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A FORMULA FOR THE INNER SPECTRAL RADIUS

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This note presents an asymptotic formula for the minimum of the moduli of the elements in the spectrum of a bounded linear operator acting on Banach space X. This minimum moduli is called the inner spectral radius, and the formula established herein is an analogue of Gelfand's spectral radius formula.

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1. Introduction. Let X be a Banach space and let $B(X)$ denote the Banach algebra of all bounded linear operators on X, and let T be an element of $B(X)$. We denote by $m(T)$ the "minimum moduli" of T and define it by

$$m(T) = \inf\{\|Tx\|,\ x \in X,\ \|x\| = 1\}. \tag{1.1}$$

In what follows, $i(T)$ and $i_{ap}(T)$ will denote, respectively, inner spectral radius and inner approximate spectral radius of T. We define $i(T)$ and $i_{ap}(T)$ by

$$\begin{aligned} i(T) &= \inf\{|\lambda| : \lambda \in \sigma(T)\}, \\ i_{ap}(T) &= \inf\{|\lambda| : \lambda \in \sigma_{ap}(T)\}. \end{aligned} \tag{1.2}$$

Makai and Zemanek [3] proved that

$$i_{ap}(T) = \lim_{n\to\infty} \left[m(T^n)\right]^{1/n}. \tag{1.3}$$

In this note, we prove the same formula for $i(T)$. The main results established herein are the following theorem and corollary.

Theorem 1.1. *Let $T \in B(X)$. Then, $r_i(T) = i(T)$ if and only if $r_i(T) \le r_i(T^*)$.*

Corollary 1.2. *It is not necessary that $r_i(T) = r_i(T^*)$ for any $T \in B(X)$.*

2. Basic concepts. Throughout, X will denote a Banach space, $B(X)$ is the Banach algebra of all bounded linear operators on X. T will denote an element of $B(X)$. We denote T^* as the transpose of T (T^* is an element of B(X^*), where X^* is dual space of X) and define

$$(T^*g)(x) = g(T(x)), \quad x \in X,\ g \in X^*. \tag{2.1}$$

If X is a Hilbert space, then T^* is the adjoint of T and $T^* \in B(X)$. We denote by $\sigma(T)$, $\sigma_{ap}(T)$, $\sigma_p(T)$, and $\sigma_c(T)$, respectively, spectrum, approximate point spectrum, point spectrum, and compression spectrum of T and define

$$\begin{aligned} \sigma(T) &= \{\lambda : (T-\lambda I) \text{ is not invertible, } \lambda \in \mathbb{C}\}, \\ \sigma_{ap}(T) &= \{\lambda : (T-\lambda I) \text{ is not bounded below, } \lambda \in \mathbb{C}\}, \\ \sigma_p(T) &= \{\lambda : \ker(T-\lambda I) \neq 0, \ \lambda \in \mathbb{C}\}, \\ \sigma_c(T) &= \{\lambda : \operatorname{ran}(T-\lambda I) \text{ is not dense in } X, \ \lambda \in \mathbb{C}\}. \end{aligned} \tag{2.2}$$

The spectral radius of T is denoted by $r(T)$ and defined by

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}. \tag{2.3}$$

We recall the following statements. One can see their proof in [1].

(1) $\|T\| = \|T^*\|$.
(2) $r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$ (Gelfand's formula).
(3) $r(T) = r(T^*)$.
(4) $\sigma(T) = \sigma(T^*)$, if X is a Hilbert space, then $\sigma(T^*) = \overline{\sigma(T)}$, where $\overline{\sigma(T)} = \{\overline{\lambda}, \lambda \in \sigma(T)\}$.

For operator $T \in B(X)$, define

$$m(T) = \inf\{\|Tx\|, \ x \in X, \ \|x\| = 1\}. \tag{2.4}$$

$m(T)$ is called the minimum moduli of T. Note that by definition of $m(T)$, we have $\|Tx\| \geq m(T)\|x\|$. It is clear that; if T is an invertible element in $B(X)$, then $m(T) = \|T^{-1}\|^{-1}$.

Definition 2.1. The inner spectral radius and inner approximate spectral radius of T are denoted, respectively, by $i(T)$ and $i_{ap}(T)$ and defined by (1.2).

