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SOME RESULTS ON A GENERALIZED ω-JACOBI TRANSFORM

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We introduce a generalized ω-Jacobi transform and obtain images of certain functions under this transform. Moreover, we define a new probability density function (pdf) involving this new generalized ω-Jacobi function. Some basic functions associated with the pdf, such as characteristic function, moments and distribution function, are evaluated.

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1. Introduction. Debnath [1] introduced the Jacobi transform of a function $g(x)$ defined in $-1 < x < 1$ by the integral

$$J[g(x)] = \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} P_n^{(\alpha,\beta)}(x) g(x)\, dx, \tag{1.1}$$

where $P_n^{(\alpha,\beta)}$ is the Jacobi function of degree n and orders $\alpha(> -1)$ and $\beta(> -1)$. Kalla et al. [3] have studied the integral

$$I_{\upsilon,\alpha,\beta}^{a,b} = \int_{-1}^{1} (1-x)^{a}(1+x)^{b} P_{\upsilon}^{(\alpha,\beta)}(x)\, dx, \tag{1.2}$$

with $\operatorname{Re} a, \operatorname{Re} b > -1$ and $P_{\upsilon}^{(\alpha,\beta)}$ is the Jacobi function, where

$$P_{\upsilon}^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_{\upsilon}}{\Gamma(\upsilon+1)}\, {}_2F_1\left(\begin{matrix} -\upsilon, \upsilon+\lambda \\ \alpha+1 \end{matrix}; \frac{1-x}{2}\right), \tag{1.3}$$

$\lambda = \alpha+\beta+1$, and ${}_2F_1$ is the classical Gauss hypergeometric function and its partial derivatives with respect to a and b. These results were extended by Sarabia [5] using the following integral,

$$I_{\upsilon,\alpha,\beta}^{a,b,c,p} = \int_{-1}^{1} (1-x)^{a}(1+x)^{b} P_{\upsilon}^{(\alpha,\beta,c,p)}(x)\, dx, \tag{1.4}$$

where $P_{\upsilon}^{(\alpha,\beta,c,p)}(x)$ is the generalized Jacobi function defined as

$$P_{\upsilon}^{(\alpha,\beta,c,p)}(x) = \frac{(\alpha+1)_{\upsilon}}{\Gamma(\upsilon+1)}\, {}_3F_2\left(\begin{matrix} -\upsilon, \upsilon+\lambda, c \\ \alpha+1, p \end{matrix}; \frac{1-x}{2}\right), \tag{1.5}$$

with $\operatorname{Re}(p-c-\beta) > 0$ and ${}_3F_2$ is generalized hypergeometric function [2].

Moreover, Sarabia and Kalla [6] defined and studied the generalized Jacobi transform as

$$J_{\alpha,\beta}^{a,b,c,p}[f(x),\upsilon]=\int_{-1}^{1}(1-x)^a(1+x)^bP_{\upsilon}^{(\alpha,\beta,c,p)}(x)f(x)dx, \tag{1.6}$$

for continuous or sectionally continuous f on $[-1,1]$. A number of integral transforms and their applications are given in [1, 7].

Throughout this sequel, we will use the following relation:

$$\int_{-1}^{1}(1-t)^n(1+t)^m dt=2^{n+m+1}\int_0^1 y^n(1-y)^m dy=2^{n+m+1}B(n+1,m+1). \tag{1.7}$$

Further, we consider *ω-Kampe de Feriet function of two variables* in the following form:

$$\begin{aligned}&{}^{\omega}R_{l:m;n}^{p:q;k}\left(\begin{matrix}(a_p):(b_q);(\underline{ck});\\(\alpha_l):(\beta_m);(\underline{\gamma n});\end{matrix}x,y\right)\\&=\sum_{r,s=0}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_{r+s}\prod_{i=1}^{q}(b_i)_r\prod_{i=1}^{k}(\Gamma(c_i+\omega s)/\Gamma(c_i))}{\prod_{i=1}^{l}(\alpha_i)_{r+s}\prod_{i=1}^{m}(\beta_i)_r\prod_{i=1}^{n}(\Gamma(\gamma_i+\omega s)/\Gamma(\gamma_i))}\frac{x^r}{r!}\frac{y^s}{s!},\quad |x|,|y|<1.\end{aligned} \tag{1.8}$$

