for all* $t \in \mathbb{R}$ *and* $x \in X$;

(iii) $\exists C_X \ge 1 : \|x+y\| \le C_X(\|x\|+\|y\|)$ *for all* $x,y \in X$,

C_X is called the modulus of concavity of $\|\cdot\|$.

We also recall that a real quasi-Banach space is a complete metrizable real vector space whose topology is given by a quasi-norm. If in addition X is a vector lattice and $\|x\| \le \|y\|$ whenever $|x| \le |y|$ ($|x| = \sup\{x,y\}$) we say that X is a quasi-Banach lattice. Note that this implies that for any $x \in X$ the elements x and $|x|$ have the same norm. For more details see [16].

If (iii) is substituted by (iii)′ $\|x+y\|^p \le \|x\|^p + \|y\|^p$, for all $x,y \in X$ and for some fixed $p \in]0,1]$, then $\|\cdot\|$ is called a p-norm on X. Note that 1-norm is the usual norm. A quasi-Banach space is isomorphic to a Banach space if and only if it is locally convex. Every p-norm is a quasi-norm with $C = 2^{1/p-1}$. Also for every quasi-Banach space X there is a number $0 < p < 1$ and an equivalent p-norm satisfying

$$\|x+y\|^p \le \|x\|^p + \|y\|^p, \tag{2.3}$$

for all $x,y \in X$.

If $\||\cdot|\|$ denotes the original quasi-norm on X with the constant C in the quasi-triangle inequality, then the p-norm ($C = 2^{1/p-1}$) can be defined as follows:

$$\|x\| = \inf\left\{\left(\sum_{i=1}^{n} \||x_i|\|^p\right)^{1/p} : n > 0,\ x = \sum_{i=1}^{n} x_i\right\}. \tag{2.4}$$

This assertation is due to Aoki and Rolewicz [5].

Definition 2.1. A mapping T from a Banach space X into a (quasi-)Banach lattice Y is said to be sublinear if for all x, y in X and λ in $\mathbb{R}_+$, there exist

(i) $T(\lambda x) = \lambda T(x)$ (i.e., positively homogeneous),

(ii) $T(x+y) \le T(x)+T(y)$ (i.e., subadditive).

Note that the sum of two sublinear operators is a sublinear operator and the multiplication by a positive number is also a sublinear operator.

Let us denote

$$SL(X,Y) = \{\text{sublinear mappings } T : X \longrightarrow Y\} \tag{2.5}$$

and equip it with the natural order induced by Y

$$T_1 \le T_2 \Longleftrightarrow T_1(x) \le T_2(x), \quad \forall x \in X, \tag{2.6}$$

$$\nabla T = \{u \in L(X,Y) : u \le T \ (\text{i.e.}, \forall x \in X, u(x) \le T(x))\}. \tag{2.7}$$

The set ∇T is not empty by Proposition 2.3 below. As a consequence,

$$u \le T \Longleftrightarrow -T(-x) \le u(x) \le T(x), \quad \forall x \in X, \tag{2.8}$$

$$\lambda T(x) \le T(\lambda x), \quad \forall \lambda \in \mathbb{R}. \tag{2.9}$$

Now, we will give the following well-known fact and we leave the details to the reader. Let T be a sublinear operator from a Banach space X into a (quasi-)Banach lattice Y. T is continuous if and only if

$$\exists C > 0 : \forall x \in X, \quad \|T(x)\| \le C\|x\|. \tag{2.10}$$

In this case, we also say that T is bounded and we put

$$\|T\| = \sup\{\|T(x)\| : \|x\|_{B_X} = 1\}. \tag{2.11}$$

We denote

$$SB(X,Y) = \{\text{bounded sublinear operators } T : X \longrightarrow Y\} \tag{2.12}$$

and

$$B(X,Y) = \{\text{bounded linear operators } u : X \longrightarrow Y\}. \tag{2.13}$$

We will need the following remark.

Remark 2.2. Let X be an arbitrary Banach space. Let Y, Z be (quasi-)Banach lattices.
(i) Consider T in $SL(X,Y)$ and u in $L(Y,Z)$. Assume that u is positive. Then, $u \circ T \in SL(X,Z)$.
(ii) Consider u in $L(X,Y)$ and T in $SL(Y,Z)$. Then, $T \circ u \in SL(X,Z)$.

The following proposition will be useful in the sequel for the proof of Theorem 3.6.

