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RATIONAL TORAL RANKS IN CERTAIN ALGEBRAS

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We calculate the rational toral ranks of two spaces whose cohomologies are isomorphic and note that rational toral rank is a rational homotopy invariant but not a cohomology invariant.

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1. Introduction. Let $\mathrm{rk}_0(Y)$ be the *rational toral rank* of a simply connected space Y, that is, the largest integer r such that an r-torus $T^r = S^1 \times \cdots \times S^1$ (r-factors) can act continuously on a CW-complex which has the rational homotopy type of Y with all its isotropy subgroups finite. For example, $\mathrm{rk}_0(Y) = 1$ if Y has the rational homotopy type of an odd-dimensional sphere S^{2n+1}.

Let $\mathbb{Q}$ be the field of the rational numbers. For a finite-dimensional $\mathbb{Q}$-commutative graded algebra A^* with $A^0 = \mathbb{Q}$ and $A^1 = 0$, we put

$$\mathfrak{M}_{A^*} = \{\text{rational homotopy type of } Y \mid H^*(Y;\mathbb{Q}) \cong A^*\},$$
$$\mathfrak{r}_{A^*} = \{\mathrm{rk}_0(Y) \mid H^*(Y;\mathbb{Q}) \cong A^*\}, \tag{1.1}$$

the set of rational toral ranks in $\mathfrak{M}_{A^*}$. For example, we see that if $A^* = A^{\text{even}}$, then the Euler characteristic is nonzero, so there must be fixed points; hence, $\mathfrak{r}_{A^*} = \{0\}$. Note that $\mathfrak{M}_{A^*}$ and $\mathfrak{r}_{A^*}$ are not empty sets since there exists the formal space whose cohomology is isomorphic to A^* (see below), and that $\mathfrak{r}_{A^*}$ is at most finite even if $\mathfrak{M}_{A^*}$ is infinite. In this paper, we calculate $\mathfrak{r}_{A^*}$ for certain commutative graded algebras A^*.

Theorem 1.1. *For the following four algebras A^*:*
(1) $A^* \cong H^*(S^2 \vee S^2 \vee S^5;\mathbb{Q})$,
(2) $A^* \cong H^*((S^3 \times S^8)\#(S^3 \times S^8);\mathbb{Q})$,
(3) $A^* \cong H^*((S^2 \vee S^2) \times S^3;\mathbb{Q})$,
(4) $A^* \cong H^*((S^2 \times S^5)\#(S^2 \times S^5);\mathbb{Q})$,
the rational toral ranks in $\mathfrak{M}_{A^*}$ are listed in Table 1.1, where $\mathfrak{M}_{A^*} = \{X, Y\}$ with a formal space X and a nonformal space Y.

Here $\vee$ and $\#$ denote a one point union (wedge) and a connected sum, respectively. For these A^*, we can check that $\mathfrak{M}_{A^*}$ is two points as in [5] or [6].

What do we know about the set $\mathfrak{r}_{A^*}$, namely, the function $\mathrm{rk}_0 : \mathfrak{M}_{A^*} \to \{0,1,2,\ldots\}$? For example, We consider the following questions.

Question 1.2. Suppose that A^* is a Poincaré duality algebra. Then, for $X, Y \in \mathfrak{M}_{A^*}$, is $\mathrm{rk}_0(X) \le \mathrm{rk}_0(Y)$ if X is formal?

TABLE 1.1. The rational toral ranks in $\mathfrak{M}_{A^*}$.

Algebra	$\mathrm{rk}_0(X)$	$\mathrm{rk}_0(Y)$
(1)	0	0
(2)	0	1
(3)	1	0
(4)	1	1

A simply connected space Y is called (rationally) elliptic if $\dim \pi_*(Y)\otimes\mathbb{Q} < \infty$ and $\dim H^*(Y;\mathbb{Q}) < \infty$.

QUESTION 1.3. For $X, Y \in \mathfrak{M}_{A^*}$, is $\mathrm{rk}_0(X) \le \mathrm{rk}_0(Y)$ if Y is elliptic?

QUESTION 1.4. Is $\mathfrak{r}_{A^*} = \{a, a+1, \dots, b-1, b\}$ for some integers $a \le b$? Namely, are there no gaps in the sequence of integers of $\mathfrak{r}_{A^*}$?

Notice that, for our examples, the answer is positive for these questions.

For the proof of Theorem 1.1, we use the *Sullivan minimal model* $M(Y)$ of a simply connected space Y of finite type. It is a free $\mathbb{Q}$-commutative differential graded algebra (d.g.a.) $(\wedge V, d)$ with a $\mathbb{Q}$-graded vector space $V = \bigoplus_{i>1} V^i$, where $\dim V^i < \infty$ and a minimal differential, that is, $d(V^i) \subset (\wedge^+ V \cdot \wedge^+ V)^{i+1}$ and $d\circ d = 0$. Here $\wedge V =$ (the $\mathbb{Q}$-polynomial algebra over V^{even}) $\otimes$ (the $\mathbb{Q}$-exterior algebra over V^{odd}) and $\wedge^+ V$ is the ideal of $\wedge V$ generated by elements of positive degree. Denote the degree of an element x of a graded algebra as $|x|$. Then $xy = (-1)^{|x||y|} yx$ and $d(xy) = d(x)y + (-1)^{|x|} x d(y)$. Notice that $M(Y)$ determines the rational homotopy type of Y. See [3] for a general introduction and notation: for example, for the notion of Koszul-Sullivan (KS) extension. Especially note that $H^*(M(Y)) \cong H^*(Y;\mathbb{Q})$ and a space Y is said to be *formal* if there is a d.g.a. map $M(Y) \to (H^*(Y;\mathbb{Q}), 0)$ which induces an isomorphism of cohomologies. The formal minimal model M_{A^*} is constructed by a free commutative resolution of the algebra A^* [5]. Throughout this paper, $\mathbb{Q}\langle x, y, \dots\rangle$ denotes the $\mathbb{Q}$-graded vector space generated by $\{x, y, \dots\}$.

