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STAGNATION-POINT FLOW OF THE WALTERS' B' FLUID WITH SLIP

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The steady two-dimensional stagnation point flow of a non-Newtonian Walters' B' fluid with slip is studied. The fluid impinges on the wall either orthogonally or obliquely. A finite difference technique is employed to obtain solutions.

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1. Introduction. Some rheologically complex fluids such as polymer solutions, blood, paints, and certain oils cannot be adequately described by the Navier-Stokes theory. For this reason, several theories of non-Newtonian fluids were developed. One important and useful model which has been used to describe the non-Newtonian behavior exhibited by certain fluids is the Walters' B' fluid [16]. The equations of motion of non-Newtonian fluids are highly nonlinear and one order higher than the Navier-Stokes equations. Due to the complexity of these equations, finding accurate solutions is not easy.

One class of flows which has received considerable attention is stagnation-point flow. In a stagnation-point flow of a Newtonian fluid, a rigid wall occupies the entire x-axis, the fluid domain is $y > 0$, and the flow impinges on the wall either orthogonally [6, 7] or obliquely [4, 14, 15]. In a study of Newtonian fluid impinging on a flat rigid wall obliquely, Dorrepaal [4] found that the slope of the dividing streamline at the wall divided by its slope at infinity is independent of the angle of incidence. Beard and Walters [2] used boundary-layer equations to study two-dimensional flow near a stagnation point of a viscoelastic fluid. Rajagopal et al. [11] have studied the Falkner-Skan flows of an incompressible second grade fluid. Dorrepaal et al. [5] investigated the behavior of a viscoelastic fluid impinging on a flat rigid wall at an arbitrary angle of incidence. Labropulu et al. [9] studied the oblique flow of a second grade fluid impinging on a porous wall with suction or blowing.

In a recent paper, Wang [17] studied stagnation-point flows with slip. This problem appears in some applications where a thin film of light oil is attached to the plate or when the plate is coated with special coatings such as a thick monolayer of hydrophobic octadecyltrichlorosilane [3]. Also, wall slip can occur if the working fluid contains concentrated suspensions [13].

When the molecular mean free path length of the fluid is comparable to the system's characteristic length, then rarefaction effects must be considered. The Knudsen number K_n, defined as the ratio of the molecular mean free path to the characteristic length of the system, is the parameter used to classify fluids that deviate from continuum behavior. If $K_n > 10$, it is free molecular flow, if $0.1 < K_n < 10$, it is transition flow, if $0.01 < K_n < 0.1$, it is slip flow, and if $K_n < 0.01$, it is viscous flow (see Wang [17], Kogan [8]). Flows in the slip-flow region have been modeled using the Navier-Stokes equations and the traditional nonslip condition is replaced by the slip condition

$$u_t = A_p \frac{\partial u_t}{\partial n}, \tag{1.1}$$

where u_t is the tangential velocity component, n is normal to the plate, and A_p is a coefficient close to $2(\text{mean free path})/\sqrt{\pi}$ (see Sharipov and Seleznev [12]). This condition was first proposed by Navier [10] nearly two hundred years ago.

In the present study, we follow Wang [17] and investigate the behavior of the Walters' B' fluid impinging on a rigid wall with slip. The fluid impinges on the wall either orthogonally or obliquely. In particular, we study the effects of the slip condition and the effects of viscoelasticity of the fluid.

2. Flow equations. The two-dimensional flow of a viscous incompressible non-Newtonian Walters' B' fluid, neglecting thermal effects and body forces, is governed by (see Beard and Walters [2]):