Proposition 2.3. *Let X be a Banach space and let Y be a complete (quasi-)Banach lattice. Let $T \in SL(X,Y)$. Then, for all x in X there is $u_x \in \nabla T$ such that, $T(x) = u_x(x)$, (i.e., the supremum is attained, $T(x) = \sup\{u(x) : u \in \nabla T\}$).*

Proof. Let x be in X. Consider $v_x : \mathbb{R} \cdot x \to Y$ defined by $v_x(\lambda x) = \lambda T(x)$, for all λ in $\mathbb{R}$. We have by (2.9), $v_x \le T$ on $\mathbb{R} \cdot x$. By Hahn-Banach theorem applied to sublinear operators, see, for example, [16, page 244], there is a linear extension of v_x noted u_x such that $u_x(\lambda x) = v_x(\lambda x)$, for all λ in $\mathbb{R}$ and $u_x(y) \le T(y)$, for all y in Y and the proof is completed. □

As an immediate consequence of Proposition 2.3, we have the following corollaries.

Corollary 2.4. *In the same conditions of the above proposition, it holds that*

(i) $\forall x \in X$, $\|T(x)\| \le \sup_{u\in\nabla T}\|u(x)\| \le \|T(x)\| + \|T(-x)\|$;

(ii) $\|T\| \le \sup_{u\in\nabla T}\|u\| \le 2\|T\|$.

Corollary 2.5. *Let $T : X \to Y$ be a sublinear operator between a Banach space X and a complete (quasi-)Banach lattice Y. Then, the following properties are equivalent:*

(i) *T is bounded;*

(ii) *$\forall u \in \nabla T$, u is bounded.*

3. Our main result. Let $1 \le p < q \le \infty$. Let X be a Banach space and let (Ω,μ) be a measure space. Consider T in $SB(X, L_p(\Omega,\mu))$. Let C be a positive constant. Suppose that T admits a factorization of the form

$$X \xrightarrow{\tilde{T}} L_q(\Omega,\mu) \xrightarrow{M_{g_T}} L_p(\Omega,\mu), \tag{3.1}$$

where $\tilde{T}$ is a bounded sublinear operator $\|\tilde{T}\| \le C$, M_{g_T} is the bounded operator of multiplication by g_T which is in $B_{L_r^+(\Omega,\mu)}$ ($1/p = 1/q + 1/r$, $B_{L_r^+(\Omega,\mu)} = \{g \in L_r^+(\Omega,\mu) : \|g\| \le 1\}$) and $T = M_{g_T} \circ \tilde{T}$. Our main objective in this section is to prove the converse (i.e., to generalize Corollary 2.5 for $p \ne \infty$ and $Y = L_p(\Omega,\mu)$). If the constant of q-convexity of u, $C_{pq}(u) \le C$ for all u in ∇T (i.e., u admits a factorization of the form

$$X \xrightarrow{\tilde{u}} L_q(\Omega,\mu) \xrightarrow{M_{g_u}} L_p(\Omega,\mu) \tag{3.2}$$

with $\|\tilde{u}\| \le C$) then T factors through $L_q(\Omega,\mu)$ as above under the supposition that $\{g_u\}_{u\in\nabla T}$ is latticially bounded in $L_r^+(\Omega,\mu)$.

We start this section by recalling the definition of the q-convexity.

Definition 3.1. Let $0 < p < q \le \infty$. Let X be a Banach space and let (Ω,μ) be a measure space. A sublinear operator $T : X \to L_p(\Omega,\mu)$ is called q-convex if there is a constant C such that, for all finite sequences $\{x_i\}_{1\le i\le n}$ in X, it holds that

$$\begin{aligned} \left\| \left(\sum_{i=1}^{n} |T(x_i)|^q \right)^{1/q} \right\|_{L_p} &\le C \left(\sum_{i=1}^{n} \|x_i\|_X^q \right)^{1/q} \quad \text{if } 1 \le q < \infty, \\ \left\| \sup_{1\le i\le n} |T(x_i)| \right\|_{L_p} &\le C \sup_{1\le i\le n} \|x_i\|_X \quad \text{if } q = \infty. \end{aligned} \tag{3.3}$$

The smallest constant C for which the above inequality holds is denoted by $C_{pq}(T)$.

Any sublinear q-convex operator is bounded and $\|T\| \le C_{pq}(T)$. We now formulate the sublinear version of the Maurey's theorem of factorization. For the ease of the reader and the coherence of this paper we give the proof which is a simple application of a theorem due to Maurey [8, Theorem 2]. It was extended by Defant [3] to the case of homogeneous operators by another method.