2. Preliminaries. Let Y be a simply connected space of finite type with minimal model $M(Y) = (\wedge V, d)$. If an r-torus T^r acts on Y, there is a KS extension, with $|t_i| = 2$ for $i = 1, \dots, r$,

$$(\mathbb{Q}[t_1, \dots, t_r], 0) \longrightarrow (\mathbb{Q}[t_1, \dots, t_r] \otimes \wedge V, D) \longrightarrow (\wedge V, d), \tag{2.1}$$

which is induced from the Borel fibration [2]

$$Y \longrightarrow ET^r \times_{T^r} Y \longrightarrow BT^r. \tag{2.2}$$

In particular, the fact that (2.1) is a KS extension entails that, $Dt_i = 0$ and for $v \in V$, $Dv \equiv dv$ modulo the ideal $(t_1, \dots, t_r)$, that is,

$$Dv = dv + \sum_{i_1 + \cdots + i_r > 0} h_{i_1,\dots,i_r} t_1{}^{i_1} \cdots t_r{}^{i_r} \tag{2.3}$$

with $h_{i_1,\dots,i_r} \in \wedge V$. The differential D also satisfies $D\circ D = 0$.

Lemma 2.1 [4, Proposition 4.2]. *Suppose that* $\dim H^*(Y;\mathbb{Q})<\infty$. *Then,* $\mathrm{rk}_0(Y)\geq r$ *if and only if there is a KS extension (2.1) satisfying* $\dim H^*(\mathbb{Q}[t_1,\ldots,t_r]\otimes\wedge V,D)<\infty$.

So we may try to construct inductively for $1,\ldots,i$, the KS extensions:

$$(\mathbb{Q}[t_i],0)\longrightarrow(\mathbb{Q}[t_1,\ldots,t_i]\otimes\wedge V,D_i)\longrightarrow(\mathbb{Q}[t_1,\ldots,t_{i-1}]\otimes\wedge V,D_{i-1}) \tag{2.4}$$

satisfying $\dim H^*(\mathbb{Q}[t_1,\ldots,t_i]\otimes\wedge V,D)<\infty$ in general. In the following, we consider the particular case of $i=1$.

Lemma 2.2. *Suppose that* $H^{n+2}(\wedge V,d)=0$ *and* $H^n(\mathbb{Q}[t]\otimes\wedge V,D)=\mathbb{Q}\langle\gamma_1,\ldots,\gamma_m\rangle$. *Then,* $H^{n+2}(\mathbb{Q}[t]\otimes\wedge V,D)\subset\mathbb{Q}\langle\gamma_1t,\ldots,\gamma_mt\rangle$. *Moreover, if* $H^{n+1}(\wedge V,d)=0$, *then the inclusion is an equality.*

Proof. Let $\alpha+\alpha't$ be a D-cocycle in $(\mathbb{Q}[t]\otimes\wedge V)^{n+2}$ with $\alpha\in(\wedge V)^{n+2}$ and $\alpha'\in(\mathbb{Q}[t]\otimes\wedge V)^n$. Then we have $D\alpha=-D(\alpha')t$, and consequently, $d\alpha=0$.

Since $H^{n+2}(\wedge V,d)=0$, there is an element $\beta\in(\wedge V)^{n+1}$ such that $d\beta=\alpha$. Let $D\beta=\alpha+\alpha''t$ for some $\alpha''\in(\mathbb{Q}[t]\otimes\wedge V)^n$. Then, since

$$0=D^2\beta=D\alpha+D(\alpha'')t=-D(\alpha'-\alpha'')t, \tag{2.5}$$

we see that $\alpha'-\alpha''$ is a D-cocycle in $(\mathbb{Q}[t]\otimes\wedge V)^n$.

Since $H^n(\mathbb{Q}[t]\otimes\wedge V,D)=\mathbb{Q}\langle\gamma_1,\ldots,\gamma_m\rangle$, we can denote $\alpha'-\alpha''=c_1\gamma_1+\cdots+c_m\gamma_m+D\beta'$ for some $c_1,\ldots,c_m\in\mathbb{Q}$ and $\beta'\in(\mathbb{Q}[t]\otimes\wedge V)^{n-1}$. Then we have

$$\begin{aligned}\alpha+\alpha't&=\alpha+(\alpha''+c_1\gamma_1+\cdots+c_m\gamma_m+D\beta')t\\&=c_1\gamma_1t+\cdots+c_m\gamma_mt+D(\beta+\beta't).\end{aligned} \tag{2.6}$$

Hence $[\alpha+\alpha't]=[c_1\gamma_1t+\cdots+c_m\gamma_mt]$ in $H^{n+2}(\mathbb{Q}[t]\otimes\wedge V,D)$. Thus we have $H^{n+2}(\mathbb{Q}[t]\otimes\wedge V,D)\subset\mathbb{Q}\langle\gamma_1t,\ldots,\gamma_mt\rangle$.