$$\begin{aligned}
&\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0,\\
&u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} + \frac{1}{\rho}\frac{\partial p^*}{\partial x^*}\\
&\quad = \nu \nabla^{*2} u^* - \frac{\alpha}{\rho}\left\{\left(u^*\frac{\partial}{\partial x^*} + v^*\frac{\partial}{\partial y^*}\right)\nabla^2 u^* - \frac{\partial u^*}{\partial x^*}\nabla^2 u^* - \frac{\partial u^*}{\partial y^*}\nabla^2 v^*\right.\\
&\qquad \left. - 2\left[\frac{\partial u^*}{\partial x^*}\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial v^*}{\partial y^*}\frac{\partial^2 u^*}{\partial y^{*2}} + \left(\frac{\partial u^*}{\partial y^*} + \frac{\partial v^*}{\partial x^*}\right)\frac{\partial^2 u^*}{\partial x^* \partial y^*}\right]\right\}, \qquad (2.1)\\
&u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} + \frac{1}{\rho}\frac{\partial p^*}{\partial y^*}\\
&\quad = \nu \nabla^{*2} v^* - \frac{\alpha}{\rho}\left\{\left(u^*\frac{\partial}{\partial x^*} + v^*\frac{\partial}{\partial y^*}\right)\nabla^2 v^* - \frac{\partial v^*}{\partial x^*}\nabla^2 u^* - \frac{\partial v^*}{\partial y^*}\nabla^2 v^*\right.\\
&\qquad \left. - 2\left[\frac{\partial u^*}{\partial x^*}\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial v^*}{\partial y^*}\frac{\partial^2 v^*}{\partial y^{*2}} + \left(\frac{\partial u^*}{\partial y^*} + \frac{\partial v^*}{\partial x^*}\right)\frac{\partial^2 v^*}{\partial x^* \partial y^*}\right]\right\},
\end{aligned}$$

where $u^* = u^*(x^*, y^*)$, $v^* = v^*(x^*, y^*)$ are the velocity components, $p^* = p^*(x^*, y^*)$ is the pressure, $\nu = \mu/\rho$ is the kinematic viscosity, and α is the viscoelasticity of the fluid. The star on a variable indicates its dimensional form. We nondimensionalize the above equations according to

$$\begin{gathered}
x = x^*\sqrt{\frac{\beta}{\nu}}, \qquad y = y^*\sqrt{\frac{\beta}{\nu}},\\
u = \frac{1}{\sqrt{\nu\beta}}u^*, \qquad v = \frac{1}{\sqrt{\nu\beta}}v^*, \qquad p = \frac{1}{\rho\nu\beta}p^*,
\end{gathered} \tag{2.2}$$

where β has the units of inverse time. The flow equations in nondimensional form are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2.3}$$

$$\begin{aligned} u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{\partial p}{\partial x} = \nabla^2 u - W_e\Big\{\Big(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\Big)\nabla^2 u-\frac{\partial u}{\partial x}\nabla^2 u-\frac{\partial u}{\partial y}\nabla^2 v \\ -2\Big[\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+\frac{\partial v}{\partial y}\frac{\partial^2 u}{\partial y^2}+\Big(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\Big)\frac{\partial^2 u}{\partial x\partial y}\Big]\Big\}, \end{aligned} \tag{2.4}$$

$$\begin{aligned} u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\frac{\partial p}{\partial y} = \nabla^2 v - W_e\Big\{\Big(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\Big)\nabla^2 v-\frac{\partial v}{\partial x}\nabla^2 u-\frac{\partial v}{\partial y}\nabla^2 v \\ -2\Big[\frac{\partial u}{\partial x}\frac{\partial^2 v}{\partial x^2}+\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial y^2}+\Big(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\Big)\frac{\partial^2 v}{\partial x\partial y}\Big]\Big\}, \end{aligned} \tag{2.5}$$

where W_e is the Weissenberg number.

Continuity equation (2.3) implies the existence of a streamfunction $\psi(x,y)$ such that

$$u=\frac{\partial \psi}{\partial y}, \qquad v=-\frac{\partial \psi}{\partial x}. \tag{2.6}$$

Substitution of (2.6) in (2.4) and (2.5) and elimination of pressure from the resulting equations using $p_{xy}=p_{yx}$ yields

$$\frac{\partial(\psi,\nabla^2\psi)}{\partial(x,y)}+W_e\frac{\partial(\psi,\nabla^4\psi)}{\partial(x,y)}+\nabla^4\psi=0. \tag{2.7}$$

Having obtained a solution of (2.7), the velocity components are given by (2.6) and the pressure can be found by integrating (2.4) and (2.5).

