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COMMON FIXED POINT THEOREMS OF CONTRACTIVE-TYPE MAPPINGS

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Using the concept of D-metric we prove some common fixed point theorems for generalized contractive mappings on a complete D-metric space. Our results extend, improve, and unify results of Fisher and Ćirić.

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1. Introduction. The Banach contraction mapping principle is well known. There are many generalizations of that principle to single- and multivalued mappings (see [1, 4, 5, 10, 11, 12]). The study of maps satisfying some contractive conditions has been the center of rigorous research activity since such mappings have many applications (see [2, 3, 9, 13, 14, 15]).

In 1998, Ćirić [6] proved a common fixed point theorem for nonlinear mappings on a complete metric space: let (X,d) be a complete metric space and $S,T : X \to X$ self-maps such that $d(STx,TSy) \le \max\{\varphi_1[(1/2)(d(x,Sy)+d(y,Tx))],\varphi_2[d(x,Tx)], \varphi_3[d(y,Sy)],\varphi_4[d(x,y)]\}$ for all x,y in X, where $\varphi_i \in \Phi$ $(i = 1,2,3,4)$. If S or T is continuous, then S and T have a unique common fixed point. This result improved and extended a theorem of Fisher [8].

In this paper, using the concept of D-metric, we prove common fixed point theorems which extend, improve, and unify the corresponding theorems of Fisher [8] and Ćirić [6].

Throughout the paper, by Φ we denote the collection of functions $\varphi : [0,\infty) \to [0,\infty)$ which are continuous from the right, nondecreasing, and which satisfy the condition $\varphi(t) < t$ for all $t > 0$. We denote by $\mathbb{N}$ the set of all positive integers.

2. Preliminaries. Before proving the main theorem, we will introduce some definitions and lemmas.

Definition 2.1 [7]. Let X be any nonempty set. A D-metric for X is a function $D : X \times X \times X \to R$ such that

(1) $D(x,y,z) \ge 0$ for all $x,y,z \in X$ and equality holds if and only if $x = y = z$,
(2) $D(x,y,z) = D(x,z,y) = D(y,x,z) = D(y,z,x) = D(z,x,y) = D(z,y,x)$ for all $x,y,z \in X$,
(3) $D(x,y,z) \le D(x,y,a)+D(x,a,z)+D(a,y,z)$ for all $x,y,z \in X$.

If D is a D-metric for X, then the ordered pair (X,D) is called a D-metric space or the set X, together with a D-metric, is called a D-metric space. We note that to a given ordinary metric space (X,d) there corresponds a D-metric space (X,D), but the converse may not be true (see Example 3.3). In this sense the D-metric spaces are the generalizations of the ordinary metric space.

Definition 2.2 [7]**.** A sequence $\{x_n\}$ of points of a D-metric space X converges to a point $x \in X$ if for an arbitrary $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that for all $n > m \geq n_0$, $D(x_m, x_n, x) < \varepsilon$.

Definition 2.3 [7]**.** A sequence $\{x_n\}$ of points of a D-metric space X is said to be a D-Cauchy sequence if for an arbitrary $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that for all $p > n > m \geq n_0$, $D(x_m, x_n, x_p) < \varepsilon$.

Definition 2.4 [7]**.** A D-metric space X is a complete D-metric space if every D-Cauchy sequence $\{x_n\}$ in X converges to a point x in X.

Definition 2.5. A real-valued function f defined on a metric space X is said to be lower semicontinuous at a point t in X if $\lim_{x \to t} \inf f(x) = \infty$ or $\lim_{x \to t} \inf f(x) \geq f(t)$.

Definition 2.6. A real-valued function f defined on a metric space X is said to be upper semicontinuous at a point t in X if $\lim_{x \to t} \sup f(x) = \infty$ or $\lim_{x \to t} \sup f(x) \leq f(t)$.

Definition 2.7. Let $x_0 \in X$ and $\varepsilon > 0$ be given. Then the open ball $B(x_0, \varepsilon)$ in X centered at x_0 of radius ε is defined by

$$B(x_0,\varepsilon) = \Big\{ y \in X \mid D(x_0,y,y) < \varepsilon \text{ if } y = x_0,\ \sup_{z \in X} D(x_0,y,z) < \varepsilon \text{ if } y \neq x_0 \Big\}. \tag{2.1}$$

Then the collection of all open balls $\{B(x,\varepsilon) : x \in X\}$ defines the topology on X denoted by τ.