For $\omega=1$, (1.8) reduces to *Kampe de Feriet function of two variables* [8]*:*

$$\begin{aligned}&F_{l:m;n}^{p:q;k}\left(\begin{matrix}(a_p):(b_q);(c_k);\\(\alpha_l):(\beta_m);(\gamma_n);\end{matrix}x,y\right)\\&=\sum_{r,s=0}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_{r+s}\prod_{i=1}^{q}(b_i)_r\prod_{i=1}^{k}(c_i)_s}{\prod_{i=1}^{l}(\alpha_i)_{r+s}\prod_{i=1}^{m}(\beta_i)_r\prod_{i=1}^{n}(\gamma_i)_s}\frac{x^r}{r!}\frac{y^s}{s!},\quad |x|,|y|<1.\end{aligned} \tag{1.9}$$

In Section 2, we obtain images of certain functions under our generalized ω-Jacobi transform. We introduce and study a new probability density function (pdf) involving the generalized ω-Jacobi function in the third section. Finally, we establish some functions associated with this pdf, such as its distribution function, survival function, characteristic function, and some other basic moments. It is interesting to observe that several results, including the formulas obtained by Sarabia and Kalla [6] follow as special case of our results in this work.

2. Generalized ω-Jacobi transform.

Let f be defined as before, then the generalized ω-Jacobi transform of $f(x)$ is defined as

$$J_{\omega;\alpha,\beta}^{a,b,c,p}[f(x),\upsilon]=\int_{-1}^{1}(1-x)^a(1+x)^bP_{\omega;\upsilon}^{(\alpha,\beta,c,p)}(x)f(x)dx, \tag{2.1}$$

where $P_{\omega;\upsilon}^{(\alpha,\beta,c,p)}(x)$ is the generalized ω-Jacobi function defined as

$$P_{\omega;\upsilon}^{(\alpha,\beta,c,p)}(x) = \frac{(\alpha+1)_\upsilon}{\Gamma(\upsilon+1)} {}_3\overset{\omega}{R}_2\left(\begin{matrix}-\upsilon,\underline{\upsilon+\lambda},c \\ \underline{\alpha+1},p\end{matrix};\frac{1-x}{2}\right), \tag{2.2}$$

and ${}_3\overset{\omega}{R}_2$ is the generalized ω-Gauss hypergeometric function defined in [4, equation (1.9)]:

$${}_3\overset{\omega}{R}_2\left(\begin{matrix}a_1,\underline{a_2},a_3 \\ \underline{b_1},b_2\end{matrix};y\right) = \frac{\Gamma(b_1)}{\Gamma(a_2)} \sum_{k=0}^{\infty} \frac{\Gamma(a_2+\omega k)(a_1)_k(a_3)_k}{\Gamma(b_1+\omega k)(b_2)_k} \frac{y^k}{k!}, \tag{2.3}$$

where $\underline{a}$ is defined to be $\Gamma(a+\omega k)/\Gamma(a)$.

For $\omega = 1$, (2.1) reduces to (1.6) which has been studied in [6]. In addition, if $c = p$, (2.1) reduces to usual Jacobi transform [1].

Now we obtain images of some functions under the generalized ω-Jacobi transform.

(1) For $f(x) = 1$, we have

$$\begin{aligned} J_{\omega;\alpha,\beta}^{a,b,c,p}[1,\upsilon] &= \int_{-1}^{1} (1-x)^a(1+x)^b P_{\omega;\upsilon}^{(\alpha,\beta,c,p)}(x)\,dx \\ &= \frac{(\alpha+1)_\upsilon}{\Gamma(\upsilon+1)} \sum_{k=0}^{\infty} B_k \int_{-1}^{1} (1-x)^a(1+x)^b\,dx, \end{aligned} \tag{2.4}$$

where

$$B_k = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1+\omega k)} \frac{\Gamma(\nu+\lambda+\omega k)}{\Gamma(\nu+\lambda)} \frac{(-\upsilon)_k(c)_k}{(p)_k k!}. \tag{2.5}$$