Suppose that $c_1\gamma_1t+\cdots+c_m\gamma_mt=D(\eta+\eta't)$ for some $\eta\in(\wedge V)^{n+1}$ and $\eta'\in(\mathbb{Q}[t]\otimes\wedge V)^{n-1}$. Then we have $d\eta=0$ since $d\eta\notin\mathrm{Ideal}(t)$. If $H^{n+1}(\wedge V,d)=0$, there is an element $\theta\in(\wedge V)^n$ such that $d\theta=\eta$. Let $D\theta=\eta+\eta''t$ for some $\eta''\in(\mathbb{Q}[t]\otimes\wedge V)^{n-1}$. Then we have

$$(c_1\gamma_1+\cdots+c_m\gamma_m)t=D(\eta+\eta't)=D(D\theta-\eta''t+\eta't)=D(\eta'-\eta'')t. \tag{2.7}$$

However, $c_1\gamma_1+\cdots+c_m\gamma_m\notin\mathrm{Im}\,D$ unless $c_1=\cdots=c_m=0$. Thus, if $H^{n+1}(\wedge V,d)=0$, $\gamma_1t,\ldots,\gamma_mt$ are linearly independent in $H^{n+2}(\mathbb{Q}[t]\otimes\wedge V,D)$. ◻

A commutative graded algebra A^* with $\dim A^* < \infty$ will be said to *have formal dimension* n if $A^n \neq 0$ and $A^i = 0$ for all $i > n$. For example, the formal dimensions of (1), (2), (3), and (4) are 5, 11, 5, and 7, respectively.

Lemma 2.3 [4, Lemma 5.4]**.** *Suppose that $H^*(\wedge V, d)$ and $H^*(\mathbb{Q}[t] \otimes \wedge V, D)$ have formal dimensions n and n', respectively. Then $n' = n - 1$. If one algebra satisfies Poincaré duality, so does the other.*

From Lemma 2.1 the following corollary may be useful to estimate a rational toral rank to be nonzero.

Corollary 2.4. *Suppose that $H^*(\wedge V, d)$ has formal dimension n. Then,* $\dim H^*(\mathbb{Q}[t] \otimes \wedge V, D) < \infty$ *if and only if $H^n(\mathbb{Q}[t] \otimes \wedge V, D) = H^{n+1}(\mathbb{Q}[t] \otimes \wedge V, D) = 0$.*

Proof. The "if" part is proved as follows. Since $H^{n+2i}(\wedge V, d) = 0$ for $i > 0$, we have $H^{n+2i}(\mathbb{Q}[t] \otimes \wedge V, D) = 0$ for $i \geq 0$ from Lemma 2.2. Similarly, since $H^{n+2i-1}(\wedge V, d) = 0$ for $i > 0$, we have $H^{n+2i-1}(\mathbb{Q}[t] \otimes \wedge V, D) = 0$ for $i > 0$ from Lemma 2.2. Hence we have $H^{n+i}(\mathbb{Q}[t] \otimes \wedge V, D) = 0$ for $i \geq 0$, that is, $\dim H^*(\mathbb{Q}[t] \otimes \wedge V, D) < \infty$.

The "only if" part follows from Lemma 2.3. □

Proposition 2.5. *Suppose that $H^*(\wedge V, d)$ has formal dimension n and $(\wedge Z, D)$ is a minimal d.g.a. Then $H^*(\wedge Z, D)$ has formal dimension $n - 1$ and $Z^{\leq n} = \mathbb{Q}\langle t \rangle \oplus V^{\leq n}$ with $D \equiv d \bmod (t)$ on $V^{\leq n}$ if and only if $Z = \mathbb{Q}\langle t \rangle \oplus V$ and $D \equiv d \bmod (t)$, that is, there is a KS extension*

$$(\mathbb{Q}[t], 0) \longrightarrow (\wedge Z, D) = (\mathbb{Q}[t] \otimes \wedge V, D) \longrightarrow (\wedge V, d) \tag{2.8}$$

such that $\dim H^*(\mathbb{Q}[t] \otimes \wedge V, D) < \infty$*.*

Proof. The "if" part is obvious from Lemma 2.3.