Proposition 2.2. *If $|\lambda| < m(T)$, then $(T-\lambda I)$ is bounded below.*

Proof. We have

$$\|(T-\lambda I)x\| \geq \|Tx\| - \|\lambda x\| \geq (m(T) - |\lambda|)\|x\|. \tag{2.5}$$

The assumption implies that $m(T) - |\lambda| > 0$ and hence $(T-\lambda I)$ is bounded below by the definition. □

Proposition 2.3. *For every operator $T \in B(X)$,*

$$\lim_{n\to\infty} [m(T^n)]^{1/n} = \sup [m(T^n)]^{1/n}. \tag{2.6}$$

Proof. For every operator T and S in $B(X)$, we have

$$m(TS) \geq m(T)m(S), \tag{2.7}$$

by definition of the minimum moduli. Therefore, for every positive integers i and j,

$$m(T^{i+j}) \geq m(T^i)m(T^j). \tag{2.8}$$

This is the crucial inequality. Let k be fixed. For every integer number n, we have $n = kq + r$, $0 \leq r < k$, where $q = q(n)$ and $r = r(n)$ are functions of n. Note that $\lim_{n\to\infty} q(n)/n = 1/k$. Thus, by (2.8) we have

$$m(T^n) \geq m(T^k)^q m(T)^r, \quad \text{for each positive integer } n. \tag{2.9}$$

Hence,

$$\liminf_{n\to\infty} [m(T^n)]^n \geq m(T^k)^{1/k}. \tag{2.10}$$

Since this equation holds for all k, we have

$$\liminf_{n\to\infty} [m(T^n)]^n \geq \sup [m(T^n)]^{1/n} \geq \limsup_{n\to\infty} [m(T^n)]^n \tag{2.11}$$

and the result follows. □

Assume that $r_i(T) = \lim_{n\to\infty}[m(T^n)]^{1/n}$. By Gelfand's formula, it is clear that if T is invertible, then $r_i(T) = [r(T^{-1})]^{-1}$.

Corollary 2.4. *Let $T \in B(X)$. Then, $0 \in \sigma_{ap}(T)$ if and only if $m(T) = 0$.*

Proof. The result follows from the facts that $0 \in \sigma_{ap}(T)$ if and only if T is not bounded below and $\|Tx\| \geq m(T)\|x\|$ for each $x \in X$. □

Proposition 2.5. *Let $T \in B(X)$. If $\lambda \in \sigma_{ap}(T)$, then $|\lambda| \geq r_i(T)$.*

Proof. Suppose $\lambda \in \sigma_{ap}(T)$. Assume, contrary to what we wish to prove, that $|\lambda| < r_i(T)$. Thus, $|\lambda|^n < m(T^n)$ for some integer n by the definition of $r_i(T)$. By Proposition 2.2, $(T^n - \lambda^n I)$ is bounded below. We have

$$T^n - \lambda^n = (T^{n-1} + T^{n-2}\lambda + \cdots + \lambda^n)(T - \lambda). \tag{2.12}$$

Hence, $(T - \lambda)$ is bounded below and so $\lambda \notin \sigma_{ap}(T)$, which is contradictory to our assumption. □

Corollary 2.6. *For each $T \in B(X)$,*

$$\sigma_{ap}(T) \subseteq \{\lambda : r_i(T) \leq |\lambda| \leq r(T)\}. \tag{2.13}$$

Makai and Zemanek in [3] proved that $i_{ap}(T) = r_i(T)$ for every $T \in B(X)$. In the next section, we will prove that $i(T) = r_i(T)$ if and only if $r_i(T) \leq r_i(T^*)$.

3. Inner spectral radius.

The purpose of this section is to prove the main result.

We know that $\partial\sigma_{ap}(T) \subseteq \sigma(T)$ and $r_i(T) = i_{ap}(T)$ and so $r_i(T) \in \sigma(T)$. Therefore, for every $T \in B(X)$, we have

$$i(T) \leq r_i(T). \tag{3.1}$$

Fact 3.1. If X is a finite-dimensional space, then $\sigma_{ap}(T) = \sigma(T)$ for each $T \in B(X)$ and hence $r_i(T) = i(T)$.