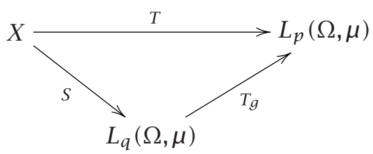
Theorem 3.2. *Let $p,q,r \in]0,+\infty]$ be such that $0 < p \le q \le +\infty$ and $1/p = 1/q + 1/r$. Let X be a Banach space and T be a sublinear operator from X into an $L_p(\Omega,\mu)$. Let C be a fixed constant. The following assertions are equivalent:*

(i) *the operator T is q-convex and $C_{pq}(T) \le C$;*

(ii) *there is a function g in $B^+_{L_r(\Omega,\mu)}$ such that for all x in X, it holds that*

$$\left(\int_\Omega \left|\frac{T(x)}{g}\right|^q d\mu\right)^{1/q} \le C\|x\|_X; \tag{3.4}$$

(iii) *there is a function g in $B^+_{L_r(\Omega,\mu)}$ and a bounded sublinear operator S from X into $L_q(\Omega,\mu)$, such that $\|S\| \le C$ and $T = T_g \circ S$*



(3.5)

Proof. (i)⇒(ii). This is elementary. It suffices to take in [8, Theorem 2] the set $\{\alpha_i = \|x_i\|_X\}_{i\in I} \in \mathbb{R}^I$, where $I = \{1,\dots,n\}$ and $f_i = T(x_i/\|x_i\|_X)$. In fact we have

$$\begin{aligned}
&\left(\int_\Omega \left[\sum_{i=1}^n |T(x_i)|^q\right]^{p/q} d\mu(\omega)\right)^{1/p} \\
&\quad = \left(\int_\Omega \left[\sum_{i=1}^n \left|\|x_i\|_X T\left(\frac{x_i}{\|x_i\|}\right)\right|^q\right]^{p/q} d\mu(\omega)\right)^{1/p} \\
&\quad \le C\left[\sum_{i=1}^n \|x_i\|_X^q\right]^{1/q}.
\end{aligned} \tag{3.6}$$

Thus there is by [8, Theorem 2] a function g in $B^+_{L_r(\Omega,\mu)}$ such that

$$\forall i \in I \quad \left(\int_\Omega \left|\frac{T(x_i/\|x_i\|_X)}{g}\right|^q d\mu(\omega)\right)^{1/q} \le C \tag{3.7}$$

this implies

$$\forall i \in I \quad \left(\int_\Omega \left|\frac{T(x_i)}{g}\right|^q d\mu(\omega)\right)^{1/q} \le C\|x_i\|_X. \tag{3.8}$$

As a consequence of the previous inequality, we have for all x in X

$$\left(\int_\Omega \left|\frac{T(x)}{g}\right|^q d\mu(\omega)\right)^{1/q} \le C\|x\|_X. \tag{3.9}$$

(ii)⇒(iii). We define S by $S(x) = T(x)/g$. S is a sublinear operator and clearly we have $T = T_g \circ S$ (i.e., the diagram is commutative by Remark 2.2(i)) and $\|S\| = \|T/g\| \le C$.

(iii)⇒(i). The Hölder inequality implies, for $1/p = 1/q + 1/r$,

$$\begin{aligned}
\int_\Omega \left(\sum_{i=1}^n |T(x_i)|^q\right)^{p/q} d\mu &= \left[\int_\Omega \left(\sum_{i=1}^n |g|^q \left|\frac{T(x_i)}{g}\right|^q\right)^{p/q} d\mu(\omega)\right] \\
&\le \left(\int_\Omega \sum_{i=1}^n \left|\frac{T(x_i)}{g}\right|^q d\mu(\omega)\right)^{p/q} \left(\int_\Omega \left(|g|^p\right)^{r/p} d\mu(\omega)\right)^{p/r} \\
&\le \left(\sum_{i=1}^n \int_\Omega \left|\frac{T(x_i)}{g}\right|^q d\mu(\omega)\right)^{p/q} \left(\int_\Omega |g|^r d\mu(\omega)\right)^{p/r} \\
&\le C^p \left(\sum_{i=1}^n ||x_i||_X^q\right)^{p/q}.
\end{aligned} \tag{3.10}$$

This proves that (iii)⇒(i). □

Proposition 3.3. *Given two sublinear operators T_1, T_2 from a Banach space X into a (quasi-)Banach lattice Y. If $T_1 \le T_2$ (in the sense of (2.6)). Then,*
(i) $|T_1(x)| \le \sup\{|T_2(x)|, |T_2(-x)|\}$;
(ii) $\|T_1(x)\| \le C_Y(\|T_2(x)\| + \|T_2(-x)\|)$.