Using (1.7), we get

$$J_{\omega;\alpha,\beta}^{a,b,c,p}[1,\upsilon] = \frac{2^{a+b+1}B(a+1,b+1)(\alpha+1)_\upsilon}{\Gamma(\upsilon+1)} {}_4\overset{\omega}{R}_3\left(\begin{matrix}-\upsilon,\underline{\upsilon+\lambda},c,a+1 \\ \underline{\alpha+1},p,a+b+2\end{matrix};1\right), \tag{2.6}$$

where

$${}_4\overset{\omega}{R}_3\left(\begin{matrix}-\upsilon,\underline{\upsilon+\lambda},c,a+1 \\ \underline{\alpha+1},p,a+b+2\end{matrix};1\right) = \sum_{k=0}^{\infty} \frac{(a+1)_k B_k}{(a+b+2)_k}. \tag{2.7}$$

(2) For $f(x) = \ln(1-x)$, we have

$$\begin{aligned} J_{\omega;\alpha,\beta}^{a,b,c,p}[\ln(1-x),\upsilon] &= \int_{-1}^{1} \ln(1-x)(1-x)^a(1+x)^b P_{\omega;\upsilon}^{(\alpha,\beta,c,p)}(x)\,dx \\ &= \frac{\partial}{\partial a} J_{\omega;\alpha,\beta}^{a,b,c,p}[1,\upsilon] \\ &= (\ln 2) J_{\omega;\alpha,\beta}^{a,b,c,p}[1,\upsilon] + \frac{2^{a+b+1}B(a+1,b+1)(\alpha+1)_\upsilon}{\Gamma(\upsilon+1)} \sum_{k=0}^{\infty} D_k, \end{aligned} \tag{2.8}$$

where

$$D_k = \frac{(a+1)_k B_k}{(a+b+2)_k}[\psi(a+k+1)-\psi(a+b+k+2)]. \tag{2.9}$$

In similar way we have, for $f(x)=\ln(1+x)$,

$$\begin{aligned}J^{a,b,c,p}_{\omega;\alpha,\beta}[\ln(1+x),\upsilon] &= \int_{-1}^{1}\ln(1+x)(1-x)^a(1+x)^b P^{(\alpha,\beta,c,p)}_{\omega;\upsilon}(x)\,dx\\ &= \frac{\partial}{\partial b}J^{a,b,c,p}_{\omega;\alpha,\beta}[1,\upsilon]\\ &= [\ln 2+\psi(b+1)]J^{a,b,c,p}_{\omega;\alpha,\beta}[1,\upsilon]\\ &\quad -\frac{2^{a+b+1}B(a+1,b+1)(\alpha+1)_\upsilon}{\Gamma(\upsilon+1)}\sum_{k=0}^{\infty}L_k,\end{aligned}\tag{2.10}$$

where

$$L_k=\frac{(a+1)_kB_k}{(a+b+2)_k}\psi(a+b+k+2).\tag{2.11}$$

(3) For $f(x)=\ln((1-x)/(1+x))$, using (2.8) and (2.10), we have

$$\begin{aligned}J^{a,b,c,p}_{\omega;\alpha,\beta}\left[\ln\left(\frac{1-x}{1+x}\right),\upsilon\right] &= \psi(b+1)\times J^{a,b,c,p}_{\omega;\alpha,\beta}[1,\upsilon]\\ &\quad -\frac{2^{a+b+1}B(a+1,b+1)(\alpha+1)_\upsilon}{\Gamma(\upsilon+1)}\sum_{k=0}^{\infty}\frac{(a+1)_kB_k}{(a+b+2)_k}\psi(a+k+1).\end{aligned}\tag{2.12}$$

(4) For $f(x)=(1-x)^A(1+x)^B$, we have

$$\begin{aligned}J^{a,b,c,p}_{\omega;\alpha,\beta}[(1-x)^A(1+x)^B,\upsilon] &= J^{a+A,b+B,c,p}_{\omega;\alpha,\beta}[1,\upsilon]\\ &= \frac{2^{a+A+b+B+1}B(a+A+1,b+B+1)(\alpha+1)_\upsilon}{\Gamma(\upsilon+1)}\\ &\quad\times {}_4\overset{\omega}{R}_3\left(\begin{matrix}-\upsilon,\underline{\upsilon+\lambda},c,a+A+1\\ \underline{\alpha+1},p,a+A+b+B+2\end{matrix};1\right).\end{aligned}\tag{2.13}$$