Lemma 2.8 [7]**.** *The D-metric for X is a continuous function on $X \times X \times X$ in the topology τ on X.*

Lemma 2.9 [6]**.** *If $\varphi_1, \varphi_2 \in \Phi$, then there is some $\varphi \in \Phi$ such that* $\max\{\varphi_1(t), \varphi_2(t)\} \leq \varphi(t)$ *for all $t > 0$.*

Lemma 2.10. *Let (X,D) be a D-metric space. Let $g : X \times X \to X$ be a mapping and let $S, T : X \to X$ be mappings such that*

$$\begin{aligned} \max\{D(STx, TSy, g(STx, TSy)), D(TSy, STx, g(TSy, STx))\} \\ \leq \max\Big\{ \varphi_1\Big[\frac{1}{2}(D(x, Sy, g(x, Sy)) + D(y, Tx, g(y, Tx)))\Big], \\ \varphi_2[D(x, Tx, g(x, Tx))], \varphi_3[D(y, Sy, g(y, Sy))], \\ \varphi_4[D(x, y, g(x, y))] \Big\} \end{aligned} \tag{2.2}$$

for all $x, y \in X$, where $\varphi_i \in \Phi$ $(i = 1, 2, 3, 4)$,

$$x = y \Longrightarrow D(x, y, g(x, y)) = 0, \tag{2.3}$$

and

$$\begin{aligned}&\max\{D(x,z,g(x,z)),D(x,y,g(x,z)),D(y,z,g(x,z))\}\\&\quad\le D(x,y,g(x,y))+D(y,z,g(y,z))\end{aligned}\tag{2.4}$$

for all $x,y,z\in X$. The sequence $\{x_n\}$ is defined by $x_0\in X$, $x_{2n+1}=Tx_{2n}$, and $x_{2n+2}=Sx_{2n+1}$ for every $n\in\mathbb{N}\cup\{0\}$. Then

(I) *for an arbitrary $\varepsilon>0$, there exists a positive integer L such that $L\le n<m$ implies $\max\{D(x_n,x_m,g(x_n,x_m)),D(x_m,x_n,g(x_m,x_n))\}<\varepsilon$,*

(II) *a sequence $\{x_n\}_{n=0}^{\infty}$ is a D-Cauchy sequence.*

Proof. Let $M=\max\{D(x_0,x_1,g(x_0,x_1)),D(x_1,x_2,g(x_1,x_2)),D(x_2,x_1,g(x_2,x_1))\}$. Since all φ_i are nondecreasing functions by (2.2), (2.3), and (2.4),

$$\begin{aligned}&\max\{D(x_2,x_3,g(x_2,x_3)),D(x_3,x_2,g(x_3,x_2))\}\\&\quad=\max\{D(STx_0,TSx_1,g(STx_0,TSx_1)),D(TSx_1,STx_0,g(TSx_1,STx_0))\}\\&\quad\le\max\Big\{\varphi_1\Big[\frac{1}{2}\big(D(x_0,Sx_1,g(x_0,Sx_1))+D(x_1,Tx_0,g(x_1,Tx_0))\big)\Big],\\&\qquad\qquad\varphi_2[D(x_0,Tx_0,g(x_0,Tx_0))],\varphi_3[D(x_1,Sx_1,g(x_1,Sx_1))],\\&\qquad\qquad\varphi_4[D(x_0,x_1,g(x_0,x_1))]\Big\}\\&\quad\le\max\{\varphi_1(M),\varphi_2(M),\varphi_3(M),\varphi_4(M)\}\\&\quad\le\varphi(M),\end{aligned}\tag{2.5}$$

where $\varphi\in\Phi$. Such φ exists from an extended version of Lemma 2.9. Therefore, we have $\max\{D(x_2,x_3,g(x_2,x_3)),D(x_3,x_2,g(x_3,x_2))\}\le\varphi(M)$. Again, from (2.2), (2.3), and (2.4), we get

$$\begin{aligned}&\max\{D(x_3,x_4,g(x_3,x_4)),D(x_4,x_3,g(x_4,x_3))\}\\&\quad=\max\{D(TSx_1,STx_2,g(TSx_1,STx_2)),D(STx_2,TSx_1,g(STx_2,TSx_1))\}\\&\quad\le\max\Big\{\varphi_1\Big[\frac{1}{2}\big(D(x_2,Sx_1,g(x_2,Sx_1))+D(x_1,Tx_2,g(x_1,Tx_2))\big)\Big],\\&\qquad\qquad\varphi_2[D(x_2,Tx_2,g(x_2,Tx_2))],\varphi_3[D(x_1,Sx_1,g(x_1,Sx_1))],\\&\qquad\qquad\varphi_4[D(x_2,x_1,g(x_2,x_1))]\Big\}\\&\quad\le\max\{\varphi_1(M),\varphi_2[\varphi(M)],\varphi_3(M),\varphi_4(M)\}\\&\quad\le\varphi(M).\end{aligned}\tag{2.6}$$