Now we show the "only if" part. For some $k \geq n$, assume that $Z^{\leq k} = \mathbb{Q}\langle t \rangle \oplus V^{\leq k}$ with $Dv \equiv dv \bmod (t)$ for $v \in V^{\leq k}$. Then an element in $H^{k+2}(\wedge Z^{\leq k}, D)$ can be written using $[\alpha + \alpha' t]$ with $\alpha \in (\wedge V^{\leq k})^{k+2}$ and $\alpha' \in (\wedge Z^{\leq k})^k$. Since $D(\alpha + \alpha' t) = 0$, we have $d\alpha = 0$. Now we give a map

$$\rho_{k+1} : H^{k+2}(\wedge Z^{\leq k}, D) \longrightarrow H^{k+2}(\wedge V^{\leq k}, d) \tag{2.9}$$

where $\rho_{k+1}([\alpha + \alpha' t]) = [\alpha]$. It is well defined. Indeed, if $[\alpha_1 + \alpha_1' t] = [\alpha_2 + \alpha_2' t]$ in $H^{k+2}(\wedge Z^{\leq k}, D)$, then $\alpha_1 + \alpha_1' t = \alpha_2 + \alpha_2' t + D(\beta + \beta' t)$ for some $\beta \in (\wedge V^{\leq k})^{k+1}$ and $\beta' \in (\wedge Z^{\leq k})^{k-1}$. Let $D\beta = d\beta + \beta'' t$. Then we have

$$(\alpha_1 - \alpha_2) + (\alpha_1' - \alpha_2')t = d\beta + (\beta'' + D(\beta'))t. \tag{2.10}$$

So $\alpha_1 - \alpha_2 = d\beta$. Hence $[\alpha_1] = [\alpha_2]$ in $H^{k+2}(\wedge V^{\leq k}, d)$.

Since ρ_{k+1} is bijective, from the following paragraphs we see that $Z^{k+1} = V^{k+1}$ with $Dv \equiv dv \bmod (t)$ for $v \in V^{k+1}$ from the construction of minimal d.g.a.'s such that $H^{>k}(\wedge Z, D) = H^{>k}(\wedge V, d) = 0$. Thus we have inductively $Z = \mathbb{Q}\langle t \rangle \oplus V$ with $Dv \equiv dv \bmod (t)$ for $v \in V$.

Now we show that ρ_{k+1} is injective. Suppose that $\rho_{k+1}([\alpha+\alpha' t]) = [\alpha] = 0$. Then there is an element $\beta \in (\wedge V^{\le k})^{k+1}$ such that $d\beta = \alpha$. Let $D\beta = \alpha + \alpha'' t$. Since $D(\alpha + \alpha' t) = 0$ and $D(\alpha+\alpha'' t) = D^2\beta = 0$, we have $D(\alpha' - \alpha'') = 0$. Since $H^k(\wedge Z^{\le k}, D) = 0$, $\alpha' - \alpha'' = D\beta'$ for some $\beta' \in (\wedge Z^{\le k})^{k-1}$. Then we have

$$\alpha + \alpha' t = \alpha + (\alpha'' + D\beta')t = D(\beta + \beta' t). \tag{2.11}$$

Hence $[\alpha + \alpha' t] = 0$.

Now we show that ρ_{k+1} is surjective. Let $[\alpha] \in H^{k+2}(\wedge V^{\le k}, d)$. Since $d\alpha = 0$, we can denote $D\alpha = \gamma t$ with $\gamma \in (\wedge Z^{\le k})^{k+1}$. Since $H^{k+1}(\wedge Z^{\le k}, D) = 0$, $\gamma = D\eta$ for some $\eta \in (\wedge Z^{\le k})^k$. Then we have

$$D(\alpha - \eta t) = D\alpha - D(\eta)t = \gamma t - \gamma t = 0. \tag{2.12}$$

Hence there is an element $[\alpha - \eta t] \in H^{k+2}(\wedge Z^{\le k}, d)$ such that $f([\alpha - \eta t]) = [\alpha]$. □

From Lemma 2.1, we have the following.

Corollary 2.6. *Let $M(Y) = (\wedge V, d)$ with cohomology of formal dimension n. If there is a minimal d.g.a. $(\wedge Z, D)$ such that $H^*(\wedge Z, D)$ has formal dimension $n-1$ and $Z^{\le n} = \mathbb{Q}\langle t\rangle \oplus V^{\le n}$ with $D \equiv d \bmod (t)$ on $V^{\le n}$, then $M(ES^1 \times_{S^1} Y) \cong (\wedge Z, D)$, that is, $\mathrm{rk}_0(Y) \ge 1$.*

In the following, X is formal and Y is nonformal.

3. Examples

Example 3.1. Let $X = S^2 \vee S^2 \vee S^5$. Then $\chi_H(X) = \sum_i (-1)^i \dim H^i(X;\mathbb{Q}) = 2 > 0$. Recall

$$\chi_H(ES^1 \times_{S^1} X) = \chi_H(X) \cdot \chi_H(BS^1) \tag{3.1}$$

for a Borel fibration $X \to ES^1 \times_{S^1} X \to BS^1$. Since $\chi_H(BS^1) = \infty$ we have $\chi_H(ES^1 \times_{S^1} X) = \infty$, that is, $\dim H^*(ES^1 \times_{S^1} X;\mathbb{Q}) = \infty$. From Lemma 2.1, $\mathrm{rk}_0(X) = 0$. By the same argument, we have $\mathrm{rk}_0(Y) = 0$.

Note that $\chi_H(X) = \chi_H(Y) = 0$ in (2), (3), and (4).

Remark 3.2. Even if X is a wedge of spaces, $\mathrm{rk}_0(X)$ may not be zero. For example, $M(S^3 \vee S^3 \vee S^4) = (\wedge V, d) = (\wedge(x, y, z, \dots), d)$ with $|x| = |y| = 3$ and $|z| = 4$ and $dx = dy = dz = 0$. On the other hand, $M(S^2 \vee S^3)^{\le 4} = (\wedge Z, D)^{\le 4} = (\wedge(t, x, y, z), D)$ with $|t| = 2$, $Dt = Dx = 0$, $Dy = t^2$, and $Dz = xt$. From Corollary 2.6, we have $\mathrm{rk}_0(S^3 \vee S^3 \vee S^4) \ge 1$.