(5) For $f(x)=P^{(\gamma,\delta,d,q)}_{\omega;\mu}(x)$, we have

$$\begin{aligned}J^{a,b,c,p}_{\omega;\alpha,\beta}[P^{(\gamma,\delta,d,q)}_{\omega;\mu}(x),\upsilon] &= \frac{2^{a+b+1}B(a+1,b+1)(\alpha+1)_\upsilon(\gamma+1)_\mu}{\Gamma(\upsilon+1)\Gamma(\mu+1)}\\ &\quad\times \overset{\omega}{R}{}^{1,3,3}_{1,2,2}\left(\begin{matrix}a+1;\ -\upsilon,\underline{\upsilon+\lambda},c;\ -\mu,\underline{\mu+\eta},d\\ a+b+2;\ \underline{\alpha+1},p;\ \underline{\gamma+1},q\end{matrix};1,1\right),\end{aligned}\tag{2.14}$$

where $\eta=\gamma+\delta+1$ and

$$\overset{\omega}{R}{}^{1,3,3}_{1,2,2}\left(\begin{matrix}a+1;\ -\upsilon,\underline{\upsilon+\lambda},c;\ -\mu,\underline{\mu+\eta},d\\ a+b+2;\ \underline{\alpha+1},p;\ \underline{\gamma+1},q\end{matrix};1,1\right)=\sum_{n,k=0}^{\infty}\frac{(a+1)_{n+k}}{(a+b+2)_{n+k}}\times N_n\times B_k,\tag{2.15}$$

with B_k given by (2.5) and

$$\begin{gathered}N_n=\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\omega n)}\frac{\Gamma(\mu+\eta+\omega n)}{\Gamma(\mu+\eta)}\frac{(-\mu)_n(d)_n}{(q)_n n!},\\ \sigma_X^2\triangleq E[X^2]-(E[X])^2.\end{gathered}\tag{2.16}$$

Remark 2.1. Letting $\omega = 1$ in previous results, we get same results presented in [6].

3. Generalized ω-Jacobi random variable.

Recently statistical distributions and their generalizations have played a significant role in application of applied statistics and reliability theory. In this section we define our ω-Jacobi random variable and its pdf, then we derive some basic functions associated with this density function.

3.1. The probability density function.

In a recent paper of Sarabia and Kalla [6], the following pdf has been studied:

$$h(x) = \frac{(1-x)^a(1+x)^b}{2^{a+b+1}B(a+1,b+1)R}\,{}_3F_2\left(\begin{matrix}-\upsilon,\upsilon+\lambda,c\\ \alpha+1,p\end{matrix};\frac{1-x}{2}\right)\times 1[-1\le x\le 1], \tag{3.1}$$

where

$$R = {}_4F_3\left(\begin{matrix}-\upsilon,\upsilon+\lambda,c,a+1\\ \alpha+1,p,a+b+2\end{matrix};1\right). \tag{3.2}$$

In the present paper, we introduce a generalization of (3.1) in the form given by (3.3). Using the condition $\int_{-1}^{1} g(t)\,dt = 1$, we obtained the pdf of a random variable X associated with (1.2) to be given by

$$\begin{aligned} g(x) &= g^{a,b,c,p;\upsilon}_{\omega;\alpha,\beta}(x)\\ &= \frac{(1-x)^a(1+x)^b}{2^{a+b+1}B(a+1,b+1)\overset{\omega}{R}}\,{}_3\overset{\omega}{R}_2\left(\begin{matrix}-\upsilon,\underline{\upsilon+\lambda},c\\ \underline{\alpha+1},p\end{matrix};\frac{1-x}{2}\right)\times 1[-1\le x\le 1], \end{aligned} \tag{3.3}$$

where

$$\overset{\omega}{R} = {}_4\overset{\omega}{R}_3\left(\begin{matrix}-\upsilon,\underline{\upsilon+\lambda},c,a+1\\ \underline{\alpha+1},p,a+b+2\end{matrix};1\right). \tag{3.4}$$

3.2. Some statistical functions.

The aim of this section is to obtain some basic functions associated with the pdf $g(x)$, such as the population moments, the cumulative distribution function (cdf), and the survivor function.