Using the obtained relations $\max\{D(x_2,x_3,g(x_2,x_3)),D(x_3,x_2,g(x_3,x_2))\}\le\varphi(M)$ and $\max\{D(x_3,x_4,g(x_3,x_4)),D(x_4,x_3,g(x_4,x_3))\}\le\varphi(M)$, from (2.2), (2.3), and (2.4), we get

$$\begin{aligned}
&\max\{D(x_4,x_5,g(x_4,x_5)),D(x_5,x_4,g(x_5,x_4))\}\\
&\quad=\max\{D(STx_2,TSx_3,g(STx_2,TSx_3)),D(TSx_3,STx_2,g(TSx_3,STx_2))\}\\
&\quad\le\max\Big\{\varphi_1\Big[\frac{1}{2}(D(x_2,Sx_3,g(x_2,Sx_3))+D(x_3,Tx_2,g(x_3,Tx_2)))\Big],\\
&\qquad\quad\varphi_2[D(x_2,Tx_2,g(x_2,Tx_2))],\varphi_3[D(x_3,Sx_3,g(x_3,Sx_3))],\\
&\qquad\quad\varphi_4[D(x_2,x_3,g(x_2,x_3))]\Big\}\\
&\quad\le\max\{\varphi_1[\varphi(M)],\varphi_2[\varphi(M)],\varphi_3[\varphi(M)],\varphi_4[\varphi(M)]\}\\
&\quad\le\varphi^2(M).
\end{aligned}\tag{2.7}$$

Similarly, again from (2.2), (2.3), and (2.4), we get

$$\begin{aligned}
&\max\{D(x_5,x_6,g(x_5,x_6)),D(x_6,x_5,g(x_6,x_5))\}\\
&\quad=\max\{D(TSx_3,STx_4,g(TSx_3,STx_4)),D(STx_4,TSx_3,g(STx_4,TSx_3))\}\\
&\quad\le\max\Big\{\varphi_1\Big[\frac{1}{2}(D(x_4,Sx_3,g(x_4,Sx_3))+D(x_3,Tx_4,g(x_3,Tx_4)))\Big],\\
&\qquad\quad\varphi_2[D(x_4,Tx_4,g(x_4,Tx_4))],\varphi_3[D(x_3,Sx_3,g(x_3,Sx_3))],\\
&\qquad\quad\varphi_4[D(x_4,x_3,g(x_4,x_3))]\Big\}\\
&\quad\le\max\{\varphi_1[\varphi(M)],\varphi_2[\varphi^2(M)],\varphi_3[\varphi(M)],\varphi_4[\varphi(M)]\}\\
&\quad\le\varphi^2(M).
\end{aligned}\tag{2.8}$$

In general, by induction, we get

$$\max\{D(x_n,x_{n+1},g(x_n,x_{n+1})),D(x_{n+1},x_n,g(x_{n+1},x_n))\}\le\varphi^{[n/2]}(M)\tag{2.9}$$

for $n\ge 2$, where $[n/2]$ stands for the greatest integer not exceeding $n/2$. Since $\varphi\in\Phi$, by Singh and Meade [13, Lemma 1], it follows that $\varphi^n(M)\to 0$ as $n\to+\infty$ for every $M>0$. Thus, we obtain

$$\max\{D(x_n,x_{n+1},g(x_n,x_{n+1})),D(x_{n+1},x_n,g(x_{n+1},x_n))\}\longrightarrow 0\quad\text{as } n\longrightarrow\infty.\tag{2.10}$$

Suppose that (I) does not hold. Then there exists an $\varepsilon>0$ such that for each $i\in\mathbb{N}$, there exist positive integers n_i, m_i, with $i\le n_i<m_i$, satisfying

$$\begin{gathered}
\varepsilon\le\max\{D(x_{n_i},x_{m_i},g(x_{n_i},x_{m_i})),D(x_{m_i},x_{n_i},g(x_{m_i},x_{n_i}))\},\\
\max\{D(x_{n_i},x_{m_i-1},g(x_{n_i},x_{m_i-1})),D(x_{m_i-1},x_{n_i},g(x_{m_i-1},x_{n_i}))\}<\varepsilon\quad\text{for } i=1,2,\ldots.
\end{gathered}\tag{2.11}$$