Example 3.3. Let $X = (S^3 \times S^8)\#(S^3 \times S^8)$. Then

$$A^* = H^*(X;\mathbb{Q}) = \frac{\wedge(x,y) \otimes \mathbb{Q}[w,u]}{(xy, xu, xw - yu, yw, w^2, wu, u^2)} \tag{3.2}$$

with $|x| = |y| = 3$, $|w| = |u| = 8$ and X has the minimal model

$$(\wedge V_X, d) = (\wedge(x, y, w, u, v_1, v_2, v_3, v_4, v_5, v_6, v_7, z_1, \dots), d) \tag{3.3}$$

with $|v_1| = 5$, $|v_2| = |v_3| = |v_4| = 10$, $|v_5| = |v_6| = |v_7| = 15$, $|z_1| = 7$ and $dx = dy = dw = du = 0$, $dv_1 = xy$, $dv_2 = xu$, $dv_3 = xw - yu$, $dv_4 = yw$, $dv_5 = w^2$, $dv_6 = wu$, $dv_7 = u^2$, $dz_1 = xv_1, \dots$.

From $D \circ D = 0$, we have $Dx = Dy = 0$, $Du = \lambda xt^3$, and $Dw = -\lambda yt^3$ for $\lambda \in \mathbb{Q}$. Assume $\dim H^*(\mathbb{Q}[t] \otimes \wedge V_X, D) < \infty$. From Lemma 2.3, $\lambda \neq 0$. Let $Dv_1 = xy + at^3$ for $a \in \mathbb{Q}$ and $Dz_1 = xv_1 + ht$ for $h \in (\mathbb{Q}[t] \otimes \wedge V_X, D)^6$. Then $0 = D^2 z_1 = -axt^3 + D(h)t$. But there is no element h such that $Dh = axt^2$. Hence we have $a = 0$. Since $H^*(X;\mathbb{Q})$ satisfies Poincaré duality with formal dimension 11, so does $H^*(\mathbb{Q}[t] \otimes \wedge V_X, D)$ with formal dimension 10 from Lemma 2.3. Since $H^3(\mathbb{Q}[y] \otimes \wedge V_X, D) = \mathbb{Q}\langle x, y\rangle$ and $H^i(\wedge V_X, d) = 0$ for $4 \le i \le 7$, we have $H^7(\mathbb{Q}[t] \otimes \wedge V_X, D) = \mathbb{Q}\langle xt^2, yt^2\rangle$ from Lemma 2.2. But

$$x \cdot xt^2 = x \cdot yt^2 = 0 \tag{3.4}$$

in $H^{10}(\mathbb{Q}[t] \otimes \wedge V_X, D)$ since $a = 0$. This contradicts Poincaré duality. Thus $\dim H^*(\mathbb{Q}[t] \otimes \wedge V_X, D) = \infty$. From Lemma 2.1, we have $\mathrm{rk}_0(X) = 0$.

Let $M(Y) = (\wedge V_Y, d) = (\wedge(x, y, z), d)$ with $|x| = |y| = 3$, $|z| = 5$ and $dx = dy = 0$, $dz = xy$. Then $H^*(Y;\mathbb{Q}) \cong A^*$.

Put $Dx = Dy = 0$ and $Dz = xy + t^3$. Then $\dim H^*(\mathbb{Q}[t] \otimes \wedge V_Y, D) < \infty$. From Lemma 2.1, we have $\mathrm{rk}_0(Y) \ge 1$. Also for any D, we have $Dx = Dy = 0$. Thus $\dim H^*(\mathbb{Q}[t_1, t_2] \otimes \wedge V_Y, D) = \infty$. From the case of $r = 2$ in Lemma 2.1, we have $\mathrm{rk}_0(Y) = 1$.

Example 3.4. Let $X = (S^2 \vee S^2) \times S^3$. Then $A^* = H^*(X;\mathbb{Q}) = \mathbb{Q}[x_1, x_2] \otimes \wedge(y)/(x_1^2, x_1x_2, x_2^2)$ with $|x_i| = 2$, $|y| = 3$. When $D = d$, except for $Dy = t^2$, $(\mathbb{Q}[t] \otimes \wedge V_X, D)$ is the minimal model of $(S^2 \vee S^2) \times S^2$. Hence $\mathrm{rk}_0(X) \ge 1$. In general, if $Dy = 0$, $[x_i y] \neq 0 \in H^5(\mathbb{Q}[t] \otimes \wedge V_X, D)$, then $\dim H^*(\mathbb{Q}[t] \otimes \wedge V_X, D) = \infty$ from Lemma 2.2. If $Dy \neq 0$, $H^{\mathrm{odd}}(\mathbb{Q}[t] \otimes \wedge V_X, D) = 0$ from Lemma 2.3. In each case, $\dim H^*(\mathbb{Q}[t_1, t_2] \otimes \wedge V_X, D)$ cannot be finite. From the case of $r = 2$ in Lemma 2.1, we have $\mathrm{rk}_0(X) = 1$.