Fact 3.2. If T is a compact operator acting on Banach space X, then $r_i(T) = i(T)$.

We begin with some general lemmas that we need in the proof of the main theorem.

Lemma 3.3. *Let $T \in B(X)$. Then, $\sigma_c(T) = \sigma_p(T^*)$. (If X is a Hilbert space, then $\sigma_c(T) = \overline{\sigma_p(T^*)}$).*

Proof. First, we show that $\sigma_c(T) \subseteq \sigma_p(T^*)$. Suppose λ is an element in $\sigma_c(T)$. Consider M the closure of $\operatorname{ran}(T-\lambda I)$. By definition of $\sigma_c(T)$, $M \neq X$. If x_0 is a nonzero element in $X - M$, then by the Hahn-Banach theorem there is $f_0 \in X^*$ such that $f_0(M) = 0$ and $f_0(x_0) = 1$. We have $((T^*-\lambda I)f_0)(x) = f_0((T-\lambda I)x) = 0$ for every $x \in X$ and hence $f_0 \in \ker(T^*-\lambda I)$, that is $\lambda \in \sigma_p(T^*)$.

Now, we prove $\sigma_p(T^*) \subseteq \sigma_c(T)$. Suppose $\lambda \in \sigma_p(T^*)$, thus, there is a nonzero functional g in X^* such that $(T^*-\lambda I)g = 0$ and so, $g((T-\lambda I)x) = 0$ for each $x \in X$ by (2.1). Hence, $g(t) = 0$ for any t in closure $\operatorname{ran}(T-\lambda I)$.

But $g \neq 0$ on X, and hence there is $x_0 \in X - M$ such that $g(x_0) \neq 0$. Therefore, $M \neq X$, that is, $\lambda \in \sigma_c(T)$.

If X is a Hilbert space, then we know that $\ker(T) = (\operatorname{ran} T^*)^\perp$ and $\operatorname{closure}(\operatorname{ran} T^*) = (\ker T)^\perp$ in [1, Theorem II.2.19]. Thus, by the definition of $\sigma_p(T)$ and $\sigma_c(T)$, we get the following result. □

Lemma 3.4. *Let $T \in B(X)$. Then, $\sigma(T) = \sigma_{ap}(T) \bigcup \sigma_c(T)$.*

Proof. It follows from [1, Proposition VII.6.4] and the definition of $\sigma_{ap}(T)$ and $\sigma_c(T)$. □

Lemma 3.5. *Let $T \in B(X)$. If $\sigma(T) \subseteq \{\lambda : r_i(T) \leq |\lambda| \leq r(T)\}$, then $r_i(T) = i(T)$.*

Proof. By assumption, we have $r_i(T) \leq i(T)$ and the result follows the fact that $i(T) \leq r_i(T)$. □

Theorem 3.6. *Let $T \in B(X)$. Then, $r_i(T) = i(T)$ if and only if $r_i(T) \leq r_i(T^*)$.*

Proof. First, suppose that $r_i(T) \le r_i(T^*)$. By Lemmas 3.3 and 3.4, $\sigma(T) = \sigma_{ap}(T) \bigcup \sigma_p(T^*)$ (if X is a Hilbert space, then $\sigma(T) = \sigma_{ap}(T) \bigcup \overline{\sigma_p(T^*)}$). We have

$$\sigma(T) \subseteq \{\lambda : r_i(T) \le |\lambda| \le r(T)\}. \tag{3.2}$$

Hence, by Lemma 3.5, $r_i(T) = i(T)$.

Conversely, suppose that $r_i(T) = i(T)$. We have $\sigma(T) = \sigma(T^*)$ (if X is a Hilbert space, then $\sigma(T^*) = \overline{\sigma(T)}$). Thus, $i(T) = i(T^*)$ by definition of $i(T)$, and, therefore,

$$r_i(T) = i(T) = i(T^*) \le r_i(T^*). \tag{3.3}$$

□

Example 3.7. Let X be a Hilbert space and $N \in B(X)$ a normal operator. Then,

$$i(N) = i(N^*) = r_i(N) = r_i(N^*). \tag{3.4}$$

Since N is normal, $\|Nx\| = \|N^*x\|$ for every x in X, and, therefore, $m(N) = m(N^*)$. Similarly, we have $m(N^n) = m(N^{*n})$ for each n, and so, $i(N) = i(N^*) = r_i(N) = r_i(N^*)$.