Set

$$\begin{aligned}
\varepsilon_i&=\max\{D(x_{n_i},x_{m_i},g(x_{n_i},x_{m_i})),D(x_{m_i},x_{n_i},g(x_{m_i},x_{n_i}))\},\\
\rho_i&=\max\{D(x_i,x_{i+1},g(x_i,x_{i+1})),D(x_{i+1},x_i,g(x_{i+1},x_i))\}\quad\text{for } i=1,2,\ldots.
\end{aligned}\tag{2.12}$$

Then we have

$$\begin{aligned}\varepsilon &\le \varepsilon_i\\ &= \max\{D(x_{n_i},x_{m_i},g(x_{n_i},x_{m_i})),D(x_{m_i},x_{n_i},g(x_{m_i},x_{n_i}))\}\\ &\le \max\{D(x_{n_i},x_{m_i-1},g(x_{n_i},x_{m_i-1})),D(x_{m_i-1},x_{n_i},g(x_{m_i-1},x_{n_i}))\}\\ &\quad+\max\{D(x_{m_i-1},x_{m_i},g(x_{m_i-1},x_{m_i})),D(x_{m_i},x_{m_i-1},g(x_{m_i},x_{m_i-1}))\}\\ &< \varepsilon+\rho_{m_i-1},\quad i=1,2,\ldots.\end{aligned} \tag{2.13}$$

Taking the limit as $i\to+\infty$, we get $\lim\varepsilon_i=\varepsilon$. On the other hand, by (2.2), (2.3), and (2.4),

$$\begin{aligned}\varepsilon_i &= \max\{D(x_{n_i},x_{m_i},g(x_{n_i},x_{m_i})),D(x_{m_i},x_{n_i},g(x_{m_i},x_{n_i}))\}\\ &\le \max\{D(x_{n_i},x_{n_i+1},g(x_{n_i},x_{n_i+1})),D(x_{n_i+1},x_{n_i},g(x_{n_i+1},x_{n_i}))\}\\ &\quad+\max\{D(x_{n_i+1},x_{n_i+2},g(x_{n_i+1},x_{n_i+2})),D(x_{n_i+2},x_{n_i+1},g(x_{n_i+2},x_{n_i+1}))\}\\ &\quad+\max\{D(x_{n_i+2},x_{m_i+2},g(x_{n_i+2},x_{m_i+2})),D(x_{m_i+2},x_{n_i+2},g(x_{m_i+2},x_{n_i+2}))\}\\ &\quad+\max\{D(x_{m_i+2},x_{m_i+1},g(x_{m_i+2},x_{m_i+1})),D(x_{m_i+1},x_{m_i+2},g(x_{m_i+1},x_{m_i+2}))\}\\ &\quad+\max\{D(x_{m_i+1},x_{m_i},g(x_{m_i+1},x_{m_i})),D(x_{m_i},x_{m_i+1},g(x_{m_i},x_{m_i+1}))\}\\ &= \rho_{n_i}+\rho_{n_i+1}+\max\{D(x_{n_i+2},x_{m_i+2},g(x_{n_i+2},x_{m_i+2})),\\ &\qquad\qquad D(x_{m_i+2},x_{n_i+2},g(x_{m_i+2},x_{n_i+2}))\}\\ &\quad+\rho_{m_i+1}+\rho_{m_i}\quad\text{for } i=1,2,\ldots.\end{aligned} \tag{2.14}$$

We will now analyze the term $\max\{D(x_{n_i+2},x_{m_i+2},g(x_{n_i+2},x_{m_i+2})),D(x_{m_i+2},x_{n_i+2},g(x_{m_i+2},x_{n_i+2}))\}$ based on the parity of the subscripts.

Case 1. n_i+2 is even and m_i+2 is odd. From (2.2), (2.3), and (2.4), we have

$$\begin{aligned}&\max\{D(x_{n_i+2},x_{m_i+2},g(x_{n_i+2},x_{m_i+2})),D(x_{m_i+2},x_{n_i+2},g(x_{m_i+2},x_{n_i+2}))\}\\ &\quad= \max\{D(STx_{n_i},TSx_{m_i},g(STx_{n_i},TSx_{m_i})),D(TSx_{m_i},STx_{n_i},g(TSx_{m_i},STx_{n_i}))\}\\ &\quad\le \max\Big\{\varphi_1\Big[\frac{1}{2}(D(x_{n_i},Sx_{m_i},g(x_{n_i},Sx_{m_i}))+D(x_{m_i},Tx_{n_i},g(x_{m_i},Tx_{n_i})))\Big],\\ &\qquad\qquad \varphi_2[D(x_{n_i},Tx_{n_i},g(x_{n_i},Tx_{n_i}))],\varphi_3[D(x_{m_i},Sx_{m_i},g(x_{m_i},Sx_{m_i}))],\\ &\qquad\qquad \varphi_4[D(x_{n_i},x_{m_i},g(x_{n_i},x_{m_i}))]\Big\}\\ &\quad\le \max\Big\{\varphi_1\Big[\frac{1}{2}(\varepsilon_i+\rho_{m_i}+\varepsilon_i+\rho_{n_i})\Big],\varphi_2(\rho_{n_i}),\varphi_3(\rho_{m_i}),\varphi_4(\varepsilon_i)\Big\}\\ &\quad\le \varphi(\varepsilon_i+\rho_{m_i}+\rho_{n_i}).\end{aligned} \tag{2.15}$$