Proof. (i). For all x in X, we have by (2.9)

$$T_1(x) \le T_2(x) \Longleftrightarrow -T_1(x) \le T_1(-x) \le T_2(-x), \tag{3.11}$$

this yields

$$|T_1(x)| \le \sup\{T_2(x), T_2(-x)\} \le \sup\{|T_2(x)|, |T_2(-x)|\}. \tag{3.12}$$

Concerning part (ii), we have from (i)

$$|T_1(x)| \le |T_2(x)| + |T_2(-x)|, \tag{3.13}$$

hence

$$||T_1(x)|| \le || |T_2(x)| + |T_2(-x)| || \le C_Y(||T_2(x)|| + ||T_2(-x)||), \tag{3.14}$$

(C_Y will appear if Y is a quasi-Banach lattice) and we obtain the announced result. □

Proposition 3.4. *Let $0 < p \le q \le \infty$ and $1/r = 1/p - 1/q$. Let X be a Banach space and T_1, T_2 be bounded sublinear operators from X into $L_p(\Omega,\mu)$ such that $T_1 \le T_2$. If T_2 factors through $L_q(\Omega,\mu)$, then T_1 factors through $L_q(\Omega,\mu)$.*

Proof. Assume that T_2 factors through $L_q(\Omega,\mu)$. From Theorem 3.2, there is a finite positive constant C_{pq} such that for all finite sequences $(x_i)_{1\le i\le n}$ in X, we have

$$\left(\int_\Omega \left[\sum_{i=1}^n |T_2(x_i)|^q\right]^{p/q} d\mu(w)\right)^{1/p} \le C_{pq}\left[\sum_{i=1}^n ||x_i||_X^q\right]^{1/q}. \tag{3.15}$$

By (3.13), we have for all x in X

$$|T_1(x)| \leq |T_2(x)| + |T_2(-x)|. \tag{3.16}$$

Then we clearly have

$$\left(\sum_{i=1}^{n}|T_1(x_i)|^q\right)^{1/q} \leq C_q\left(\sum_{i=1}^{n}\left(|T_2(x_i)|^q\right)^{1/q} + \left(\sum_{i=1}^{n}|T_2(-x_i)|^q\right)^{1/q}\right), \tag{3.17}$$

and therefore,

$$\left\|\left(\sum_{i=1}^{n}|T_1(x_i)|^q\right)^{1/q}\right\|_p \leq C_qC_p\left(\left\|\left(\sum_{i=1}^{n}|T_2(x_i)|^q\right)^{1/q}\right\|_p + \left\|\left(\sum_{i=1}^{n}|T_2(-x_i)|^q\right)^{1/q}\right\|_p\right). \tag{3.18}$$

Finally, for all finite sequences $(x_i)_{1\leq i\leq n}$ in X, we have

$$\left(\int_\Omega\left(\sum_{i=1}^{n}|T_1(x_i)|^q\right)^{p/q}d\mu\right)^{1/p} \leq C_1\left[\sum_{i=1}^{n}\|x_i\|_X^q\right]^{1/q}, \tag{3.19}$$

where $C_1 = 2C_{pq}(T_2)C_qC_p$ is an absolute constant depending only on p and q ($C_p = C_{L_p} = 2^{1/p-1}$ and $C_q = C_{L_q} = 2^{1/q-1}$ if $p,q < 1$). By Theorem 3.2, we deduce that T_1 factors by $L_q(\Omega,\mu)$ and this concludes the proof. □

Corollary 3.5. *Let p,q,r be the same as in Proposition 3.4 above. Let X be a Banach space, (Ω,μ) be any measure space and T in $SB(X,L_p(\Omega,\mu))$. If T factors by $L_q(\Omega,\mu)$, then for all u in ∇T, u factors by $L_q(\Omega,\mu)$.*

Let p, q, r be such that $0 < p \leq q \leq \infty$ and $1/p = 1/q + 1/r$. Let X be a Banach space, (Ω,μ) be an arbitrary measure space and $T: X \to L_p(\Omega,\mu)$ be a bounded sublinear operator. Assume that u factors by $L_q(\Omega,\mu)$ in the sense of Theorem 3.2 for all u in ∇T. Is it true that T factors by $L_q(\Omega,\mu)$? In other words, is it true the converse of Corollary 3.5 holds? In the next theorem, we give an answer with the supposition that $\{g_u\}_{u\in\nabla T}$ is latticially bounded. It is the main result of this paper.