Let Y be the nonformal space with $H^*(Y;\mathbb{Q}) \cong A^*$. Then $M(Y) = (\wedge V_Y, d)$ is given by

$$V_Y{}^{\le 5} = \mathbb{Q}\langle x_1, x_2, y, z_1, z_2, z_3, u_1, u_2, v_1, v_2, v_3\rangle \tag{3.5}$$

with $|x_i| = 2$, $|y| = |z_i| = 3$, $|u_i| = 4$, $|v_i| = 5$ and $dx_1 = dx_2 = dy = 0$, $dz_1 = x_1^2$, $dz_2 = x_1x_2$, $dz_3 = x_2^2$, $du_1 = x_1z_2 - x_2z_1$, $du_2 = x_1z_3 - x_2z_2 - x_2y$, $dv_1 = x_1u_1 - z_1z_2$, $dv_2 = x_1u_2 + x_2u_1 - z_1z_3 + z_2y$, $dv_3 = x_2u_2 - z_2z_3 + z_3y$. Here $H^5(\wedge V_Y, d) = \mathbb{Q}\langle x_1y, x_2y\rangle$.

Now we show that $t^3 \neq 0$ in $H^6(\mathbb{Q}[t] \otimes \wedge V_Y, D)$. Let $Dx_1 = Dx_2 = 0$, $Dy = ax_1t + bx_2t + ct^2$ for $a, b, c \in \mathbb{Q}$ and $Dz_i = dz_i + a_ix_1t + b_ix_2t + c_it^2$ for $a_i, b_i, c_i \in \mathbb{Q}$. Assume that $t^3 = D(px_1y + qx_2y + eyt + fz_1t + gz_2t + hz_3t)$ for some $p, q, e, f, g, h \in \mathbb{Q}$. Since the right-hand side is equal to

$$\begin{aligned}&(pa + f)x_1{}^2t + (pb + qa + g)x_1x_2t + (qb + h)x_2{}^2t \\ &\quad + (pc + ea + fa_1 + ga_2 + ha_3)x_1t^2 + (qc + eb + fb_1 + gb_2 + hb_3)x_2t^2 \\ &\quad + (ec + fc_1 + gc_2 + hc_3)t^3,\end{aligned} \tag{3.6}$$

we have

$$\begin{aligned} pc+ea-paa_1-pba_2-qaa_2-qba_3&=0,\\ qc+eb-pab_1-pbb_2-qab_2-qbb_3&=0,\\ ec-pac_1-pbc_2-qac_2-qbc_3&=1. \end{aligned} \tag{3.7}$$

On the other hand, let $Du_i = du_i + e_i yt + f_i z_1 t + g_i z_2 t + h_i z_3 t$ for $e_i, f_i, g_i, h_i \in \mathbb{Q}$ and $Dv_i = dv_i + l_i u_1 t + m_i u_2 t$ for $l_i, m_i \in \mathbb{Q}$. Since

$$\begin{aligned}
0 &= D^2u_1\\
&= (a_2+f_1){x_1}^2t+(b_2-a_1+g_1)x_1x_2t+(-b_1+h_1){x_2}^2t\\
&\quad+(c_2+e_1a+f_1a_1+g_1a_2+h_1a_3)x_1t^2\\
&\quad+(-c_1+e_1b+f_1b_1+g_1b_2+h_1b_3)x_2t^2\\
&\quad+(e_1c+f_1c_1+g_1c_2+h_1c_3)t^3,\\
0 &= D^2u_2\\
&= (a_3+f_2){x_1}^2t+(b_3-a_2-a+g_2)x_1x_2t+(-b_2-b+h_2){x_2}^2t\\
&\quad+(c_3+e_2a+f_2a_1+g_2a_2+h_2a_3)x_1t^2\\
&\quad+(-c_2-c+e_2b+f_2b_1+g_2b_2+h_2b_3)x_2t^2\\
&\quad+(e_2c+f_2c_1+g_2c_2+h_2c_3)t^3,\\
0 &= D^2v_1\\
&= e_1x_1yt+(f_1+a_2)x_1z_1t+(g_1-a_1+l_1)x_1z_2t+(h_1+m_1)x_1z_3t\\
&\quad-m_1x_2yt+(b_2-l_1)x_2z_1t+(-b_1-m_1)x_2z_2t\\
&\quad+(l_1e_1+m_1e_2)yt^2+(c_2+l_1f_1+m_1f_2)z_1t^2\\
&\quad+(-c_1+l_1g_1+m_1g_2)z_2t^2+(l_1h_1+m_1h_2)z_3t^2,\\
0 &= D^2v_2\\
&= (e_2+a_2)x_1yt+(f_2+a_3)x_1z_1t\\
&\quad+(g_2-a+l_2)x_1z_2t+(h_2-a_1+m_2)x_1z_3t\\
&\quad+(e_1+b_2-m_2)x_2yt+(f_1+b_3-l_2)x_2z_1t\\
&\quad+(g_1-b-m_2)x_2z_2t+(h_1-b_1)x_2z_3t\\
&\quad+(c_2+l_2e_1+m_2e_2)yt^2+(c_3+l_2f_1+m_2f_2)z_1t^2\\
&\quad+(-c+l_2g_1+m_2g_2)z_2t^2+(-c_1+l_2h_1+m_2h_2)z_3t^2,\\
0 &= D^2v_3\\
&= a_3x_1yt+(a_3+l_3)x_1z_2t+(-a_2-a+m_3)x_1z_3t\\
&\quad+(e_2+b_3-m_3)x_2yt+(f_2-l_3)x_2z_1t\\
&\quad+(g_2+b_3-m_3)x_2z_2t+(h_2-b_2-b)x_2z_3t\\
&\quad+(c_3+l_3e_1+m_3e_2)yt^2+(l_3f_1+m_3f_2)z_1t^2\\
&\quad+(c_3+l_3g_1+m_3g_2)z_2t^2+(-c_2-c+l_3h_1+m_3h_2)z_3t^2,
\end{aligned} \tag{3.8}$$