3.2.1. The survivor and distribution functions.

We derive the cdf $G(x)$ of the random variable X after computing its survivor function $S(x)$.

Theorem 3.1. *The survivor function $S(x)$ of the random variable X is given by*

$$S(x) = \frac{((1-x)/2)^{a+1}}{(a+1)B(a+1,b+1)\overset{\omega}{R}}\,R^{1,3,1}_{1,2,0}\left(\begin{matrix}a+1;\ -\upsilon,\underline{\upsilon+\lambda},c;\ -b;\ \dfrac{1-x}{2}\\ a+2;\ \underline{\alpha+1},p;\ -;\ \dfrac{1-x}{2}\end{matrix}\right). \tag{3.5}$$

Proof. Using (3.3) the survivor function $S(x)$ of X becomes

$$\begin{aligned} S(x) &\triangleq P(X \geq x) = \int_x^1 g(t)\,dt \\ &= 2\int_0^{(1-x)/2} g(1-2y)\,dy = \frac{1}{B(a+1,b+1)\overset{\omega}{R}} \sum_{k=0}^{\infty}\left(B_k \times \int_0^{(1-x)/2} y^{a+k}(1-y)^b\,dy\right) \\ &= \frac{1}{B(a+1,b+1)\overset{\omega}{R}} \sum_{k=0}^{\infty} B_{(1-x)/2}(a+k+1,b+1)\times B_k, \end{aligned} \tag{3.6}$$

where $B_x(a,b)$ is the incomplete beta function [2].

$$B_x(a,b) = \int_0^x y^a(1-y)^b\,dy = \frac{x^a}{a}\,{}_2F_1\left(\begin{matrix} a, 1-b \\ a+1 \end{matrix}; x\right), \quad a,b>0,\ 0<x<1, \tag{3.7}$$

and ${}_2F_1$ is Gauss *hypergeometric function.* Using this and (1.8) we get

$$\begin{aligned} S(x) &= \frac{((1-x)/2)^{a+1}}{(a+1)B(a+1,b+1)\overset{\omega}{R}} \sum_{n,k=0}^{\infty} \frac{(a+1)_{n+k}}{(a+2)_{n+k}} \frac{(-b)_n}{n!}\left(\frac{1-x}{2}\right)^{n+k} \times B_k \\ &= \frac{((1-x)/2)^{a+1}}{(a+1)B(a+1,b+1)\overset{\omega}{R}} R^{1,3,1}_{1,2,0}\left(\begin{matrix} a+1;\ -\upsilon, \underline{\upsilon+\lambda}, c;\ -b;\ \dfrac{1-x}{2} \\ a+2;\ \underline{\alpha+1}, p;\ -;\ \dfrac{1-x}{2} \end{matrix}\right), \end{aligned} \tag{3.8}$$

which completes the proof. □

Using the previous result, the cdf $G(x)$ can be expressed as

$$G(x) \triangleq P(X \leq x) = \int_{-1}^x g(t)\,dt = \int_{-1}^1 g(t)\,dt - \int_x^1 g(t)\,dt = 1 - S(x). \tag{3.9}$$

3.2.2. Population moments. In this subsection, we begin by evaluating the characteristic function, then we obtain the basic moments, such as the moment generating function, kth moment, and the mean. The following result will be used in obtaining the basic moments.