Therefore, we have

$$\max\{D(x_{n_i+2},x_{m_i+2},g(x_{n_i+2},x_{m_i+2})),D(x_{m_i+2},x_{n_i+2},g(x_{m_i+2},x_{n_i+2}))\}\le\varphi(k_i), \tag{2.16}$$

where $k_i = \varepsilon_i + \rho_{m_i} + \rho_{n_i}$. Substituting (2.16) into (2.14), taking the limit as $i \to +\infty$, and using the right continuity of φ, we get

$$\varepsilon = \lim_{i\to\infty} \varepsilon_i \le \lim_{k_i\to\varepsilon^+} \varphi(k_i) = \varphi(\varepsilon) < \varepsilon, \tag{2.17}$$

which is a contradiction.

Case 2. Both $n_i + 2$ and $m_i + 2$ are odd. Then, we have

$$\begin{aligned}
&\max\{D(x_{n_i+2}, x_{m_i+2}, g(x_{n_i+2}, x_{m_i+2})), D(x_{m_i+2}, x_{n_i+2}, g(x_{m_i+2}, x_{n_i+2}))\}\\
&\quad \le \max\{D(x_{n_i+2}, x_{n_i+1}, g(x_{n_i+2}, x_{n_i+1})), D(x_{n_i+1}, x_{n_i+2}, g(x_{n_i+1}, x_{n_i+2}))\}\\
&\qquad + \max\{D(x_{n_i+1}, x_{m_i+2}, g(x_{n_i+1}, x_{m_i+2})), D(x_{m_i+2}, x_{n_i+1}, g(x_{m_i+2}, x_{n_i+1}))\}\\
&\quad = \rho_{n_i+1} + \max\{D(x_{n_i+1}, x_{m_i+2}, g(x_{n_i+1}, x_{m_i+2})), D(x_{m_i+2}, x_{n_i+1}, g(x_{m_i+2}, x_{n_i+1}))\}.
\end{aligned} \tag{2.18}$$

Since $n_i + 1$ is even and $m_i + 2$ is odd, from Case 1, we have

$$\begin{aligned}
&\max\{D(x_{n_i+1}, x_{m_i+2}, g(x_{n_i+1}, x_{m_i+2})), D(x_{m_i+2}, x_{n_i+1}, g(x_{m_i+2}, x_{n_i+1}))\}\\
&\quad = \max\{D(STx_{n_i-1}, TSx_{m_i}, g(STx_{n_i-1}, TSx_{m_i})),\\
&\qquad\qquad D(TSx_{m_i}, STx_{n_i-1}, g(TSx_{m_i}, STx_{n_i-1}))\}\\
&\quad \le \max\Big\{\varphi_1\Big[\frac{1}{2}(D(x_{n_i-1}, Sx_{m_i}, g(x_{n_i-1}, Sx_{m_i})) + D(x_{m_i}, Tx_{n_i-1}, g(x_{m_i}, Tx_{n_i-1})))\Big],\\
&\qquad\qquad \varphi_2[D(x_{n_i-1}, Tx_{n_i-1}, g(x_{n_i-1}, Tx_{n_i-1}))], \varphi_3[D(x_{m_i}, Sx_{m_i}, g(x_{m_i}, Sx_{m_i}))],\\
&\qquad\qquad \varphi_4[D(x_{n_i-1}, x_{m_i}, g(x_{n_i-1}, x_{m_i}))]\Big\}\\
&\quad \le \max\Big\{\varphi_1\Big[\frac{1}{2}(\rho_{n_i-1} + \varepsilon_i + \rho_{m_i} + \varepsilon_i)\Big], \varphi_2(\rho_{n_i-1}), \varphi_3(\rho_{m_i}), \varphi_4(\rho_{n_i-1} + \varepsilon_i)\Big\}\\
&\quad \le \varphi(\varepsilon_i + \rho_{m_i} + \rho_{n_i-1}).
\end{aligned} \tag{2.19}$$