The shear stress component τ_{12} is given by

$$\begin{aligned} \tau_{12}=\mu\beta\Big\{\Big[\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial x^2}\Big]-W_e\Big[\frac{\partial\psi}{\partial y}\Big(\frac{\partial^3\psi}{\partial x\partial y^2}-\frac{\partial^3\psi}{\partial x^3}\Big)-\frac{\partial\psi}{\partial x}\Big(\frac{\partial^3\psi}{\partial y^3}-\frac{\partial^3\psi}{\partial x^2\partial y}\Big) \\ +2\frac{\partial^2\psi}{\partial x\partial y}\frac{\partial^2\psi}{\partial y^2}+2\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\psi}{\partial x\partial y}\Big]\Big\}. \end{aligned} \tag{2.8}$$

3. Orthogonal flow. We assume that the infinite plate is at $y=0$ and that the fluid occupies the entire upper half-plane $y>0$. Furthermore, we assume that the streamfunction far from the wall is given by $\psi=xy$ (see Hiemenz [7]). Thus, the nondimensional boundary conditions are given by

$$\frac{\partial\psi}{\partial x}=0 \quad \text{at } y=0, \qquad \psi(x,y)\sim y \quad \text{as } y\longrightarrow\infty. \tag{3.1}$$

The slip condition in (1.1) is

$$\frac{\partial\psi}{\partial y}=\gamma\frac{\partial^2\psi}{\partial y^2}, \tag{3.2}$$

where $\gamma=A_p\sqrt{\beta}\nu$ is a parameter representing the slip to viscous effects.

TABLE 3.1. Numerical values of $F''(0)$ for various values of W_e and γ.

γ	W_e 0	0.1	0.2	0.3
0	1.23259	1.36954	1.58730	2.11092
0.2	1.04258	1.14323	1.29803	1.63401
0.4	0.88634	0.95916	1.06657	1.27238
0.6	0.76428	0.81807	0.89459	1.02828
0.8	0.66896	0.70984	0.76634	0.85879
1	0.59346	0.62537	0.66850	0.73581
2	0.37588	0.38834	0.40415	0.42609
5	0.17726	0.17995	0.18319	0.18731
10	0.09402	0.094776	0.09565	0.09674
20	0.04847	0.04866	0.04889	0.04917

Following Wang [17], we assume that

$$\psi = xF(y). \tag{3.3}$$

Using (3.3) in (2.7) and the boundary conditions (3.1) and (3.2), we obtain

$$F^{(iv)} + FF''' - F'F'' + W_e(FF^{(v)} - F'F^{(iv)}) = 0, \tag{3.4}$$

$$F(0) = 0, \qquad F'(0) = \gamma F''(0), \qquad F'(\infty) = 0, \tag{3.5}$$

where the prime denotes differentiation with respect to y. Integration of (3.4) once with respect to y and use of the condition at infinity yields

$$\begin{gathered} F''' + FF'' - F'^2 + W_e(FF^{(iv)} - 2F'F''' + F''^2) = -1, \\ F(0) = 0, \qquad F'(0) = \gamma F''(0), \qquad F'(\infty) = 0. \end{gathered} \tag{3.6}$$

The above system with $\gamma = 0$ has been solved by many authors for various values of W_e (see Beard and Walters [2], Ariel [1]). When $W_e = 0$, the above system has been solved numerically by Wang [17] for various values of γ. Using the shooting method with the finite difference technique described by Ariel [1], we find that $F''(0) = 1.23259$ when $W_e = 0$ and $\gamma = 0$. Numerical values of $F''(0)$ for different values of W_e and γ are shown in Table 3.1. Figure 3.1 shows the profiles of F' for $\gamma = 0$ and various values of W_e. Figure 3.2 depicts the profiles of F' for $\gamma = 1$ and various values of W_e. Figure 3.3 shows the profiles of F' for $W_e = 0.2$ and various values of γ. Figure 3.4 depicts the profiles of F for $W_e = 0.2$ and various values of γ. We observe that as the elasticity of the fluid increases, the velocity near the wall increases and as the slip parameter γ increases the velocity near the wall increases as well.

For large y, we find that

$$F \approx y + C, \tag{3.7}$$

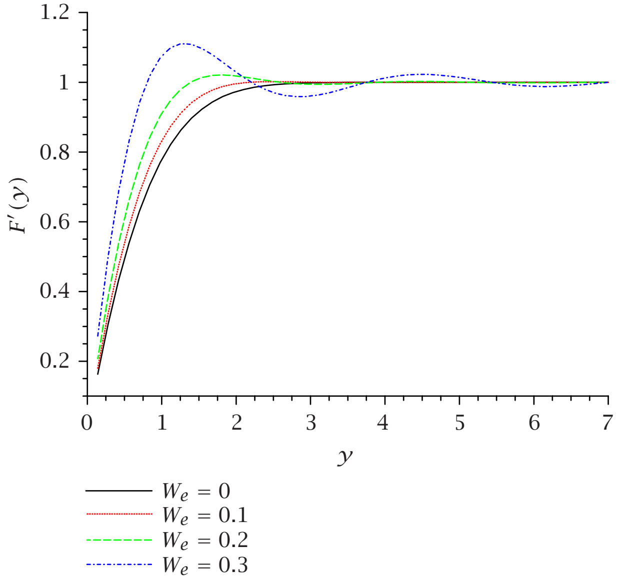

FIGURE 3.1. Variation of $F'(y)$ for $\gamma = 0$ and various values of W_e.