Theorem 3.2. *For any $A, B > 0$,*

$$\begin{aligned} E\left[(1-X)^A(1+X)^B\right] &= \frac{2^{A+B}B(a+A+1,b+B+1)}{B(a+1,b+1)\overset{\omega}{R}} \\ &\quad \times {}_4\overset{\omega}{R}_3\left(\begin{matrix} -\upsilon, \underline{\upsilon+\lambda}, c, a+A+1 \\ \underline{\alpha+1}, p, a+A+b+B+2 \end{matrix}; 1\right), \end{aligned} \tag{3.10}$$

and, in particular,

$$E\left[(1-X)^n\right] = \frac{2^n B(a+n+1,b+1)}{B(a+1,b+1)\overset{\omega}{R}} \times {}_4\overset{\omega}{R}_3\left(\begin{matrix} -\upsilon, \underline{\upsilon+\lambda}, c, a+n+1 \\ \underline{\alpha+1}, p, a+n+b+2 \end{matrix}; 1\right). \tag{3.11}$$

Proof. Using (1.2) and (1.4), for $K = (2^{a+b+1}B(a+1,b+1)\overset{\omega}{R})^{-1}$, we have

$$\begin{aligned}
E[(1-X)^A(1+X)^B] &= \int_{-1}^{1}(1-t)^A(1+t)^B g(t)\,dt \\
&= K\int_{-1}^{1}(1-t)^{a+A}(1+t)^{b+B}\,{}_3\overset{\omega}{R}_2\left(\begin{matrix}-\upsilon,\underline{\upsilon+\lambda},c\\ \underline{\alpha+1},p\end{matrix};\frac{1-t}{2}\right)dt \\
&= K\sum_{k=0}^{\infty}\frac{B_k}{2^k}\int_{-1}^{1}(1-t)^{a+A+k}(1+t)^{b+B}\,dt \\
&= K\sum_{k=0}^{\infty}2^{a+A+b+B+1}B(a+A+k+1,b+B+1)\times B_k \\
&= \frac{2^{A+B}B(a+A+1,b+B+1)}{B(a+1,b+1)\overset{\omega}{R}}\sum_{k=0}^{\infty}\frac{(a+A+1)_k B_k}{(a+A+b+B+2)_k} \\
&= \frac{2^{A+B}B(a+A+1,b+B+1)}{B(a+1,b+1)\overset{\omega}{R}}\,{}_4\overset{\omega}{R}_3\left(\begin{matrix}-\upsilon,\underline{\upsilon+\lambda},c,a+A+1\\ \underline{\alpha+1},p,a+A+b+B+2\end{matrix};1\right),
\end{aligned} \tag{3.12}$$

which completes the proof. □

Now we state and prove our main theorem.

Theorem 3.3. *The characteristic function of X, for any real t, is given by*

$$\varphi_X(t) = \frac{e^{it}}{\overset{\omega}{R}}\times \overset{\omega}{R}{}^{1,3,0}_{1,2,0}\left(\begin{matrix}a+1;\ -\upsilon,\underline{\upsilon+\lambda},c;\ -\\ a+b+2;\ \underline{\alpha+1},p;\ -\end{matrix};1,-2it\right). \tag{3.13}$$

Its moment generating function of X, for any real τ, is

$$M_X(\tau) = \frac{e^{\tau}}{\overset{\omega}{R}}\times \overset{\omega}{R}{}^{1,3,0}_{1,2,0}\left(\begin{matrix}a+1;\ -\upsilon,\underline{\upsilon+\lambda},c;\ -\\ a+b+2;\ \underline{\alpha+1},p;\ -\end{matrix};1,-2\tau\right), \tag{3.14}$$

and its rth moment is

$$E[X^r] = \frac{1}{\overset{\omega}{R}}\sum_{n=0}^{r}\binom{r}{n}\frac{(-2)^n(a+1)_n}{(a+b+2)_n}\times {}_4\overset{\omega}{R}_3\left(\begin{matrix}-\upsilon,\underline{\upsilon+\lambda},c,a+n+1\\ \underline{\alpha+1},p,a+n+b+2\end{matrix};1\right), \tag{3.15}$$

$$E[X^r] = \frac{1}{\overset{\omega}{R}}\frac{\Gamma(\alpha+1)}{\Gamma(\nu+\lambda)}\sum_{n,k=0}^{\infty}\frac{(a+1)_{n+k}}{(a+b+2)_{n+k}}\frac{\Gamma(r+1)}{\Gamma(r+1-n)}\frac{\Gamma(\nu+\lambda+\omega k)}{\Gamma(\alpha+1+\omega k)}\frac{(-\upsilon)_k(c)_k}{(p)_k k!}\frac{(-2)^n}{n!}. \tag{3.16}$$