Therefore, we get

$$\max\{D(x_{n_i+1}, x_{m_i+2}, g(x_{n_i+1}, x_{m_i+2})), D(x_{m_i+2}, x_{n_i+1}, g(x_{m_i+2}, x_{n_i+1}))\} \le \varphi(l_i), \tag{2.20}$$

where $l_i = \varepsilon_i + \rho_{m_i} + \rho_{n_i-1}$. Hence, substituting (2.20) into (2.18), then putting (2.18) into (2.14), and taking the limit as $i \to +\infty$, we have

$$\varepsilon = \lim_{i\to\infty} \varepsilon_i \le \lim_{l_i\to\varepsilon^+} \varphi(l_i) = \varphi(\varepsilon) < \varepsilon, \tag{2.21}$$

which is a contradiction. In a similar manner, we get (2.17) and (2.21) for the cases in which $n_i + 2$ and $m_i + 2$ are both even, and $n_i + 2$ is odd and $m_i + 2$ is even. That is, all cases lead to a contradiction. Therefore (I) holds.

We claim that $\{x_n\}$ is D-Cauchy. Let n,m,p $(n<m<p)$ be any positive integers. Then, by Definition 2.1 and (2.4),

$$\begin{aligned} D(x_n,x_m,x_p) &\le D(x_n,x_m,g(x_n,x_m))+D(x_n,x_p,g(x_n,x_m))+D(x_m,x_p,g(x_n,x_m)) \\ &\le D(x_n,x_m,g(x_n,x_m))+2D(x_n,x_m,g(x_n,x_m))+2D(x_m,x_p,g(x_m,x_p)) \\ &= 3D(x_n,x_m,g(x_n,x_m))+2D(x_m,x_p,g(x_m,x_p)). \end{aligned} \tag{2.22}$$

Since $\lim_{n\to\infty} D(x_n,x_m,g(x_n,x_m))=0$, we have $\lim_{n\to\infty} D(x_n,x_m,x_p)=0$. Thus $\{x_n\}$ is a D-Cauchy sequence. □

3. Main results.

Now we will prove the following fixed point theorems for a complete D-metric space.

Theorem 3.1. *Let (X,D) be a complete D-metric space. Let $g: X\times X\to X$ be a function and let S and T be self-maps on X satisfying (2.2), (2.3), and (2.4) of Lemma 2.10. For any sequences $\{u_n\},\{v_n\}$ in X such that $\lim_{n\to\infty} u_n=\alpha$ and $\lim_{n\to\infty} v_n=\beta$, $\lim_{n\to\infty} D(u_n,v_n,g(u_n,v_n))=D(\alpha,\beta,g(\alpha,\beta))$ for some α,β in X.*

If S or T is continuous, then S and T have a unique common fixed point.

Proof. Let the sequence $\{x_n\}$ be defined by $x_0\in X$, $x_{2n+1}=Tx_{2n}$, and $x_{2n+2}=Sx_{2n+1}$ for every $n\in\mathbb{N}\cup\{0\}$. Then, by Lemma 2.10(II), it follows that $\{x_n\}$ is a D-Cauchy sequence. Since X is a complete D-metric space, $\{x_n\}$ is convergent to a limit u in X. Suppose that S is continuous. Then

$$u=\lim_{n\to\infty} x_{2n+2}=\lim_{n\to\infty} Sx_{2n+1}=S\Big(\lim_{n\to\infty} x_{2n+1}\Big)=Su. \tag{3.1}$$

This implies that u is a fixed point of S. From (2.2), (2.3), and (2.4), we get $D(u,Su,g(u,Su))=0$ and

$$\begin{aligned} D(u,Tu,g(u,Tu)) &= D(u,TSu,g(u,TSu)) \\ &\le D(u,x_{2n+2},g(u,x_{2n+2}))+D(STx_{2n},TSu,g(STx_{2n},TSu)) \\ &\le D(u,x_{2n+2},g(u,x_{2n+2})) \\ &\quad +\max\Big\{\varphi_1\Big[\frac{1}{2}\big(D(x_{2n},Su,g(x_{2n},Su))+D(u,Tx_{2n},g(u,Tx_{2n}))\big)\Big], \\ &\qquad \varphi_2[D(x_{2n},Tx_{2n},g(x_{2n},Tx_{2n}))],\varphi_3[D(u,Su,g(u,Su))], \\ &\qquad \varphi_4[D(x_{2n},u,g(x_{2n},u))]\Big\}. \end{aligned} \tag{3.2}$$

Taking the limit when n tends to infinity, by hypothesis, we get $D(u,Tu,g(u,Tu))=0$. Thus, we have $u=Su=Tu$. Therefore, u is the common fixed point of S and T. The proof for T continuous is similar.