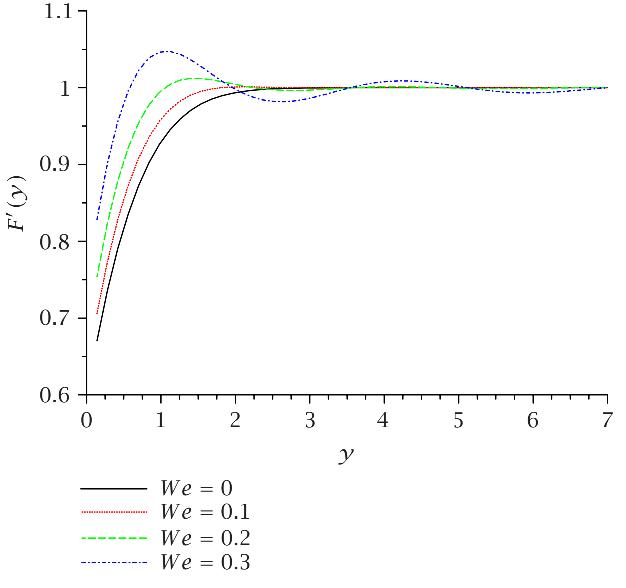

FIGURE 3.2. Variation of $F'(y)$ for $\gamma = 1$ and various values of W_e.

where the numerical values of C are shown in Table 3.2 for various values of W_e and γ. The numerical results are in good agreement with those of Wang [17] if $W_e = 0$ and those of Ariel [1] if $\gamma = 0$.

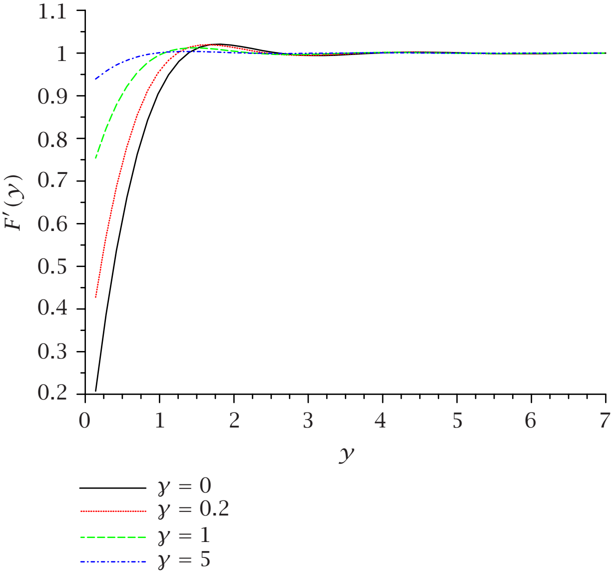

FIGURE 3.3. Variation of $F'(y)$ for $W_e = 0.2$ and various values of γ.

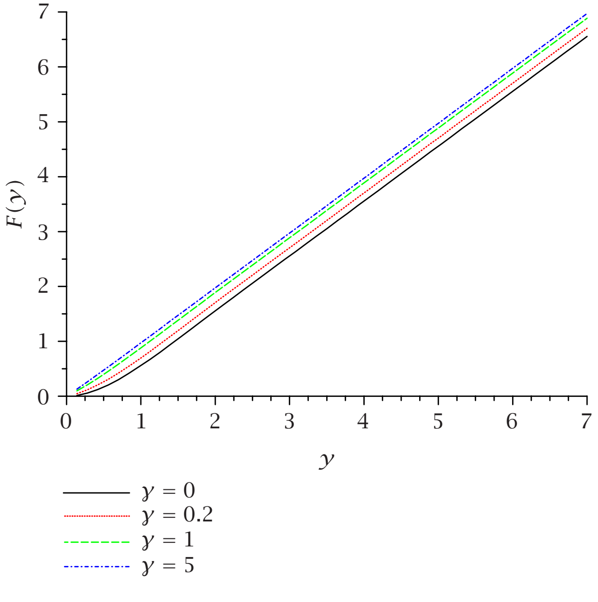

FIGURE 3.4. Variation of $F(y)$ for $W_e = 0.2$ and various values of γ.