Theorem 3.6. *Let T be a sublinear operator from a Banach space X into an $L_p(\Omega,\mu)$ (which is a complete (quasi-)Banach lattice). Assume that there is a positive constant C such that for all u in ∇T, $C_{pq}(u) \leq C$ and $\{g_u\}_{u\in\nabla T}$ is latticially bounded in $L_r(\Omega,\mu)$ (i.e., $\exists g_0 \in L_r^+(\Omega,\mu) : \forall u \in \nabla T$, $g_u \leq g_0$). Then, T factors through $L_q(\Omega,\mu)$ (as in Theorem 3.2).*

Proof. We put $\widetilde{T}(x) = T(x)/g$, where $g = g_0/\|g_0\|$. Then g_0 is in $L_r^+(\Omega,\mu)$ and by Proposition 2.3, we have

$$\|\widetilde{T}(x)\| = \left\|\frac{T(x)}{g}\right\| = \left\|\frac{u_x(x)}{g}\right\| \leq \|g_0\|\left\|\frac{u_x(x)}{g_0}\right\| \leq \|g_0\|\left\|\frac{u_x(x)}{g_{u_x}}\right\| \leq C\|g_0\| \tag{3.20}$$

and this completes the proof. □

Remark 3.7. Without the additional condition the last theorem does not hold in general as shown by this counterexample (communicated by Gilles Godefroy, 2002). We take as measure space (Ω,μ) the torus $\mathbb{T}=\mathbb{R}/2\pi\mathbb{Z}$, equipped with the invariant measure $d\theta$. Let X be the Hilbert space $H=L_2(\Omega,\mu)=L_2(\mathbb{T})$.

For all r such that $0<r\le\pi$ and for all $f\in L_2(\mathbb{T})$, we define a function 2π-periodic $S_rf\ge 0$ by

$$\forall x\in\mathbb{R},\quad S_rf(x)=\frac{1}{2r}\int_{x-r}^{x+r}|f(y)|^2dy. \tag{3.21}$$

We put $T_rf=\sqrt{S_rf}$. For all x, the expression $(T_rf)(x)$ is the L_2-norm of the function $\mathbf{1}_{(x-r,x+r)}f$, hence the operator T_r is sublinear and the operator T defined by $Tf=\sup\{T_rf:0<r<\pi\}$, is also sublinear from $L_2(\Omega,\mu)$ into $L_1(\Omega,\mu)$. Tf is the square root of the maximal function Mf^2 (the Hardy-Littlewood maximal operator) of the function $f^2\in L_1$, we know that Mf^2 is in weak-L_1, therefore Tf is in weak-L_2. Consider n in $\mathbb{N}$. We can partition $\mathbb{T}$ in n-intervals with the same length and we take $x_1,\dots,x_n$ in $H=L_2(\Omega,\mu)$, the characteristic functions. We have $\|x_i\|^2=2\pi/n$, for all $i=1,\dots,n$, but every function $T(x_i)$ worth at least $C/\sqrt{1+(i-j)}$ on the support of x_i, for all $j=1,\dots,n$ with $C=(4\pi)^{-1}$, hence it results that

$$\int_\Omega\left(\sum_{i=1}^{n}|T(x_i)(\omega)|^2\right)^{1/2}d\mu(\omega)\ge C\sqrt{1+\frac{1}{2}+\cdots+\frac{1}{n}}\ge C\sqrt{\log n} \tag{3.22}$$

and $\sum_{i=1}^{n}\|x_i\|^2=2\pi$.

But we have, from the "little Grothendieck's Theorem, G.T. in short" (see [14, Theorem 5.4(a)]), for all u in ∇T,

$$\begin{aligned}\int_\Omega&\left(\sum_{i=1}^{n}|u(x_i)(\omega)|^2\right)^{1/2}d\mu(\omega)\\&\le\sqrt{\frac{\pi}{2}}\|u\|\left(\sum_{i=1}^{n}\|x_i\|^2\right)^{1/2}\le 2\sqrt{\frac{\pi}{2}}\|T\|\left(\sum_{i=1}^{n}\|x_i\|^2\right)^{1/2}\quad\text{(by Corollary 2.4(ii))}.\end{aligned} \tag{3.23}$$

In conclusion, we have $C_{pq}(u)\le 2\sqrt{\pi/2}\|T\|$ and $C_{pq}(T)=\infty$ for $p=1$ and $q=2$. We deduce that for all u in ∇T, u factors through $L_2(\Omega,\mu)$ but T cannot factors by $L_2(\Omega,\mu)$.

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