We will now show that u is unique. Suppose that v is also a common fixed point of S and T. Then, from (2.2), (2.3), and (2.4),

$$\begin{aligned}
&\max\{D(u,v,g(u,v)),D(v,u,g(v,u))\}\\
&\quad=\max\{D(STu,TSv,g(STu,TSv)),D(TSv,STu,g(TSv,STu))\}\\
&\quad\le\max\Big\{\varphi_1\Big[\frac{1}{2}(D(u,Sv,g(u,Sv))+D(v,Tu,g(v,Tu)))\Big],\\
&\qquad\qquad\varphi_2[D(u,Tu,g(u,Tu))],\varphi_3[D(v,Sv,g(v,Sv))],\varphi_4[D(u,v,g(u,v))]\Big\}\\
&\quad=\max\Big\{\varphi_1\Big[\frac{1}{2}(D(u,v,g(u,v))+D(v,u,g(v,u)))\Big],\\
&\qquad\qquad\varphi_2[D(u,u,g(u,u))],\varphi_3[D(v,v,g(v,v))],\varphi_4[D(u,v,g(u,v))]\Big\}\\
&\quad\le\varphi(\max\{D(u,v,g(u,v)),D(v,u,g(v,u))\}).
\end{aligned}\tag{3.3}$$

We write $\max\{D(u,v,g(u,v)),D(v,u,g(v,u))\}\le\varphi(\max\{D(u,v,g(u,v)),D(v,u,g(v,u))\})$, which implies that $\max\{D(u,v,g(u,v)),D(v,u,g(v,u))\}=0$, that is, $u=v$. Therefore, the common fixed point of S and T is unique. □

Remark 3.2. Let X be a complete metric space with a metric d. If we take $D(x,y,z)=\max\{d(x,y),d(x,z),d(y,z)\}$ and $g(x,y)=x$ for all $x,y,z\in X$, then Theorem 3.1 is Ćirić's [6, Theorem 2] which has extended a theorem of Fisher [8].

The following example shows that a D-metric is a proper extension of a metric d.

Example 3.3. Let d be a metric on $\mathbb{R}$. Define the function $\varphi:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ by $\varphi(x,y)=(x-y)^2$ for all $x,y\in\mathbb{R}$. Then, clearly, φ is not metric since $\varphi(2,1/2)>\varphi(2,1)+\varphi(1,1/2)$. Let $G,H:\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be functions such that $G(x,y,z)=\max\{d(x,y),d(x,z),d(y,z)\}$ and $H(x,y,z)=\max\{\varphi(x,y),\varphi(x,z),\varphi(y,z)\}$ for all $x,y,z\in\mathbb{R}$. Then, clearly, G and H are D-metric for $\mathbb{R}$. But H is a D-metric that is a proper extension of the metric d. Therefore, a D-metric space is a proper extension of a metric space.

Corollary 3.4. *Let (X,D) be a complete D-metric space. Let $g:X\times X\to X$ be a function and let S and T be self-maps on X satisfying*

$$\begin{aligned}
&\max\{D(STx,TSy,g(STx,TSy)),D(TSy,STx,g(TSy,STx))\}\\
&\quad\le c\cdot\max\Big\{\frac{1}{2}[D(x,Sy,g(x,Sy))+D(y,Tx,g(y,Tx))],\\
&\qquad\qquad D(x,Tx,g(x,Tx)),D(y,Sy,g(y,Sy)),D(x,y,g(x,y))\Big\}
\end{aligned}\tag{3.4}$$

for all $x,y\in X$, where $x=y$ implies $D(x,y,g(x,y))=0$ and $\max\{D(x,z,g(x,z)),D(x,y,g(x,z)),D(y,z,g(x,z))\}\le D(x,y,g(x,y))+D(y,z,g(y,z))$ for all $x,y,z\in X$.

For any sequences $\{u_n\},\{v_n\}$ *in* X *such that* $\lim_{n\to\infty} u_n = \alpha$ *and* $\lim_{n\to\infty} v_n = \beta$, $\lim_{n\to\infty} D(u_n,v_n,g(u_n,v_n)) = D(\alpha,\beta,g(\alpha,\beta))$ *for some* α,β *in* X.

If S *or* T *is continuous, then* S *and* T *have a unique common fixed point.*

Proof. The proof follows by taking $\varphi_i(t) = c\cdot t$ with $0 < c < 1$ $(i = 1,2,3,4)$ in Theorem 3.1. □

We will prove the following corollary using another condition instead of continuity in Theorem 3.1.