Special case. *The mean, expected value of the random variable X is given by*

$$E[X] = 1-\frac{2(a+1)}{(a+b+2)\overset{\omega}{R}}\times {}_4\overset{\omega}{R}_3\left(\begin{matrix}-\upsilon,\underline{\upsilon+\lambda},c,a+2\\ \underline{\alpha+1},p,a+b+3\end{matrix};1\right). \tag{3.17}$$

Proof. We have

$$\varphi_X(t) \triangleq E[e^{itX}] = E[e^{it(1-(1-X))}] = e^{it}\sum_{n=0}^{\infty}\frac{(-it)^n}{n!}E[(1-X)^n]; \tag{3.18}$$

using (1.8) and (3.11),

$$\begin{aligned}\varphi_X(t) &= \frac{e^{it}}{\overset{\omega}{R}}\sum_{n=0}^{\infty}\frac{(-2it)^n}{n!}\frac{(a+1)_n}{(a+b+2)_n}\,{}_4\overset{\omega}{R}_3\left(\begin{matrix}-\upsilon,\underline{\upsilon+\lambda},c,a+n+1\\ \underline{\alpha+1},p,a+n+b+2\end{matrix};1\right)\\ &= \frac{e^{it}}{\overset{\omega}{R}}\times\sum_{n,k=0}^{\infty}\frac{(a+1)_{n+k}}{(a+b+2)_{n+k}}\frac{(-2it)^n}{n!}\times B_k\\ &= \frac{e^{it}}{\overset{\omega}{R}}\times\overset{\omega}{R}{}^{1,3,0}_{1,2,0}\left(\begin{matrix}a+1;\ -\upsilon,\underline{\upsilon+\lambda},c;\ -\\ a+b+2;\ \underline{\alpha+1},p;\ -\end{matrix};1,-2it\right).\end{aligned} \tag{3.19}$$

Observing that, with $\tau = it$ in (3.13), we get moment generating function of X. Indeed

$$M_X(\tau) \triangleq E[e^{\tau X}] = \frac{e^{\tau}}{\overset{\omega}{R}}\times\overset{\omega}{R}{}^{1,3,0}_{1,2,0}\left(\begin{matrix}a+1;\ -\upsilon,\underline{\upsilon+\lambda},c;\ -\\ a+b+2;\ \underline{\alpha+1},p;\ -\end{matrix};1,-2\tau\right). \tag{3.20}$$

To obtain the rth moment, we notice that for positive integer r,

$$E[X^r] = E[(1-(1-X))^r] = \sum_{n=0}^{r}\binom{r}{n}(-1)^nE[(1-X)^n]; \tag{3.21}$$

using (3.11),

$$E[X^r] = \frac{1}{\overset{\omega}{R}}\sum_{n=0}^{r}\binom{r}{n}\frac{(-2)^n(a+1)_n}{(a+b+2)_n}\times{}_4\overset{\omega}{R}_3\left(\begin{matrix}-\upsilon,\underline{\upsilon+\lambda},c,a+n+1\\ \underline{\alpha+1},p,a+n+b+2\end{matrix};1\right). \tag{3.22}$$

Now since the mean, expected value of the random variable X is a special case of this moment, namely, the mean is the 1st moment,

$$E[X] = 1-E[1-X] = 1-\frac{2(a+1)}{(a+b+2)\overset{\omega}{R}}\times{}_4\overset{\omega}{R}_3\left(\begin{matrix}-\upsilon,\underline{\upsilon+\lambda},c,a+2\\ \underline{\alpha+1},p,a+b+3\end{matrix};1\right), \tag{3.23}$$

which completes the proof. □

Similarly, we can obtain the variance of the random variable X, σ_X^2, using (3.15) with $k = 2$, besides (3.17), since it is defined as

$$\sigma_X^2 \triangleq E[X^2]-(E[X])^2. \tag{3.24}$$

Remark 3.4. Letting $\omega = 1$ in the previous theorem, we get the results presented in [6].

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