TABLE 3.2. Numerical values of C for various values of W_e and γ.

γ	W_e			
	0	0.1	0.2	0.3
0	−0.65086	−0.55952	−0.44641	−0.26445
0.2	−0.49199	−0.40368	−0.29665	−0.13110
0.4	−0.39045	−0.30926	−0.21462	−0.08016
0.6	−0.32210	−0.24848	−0.16585	−0.05625
0.8	−0.27359	−0.20683	−0.13433	−0.04289
1	−0.23757	−0.17677	−0.11255	−0.03449
2	−0.14297	−0.10161	−0.06152	−0.01716
5	−0.06513	−0.04433	−0.02583	−0.00675
10	−0.03416	−0.02282	−0.01311	−0.00334
20	−0.01752	−0.01157	−0.00660	−0.00166

The Maclaurin series expansion for $F(y)$ is given by

$$F(y) = \gamma s y + \frac{1}{2} s y^2 - \frac{1}{6} \frac{1 + W_e s^2 - \gamma^2 s^2}{1 - 2W_e \gamma s} y^3, \tag{3.8}$$

where the values of $F''(0) = s$ are given in Table 3.1.

4. Oblique flow. Following Stuart [14], we assume that the streamfunction far from the wall is given by

$$\psi(x,y) \sim k y^2 + x y, \tag{4.1}$$

where k is a constant. The dividing streamline which comes from the wall from infinity is defined by $\psi(x,y) = 0$ and its slope at infinity is $-1/k$. Equation (4.1) suggests that $\psi(x,y)$ has the form

$$\psi(x,y) = xF(y) + G(y). \tag{4.2}$$

The boundary conditions for $F(y)$ and $G(y)$ are

$$\begin{gathered} F(0) = 0, \qquad F'(0) = \gamma F''(0), \qquad G(0) = 0, \qquad G'(0) = \gamma G''(0), \\ F(y) \sim y, \quad G(y) \sim k y^2 \quad \text{as } y \longrightarrow \infty. \end{gathered} \tag{4.3}$$

Employing (4.2) in (2.7), we obtain an equation which contains terms of $O(x)$ and $O(1)$. The terms of $O(x)$ yield an ordinary differential equation for $F(y)$ and the terms of $O(1)$ yield an equation for $G(y)$.

After one integration the boundary-value problem for $F(y)$ is

$$\begin{gathered} F''' + FF'' - F'^2 + W_e\left(FF^{(iv)} - 2F'F''' + F''^2\right) = -1, \\ F(0) = 0, \qquad F'(0) = \gamma F''(0), \qquad F'(\infty) = 1. \end{gathered} \tag{4.4}$$

Numerical solutions of this system were obtained in the previous section for various values of W_e and γ.

TABLE 4.1. Numerical values of $H'(0)$ for various values of W_e and γ.

γ	W_e			
	0	0.1	0.2	0.3
0	1.40651	1.46151	1.55278	1.70295
0.2	1.09256	1.11018	1.14278	1.18676
0.4	0.87851	0.87714	0.88073	0.87935
0.6	0.72934	0.71877	0.70902	0.69089
0.8	0.62139	0.60642	0.59064	0.56662
1	0.54037	0.52337	0.50500	0.47934
2	0.32534	0.30824	0.29044	0.26913
5	0.14793	0.13698	0.12677	0.11561
10	0.07757	0.07100	0.06527	0.05919
20	0.03977	0.03615	0.03311	0.02995

The boundary-value problem for $G(y)$ is given by

$$G^{(iv)} + FG''' - F''G' + W_e(FG^{(v)} - F^{(iv)}G') = 0, \tag{4.5}$$

$$G(0) = 0, \qquad G'(0) = \gamma G''(0), \qquad G'(\infty) = 2ky. \tag{4.6}$$

Integration of (4.5) once with respect to y using the conditions at infinity yields

$$G''' + FG'' - F'G' + W_e(FG^{(iv)} - F'G''' + F''G'' - F'''G') = 2kC, \tag{4.7}$$

where the values of C are given in Table 3.2.