Corollary 3.5. *Let* (X,D) *be a complete* D*-metric space. Let* $g : X\times X \to X$ *be a function, let* S *and* T *be self-maps on* X *satisfying (2.2), (2.3), and (2.4) of Lemma 2.10, and, for each* $u \in X$ *with* $u \neq Su$ *or* $u \neq Tu$, *let*

$$\inf\{D(x,u,g(x,u)) + D(x,Sx,g(x,Sx)) + D(y,Ty,g(y,Ty)) : x,y \in X\} > 0. \tag{3.5}$$

For any sequences $\{a_n\}$ *and* $\{b_n\}$ *in* X *such that* $\lim_{n\to\infty} a_n = u$ *and* $\lim_{n\to\infty} b_n = v$, *the following conditions hold:*

(1) $\lim_{n\to\infty} D(a_n,b_n,g(a_n,b_n)) = D(u,v,g(u,v))$,

(2) $\lim_{m\to\infty} D(a_n,b_m,g(a_n,b_m)) = D(a_n,v,g(a_n,v))$ *for each* $n \in \mathbb{N}$,

(3) $\lim_{m\to\infty} D(b_m,a_n,g(b_m,a_n)) = D(v,a_n,g(v,a_n))$ *for each* $n \in \mathbb{N}$.

Then S *and* T *have a unique common fixed point.*

Proof. From Lemma 2.10(I) and (II), the sequence $\{x_n\}$ defined by $x_0 \in X$, $x_{2n+1} = Tx_{2n}$, and $x_{2n+2} = Sx_{2n+1}$ for every $x \in \mathbb{N}\cup\{0\}$ is a D-Cauchy sequence. Since X is a complete D-metric space, there exists $u \in X$ such that $\{x_n\}$ converges to u. Then we have

$$\begin{aligned}
&D(x_{2n+1},x_{2m+2},g(x_{2n+1},x_{2m+2}))\\
&\quad= D(TSx_{2n-1},STx_{2m},g(TSx_{2n-1},STx_{2m}))\\
&\quad\le \max\Big\{\varphi_1\Big[\frac{1}{2}(D(x_{2m},Sx_{2n-1},g(x_{2m},Sx_{2n-1}))+D(x_{2n-1},Tx_{2m},g(x_{2n-1},Tx_{2m})))\Big],\\
&\qquad\qquad \varphi_2[D(x_{2m},Tx_{2m},g(x_{2m},Tx_{2m}))],\varphi_3[D(x_{2n-1},Sx_{2n-1},g(x_{2n-1},Sx_{2n-1}))],\\
&\qquad\qquad \varphi_4[D(x_{2m},x_{2n-1},g(x_{2m},x_{2n-1}))]\Big\}\\
&\quad\le \max\Big\{\varphi_1\Big[\frac{1}{2}(D(x_{2m},x_{2n},g(x_{2m},x_{2n}))+D(x_{2n-1},x_{2m+1},g(x_{2n-1},x_{2m+1})))\Big],\\
&\qquad\qquad \varphi_2[D(x_{2m},x_{2m+1},g(x_{2m},x_{2m+1}))],\varphi_3[D(x_{2n-1},x_{2n},g(x_{2n-1},x_{2n}))],\\
&\qquad\qquad \varphi_4[D(x_{2m},x_{2n-1},g(x_{2m},x_{2n-1}))]\Big\}.
\end{aligned} \tag{3.6}$$

Thus, we obtain $\lim_{n\to\infty} D(x_{2n+1},u,g(x_{2n+1},u)) = 0$. Assume that $u \neq Su$ or $u \neq Tu$.

Then, by hypothesis, we have

$$\begin{aligned}
0 &< \inf\{D(x,u,g(x,u))+D(x,Sx,g(x,Sx))+D(y,Ty,g(y,Ty)):x,y\in X\}\\
&\le \inf\{D(x_{2n+1},u,g(x_{2n+1},u))+D(x_{2n+1},Sx_{2n+1},g(x_{2n+1},Sx_{2n+1}))\\
&\qquad +D(x_{2n+2},Tx_{2n+2},g(x_{2n+2},Tx_{2n+2})):n\in\mathbb{N}\}\\
&= \inf\{D(x_{2n+1},u,g(x_{2n+1},u))+D(x_{2n+1},x_{2n+2},g(x_{2n+1},x_{2n+2}))\\
&\qquad +D(x_{2n+2},x_{2n+3},g(x_{2n+2},x_{2n+3})):n\in\mathbb{N}\}\\
&= 0.
\end{aligned} \tag{3.7}$$

This is a contradiction. Therefore, we have $u = Su = Tu$.

On the other hand, we can prove the existence of a unique common fixed point of S and T by a method similar to that of Theorem 3.1. □

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