we have

$$\begin{aligned} a &= -2a_2+b_3, \qquad b = a_1-2b_2, \qquad c = -a_1a_2+a_1b_3-b_2b_3, \\ a_3 &= b_1 = 0, \qquad c_1 = (a_1-b_2)b_2, \qquad c_2 = a_2b_2, \qquad c_3 = -(a_2-b_3)a_2. \end{aligned} \tag{3.9}$$

Hence (3.7) will be

$$(-2a_2+b_3)(e-pb_2-qa_2) = 0, \tag{3.10}$$
$$(a_1-2b_2)(e-pb_2-qa_2) = 0, \tag{3.11}$$
$$(-a_1a_2+a_1b_3-b_2b_3)(e-pb_2-qa_2) = 1, \tag{3.12}$$

respectively. By (3.12), $e-pb_2-qa_2 \neq 0$ and $-a_1a_2+a_1b_3-b_2b_3 \neq 0$. Then, by (3.10) and (3.11), $b_3 = 2a_2$ and $a_1 = 2b_2$, respectively. But this contradicts $-a_1a_2+a_1b_3-b_2b_3 \neq 0$. Thus $t^3 \neq 0$ in $H^6(\mathbb{Q}[t]\otimes\wedge V_Y, D)$.

Since $H^*(\wedge V_Y, d)$ has formal dimension 5, from Lemma 2.3, we have $\dim H^*(\mathbb{Q}[t]\otimes\wedge V_Y, D) = \infty$. From Lemma 2.1, we have $\mathrm{rk}_0(Y) = 0$.

Example 3.5. Let $X = (S^2\times S^5)\#(S^2\times S^5)$. Then

$$A^* = H^*(X;\mathbb{Q}) = \frac{\mathbb{Q}[x_1,x_2]\otimes\wedge(y_1,y_2)}{(x_1^2,x_1x_2,x_2^2,x_1y_1-x_2y_2,x_1y_2,x_2y_1,y_1y_2)} \tag{3.13}$$

with $|x_i| = 2$, $|y_i| = 5$ and X has a minimal model $M(X) = M_{A^*} = (\wedge V_X, d)$ where

$$V_X{}^{\leq 7} = \mathbb{Q}\langle x_1,x_2,z_1,z_2,z_3,u_1,u_2,y_1,y_2,v_1,v_2,v_3,w_1,\dots,w_9,s_1,\dots,s_{18}\rangle \tag{3.14}$$

with $|x_i| = 2$, $|z_i| = 3$, $|u_i| = 4$, $|y_i| = |v_i| = 5$, $|w_i| = 6$, $|s_i| = 7$ and

$$\begin{gathered} dx_1 = dx_2 = dy_1 = dy_2 = 0, \\ dz_1 = x_1^2, \qquad dz_2 = x_1x_2, \qquad dz_3 = x_2^2, \\ du_1 = x_1z_2-x_2z_1, \qquad du_2 = x_1z_3-x_2z_2, \\ dv_1 = x_1u_1-z_1z_2, \qquad dv_2 = x_1u_2+x_2u_1-z_1z_3, \qquad dv_3 = x_2u_2-z_2z_3, \\ dw_1 = x_1y_1-x_2y_2, \qquad dw_2 = x_1y_2, \qquad dw_3 = x_2y_1, \\ dw_4 = x_1v_1-z_1u_1, \qquad dw_5 = x_1v_2-z_1u_2-z_2u_1, \qquad dw_6 = x_1v_3-z_2u_2, \\ dw_7 = x_2v_1-z_2u_1, \qquad dw_8 = x_2v_2-z_2u_2-z_3u_1, \qquad dw_9 = x_2v_3-z_3u_2, \\ ds_1 = x_1w_1-z_1y_1+z_2y_2, \qquad ds_2 = x_1w_2-z_1y_2, \qquad ds_3 = x_1w_3-z_2y_1, \\ ds_4 = x_1w_4-z_1v_1, \qquad ds_5 = x_1w_5-z_1v_2+\frac{1}{2}u_1^2, \end{gathered}$$