Letting $G'(y) = 2kH(y)$, we obtain

$$H'' + FH' - F'H + W_e(FH''' - F'H'' + F''H' - F'''H) = C. \tag{4.8}$$

The boundary conditions for $H(y)$ are

$$H(0) = \gamma H'(0), \qquad H'(\infty) = 1. \tag{4.9}$$

Equation (4.8) with boundary conditions (4.9) is solved numerically using the same numerical technique as in the previous section. The numerical values of $H'(0)$ are given in Table 4.1 for various values of W_e and γ. These values are in good agreement with those obtained by Wang [17] for $W_e = 0$. Figure 4.1 shows the profiles of H' for $\gamma = 1$ and various values of W_e. Figure 4.2 depicts the profiles of H' for $W_e = 0.2$ and various values of γ. It can be observed that as the slip parameter γ increases the values of H' near the wall decreases.

The Maclaurin series for $G(y)$ is given by

$$G(y) = 2k\gamma\lambda y + k\lambda y^2 + \frac{k}{3(1-W_e\gamma s)}\left[C + \gamma^2 s\lambda - W_e\lambda\left(s + \frac{\gamma(1-\gamma^2 s^2 + W_e s^2)}{1-2W_e\gamma s}\right)\right]y^3, \tag{4.10}$$

where $H'(0) = \lambda$ are given in Table 4.1 for various values of γ and W_e.

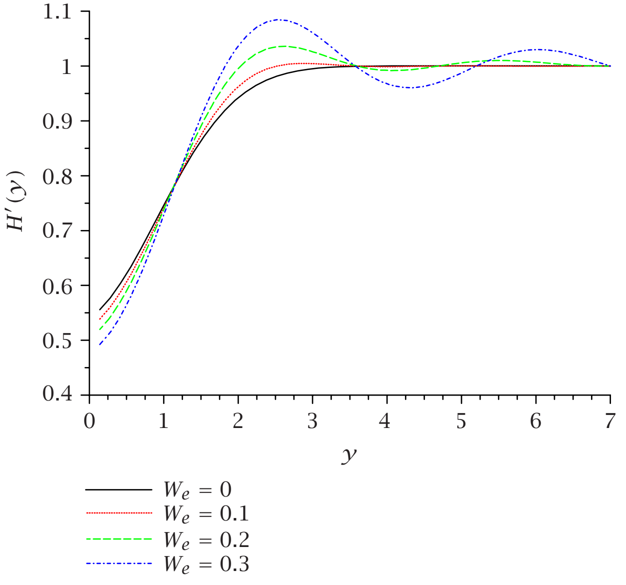

FIGURE 4.1. Variation of $H'(y)$ for $\gamma = 1$ and various values of W_e.

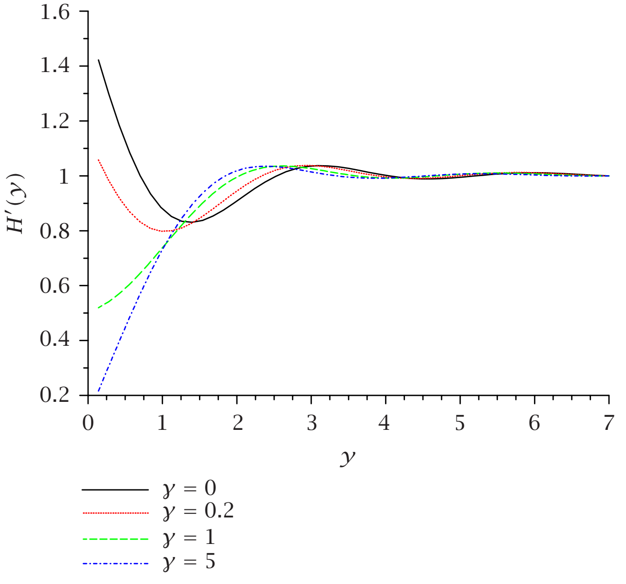

FIGURE 4.2. Variation of $H'(y)$ for $W_e = 0.2$ and various values of γ.

5. Conclusions. The behavior of the Walters' B' fluid impinging on a rigid wall with slip was examined. The fluid impinges on the wall either orthogonally or obliquely. It was found that the effects of the slip condition and the viscoelasticity were to increase the velocity near the wall.

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