If X is a Hilbert space and $N \in B(X)$ is a normal operator, then $r(N) = \|N\|$. In the next proposition, we prove that $r_i(N) = m(N)$ for the normal operator N in $B(X)$.

Recall that for each operator $T \in B(X)$ the numerical range of T is defined and denoted as follows:

$$W(T) = \{\lambda \in \mathbb{C} : \lambda = \langle Tx, x\rangle,\ x \in X \text{ with } \|x\| = 1\}. \tag{3.5}$$

The following interesting theorem was proved in [2, Theorem 27.9].

Theorem 3.8. *If T is a selfadjoint operator in $B(X)$, M_1 and M_2 denote, respectively, the infimum and the supremum of the numerical range of T, then M_1 and M_2 are approximate eigenvalues of T, and the spectrum of T is contained in the interval $[M_1, M_2]$.*

By this theorem, for each positive operator $T \in B(X)$ we have

$$i(T) = r_i(T) = \inf\{\langle Tx, x\rangle,\ x \in X \text{ with } \|x\| = 1\}. \tag{3.6}$$

Proposition 3.9. *If N is normal operator acting on Hilbert space X, then $i(N) = r_i(N) = m(N)$.*

Proof. As shown in Example 3.7, we have $m(N) = m(N^*)$. Now, we prove that $m(NN^*) = m(N)^2$. Since NN^* is positive, by (3.6) and Proposition 2.3, we have

$$\begin{aligned} m(NN^*) \le r_i(NN^*) &= \inf\{\langle NN^*x, x\rangle,\ x \in X \text{ with } \|x\| = 1\} \\ &= \inf\{\|Nx\|^2,\ x \in X \text{ with } \|x\| = 1\} = m(N)^2. \end{aligned} \tag{3.7}$$

By (2.8), we get

$$m(NN^*) \ge m(N)m(N^*) = m(N)^2. \tag{3.8}$$

Hence,

$$m(NN^*) = m(N)^2. \tag{3.9}$$

By induction, we show that if $j = 2^n$, $n = 0,1,2,\ldots$, then $m(N^j) = m(N)^j$. This is clearly true for $n = 0$. Assume it to be true for some n, then for all $x \in \mathfrak{H}$, we have

$$||N^{2^{n+1}}(x)|| = ||N^{2^n}(N^{2^n}(x))|| = ||(N^{2^n})^*(N^{2^n}(x))||, \tag{3.10}$$

because N^{2^n} is normal. This shows that $m(N^{2^{n+1}}) = m((N^{2^n})^* N^{2^n})$, which is equal to $m(N^{2^n})^2$. Thus, $m(N^{2^{n+1}}) = (m(N)^{2^n})^2 = m(N)^{2^{n+1}}$. Therefore,

$$r_i(N) = \lim_{n\to\infty}[m(N^n)]^{1/n} = \lim_{n\to\infty}[m(N^{2^n})]^{1/2^n} = m(N). \tag{3.11}$$

□

Example 3.10. Suppose U is a unilateral weighted shift with weights $(1,2,1,\ldots)$ acting on separable Hilbert space $\mathfrak{H}$. William Ridge [4] proved that $\sigma_{ap}(U) = \{\lambda : |\lambda| = \sqrt{2}\}$, $\sigma(U) = \{\lambda : |\lambda| \le \sqrt{2}\}$, and $\sigma_{ap}(U^*) = \sigma(U^*) = \sigma(U)$. Hence, $r_i(U) = r(U) = \sqrt{2}$, $r_i(U^*) = i(U^*) = 0$, and $i(U) = 0$. Therefore, we have $i(U) \ne r_i(U)$ and $r_i(U^*) < r_i(U)$.

We know that $r(T) = r(T^*)$ for any $T \in B(X)$. But in the above example $r_i(U^*) < r_i(U)$ so, we can write the next corollary.

Corollary 3.11. *It is not necessary that $r_i(T) = r_i(T^*)$ for any $T \in B(X)$.*

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