$$ds_6 = x_1w_6 + x_1w_8 - z_1v_3 - z_2v_2 + u_1u_2, \qquad ds_7 = x_1w_7 - x_2w_4 + \frac{1}{2}u_1^2,$$
$$ds_8 = x_1w_8 - x_2w_5 + u_1u_2, \qquad ds_9 = x_1w_9 - x_2w_6 + \frac{1}{2}u_2^2,$$
$$ds_{10} = x_2w_1 - z_2y_1 + z_3y_2, \qquad ds_{11} = x_2w_2 - z_2y_2, \qquad ds_{12} = x_2w_3 - z_3y_1,$$
$$ds_{13} = x_2w_4 - z_2v_1 - \frac{1}{2}u_1^2, \qquad ds_{14} = x_2w_5 + x_2w_7 - z_2v_2 - z_3v_1 - u_1u_2,$$
$$ds_{15} = x_2w_6 - z_2v_3, \qquad ds_{16} = x_2w_7 - x_1w_6 + z_1v_3 - z_3v_1 - u_1u_2,$$
$$ds_{17} = x_2w_8 - z_3v_2 - \frac{1}{2}u_2^2, \qquad ds_{18} = x_2w_9 - z_3v_3. \tag{3.15}$$

Let $(\wedge Z, D)$ be the formal minimal model M_{B^*} for the Poincaré duality algebra

$$B^* = \frac{\mathbb{Q}[t, x_1, x_2]}{(x_1t^2, x_2t^2, x_1^2 + x_2t, x_1x_2 - t^2, x_2^2 + x_1t)} \tag{3.16}$$

with $|t| = |x_i| = 2$. Note B^* has formal dimension 6. Then

$$Z^{\leq 7} = \mathbb{Q}\langle t\rangle \oplus {V_X}^{\leq 7} \tag{3.17}$$

with

$$Dt = Dx_1 = Dx_2 = 0, \qquad Dy_1 = x_2t^2, \qquad Dy_2 = x_1t^2,$$
$$Dz_1 = dz_1 + x_2t, \qquad Dz_2 = dz_2 - t^2, \qquad Dz_3 = dz_3 + x_1t,$$
$$Du_1 = du_1 + z_3t, \qquad Du_2 = du_2 - z_1t,$$
$$Dv_1 = dv_1 - u_2t, \qquad Dv_2 = dv_2, \qquad Dv_3 = dv_3 - u_1t,$$
$$Dw_1 = dw_1, \qquad Dw_2 = dw_2 + y_1t - z_1t^2, \qquad Dw_3 = dw_3 + y_2t - z_3t^2,$$
$$Dw_4 = dw_4 + v_2t, \qquad Dw_5 = dw_5 + v_3t, \qquad Dw_6 = dw_6 + v_1t,$$
$$Dw_7 = dw_7 + v_3t, \qquad Dw_8 = dw_8 + v_1t, \qquad Dw_9 = dw_9 + v_2t,$$
$$Ds_1 = ds_1 + w_3t + u_1t^2, \qquad Ds_2 = ds_2 - w_1t, \qquad Ds_3 = ds_3 - w_2t + u_2t^2,$$
$$Ds_4 = ds_4 - w_5t + w_7t, \qquad Ds_5 = ds_5 - w_6t + w_8t, \qquad Ds_6 = ds_6 - 2w_4t + w_9t,$$
$$Ds_7 = ds_7 - w_6t + w_8t, \qquad Ds_8 = ds_8 - w_4t + w_9t, \qquad Ds_9 = ds_9 - w_5t + w_7t,$$
$$Ds_{10} = ds_{10} - w_2t + u_2t^2, \qquad Ds_{11} = ds_{11} - w_3t - u_1t^2, \qquad Ds_{12} = ds_{12} + w_1t,$$
$$Ds_{13} = ds_{13} - w_8t, \qquad Ds_{14} = ds_{14} + w_4t - 2w_9t, \qquad Ds_{15} = ds_{15} - w_7t,$$
$$Ds_{16} = ds_{16} + 2w_4t - 2w_9t, \qquad Ds_{17} = ds_{17} + w_5t - w_7t, \qquad Ds_{18} = ds_{18} + w_6t - w_8t, \tag{3.18}$$

that is, $D \equiv d \bmod (t)$ on ${V_X}^{\leq 7}$. From Corollary 2.6, we have $\mathrm{rk}_0(X) \geq 1$. Also for any D satisfying $\dim H^*(\mathbb{Q}[t] \otimes \wedge V_X, D) < \infty$, we see $H^{\mathrm{odd}}(\mathbb{Q}[t] \otimes \wedge V_X, D) = 0$ from Lemma 2.3. From the case of $r = 2$ in Lemma 2.1, we have $\mathrm{rk}_0(X) = 1$.

Let $M(Y) = (\wedge V_Y, d) = (\wedge(x_1, x_2, z_1, z_2, z_3), d)$ with $|x_i| = 2$, $|z_i| = 3$ and $dx_1 = dx_2 = 0$, $dz_1 = x_1^2$, $dz_2 = x_1x_2$, $dz_3 = x_2^2$. Then $H^*(Y; \mathbb{Q}) \cong A^*$.

Put $D = d$ except for $Dz_2 = x_1x_2 - t^2$. Then we have $\dim H^*(\mathbb{Q}[t] \otimes \wedge V_Y, D) < \infty$. From the case of $r = 1$ in Lemma 2.1, $\mathrm{rk}_0(Y) \geq 1$. From [1], we have $\mathrm{rk}_0(Y) = 1$. Indeed,

$$\mathrm{rk}_0(Y) \leq -\chi_\pi(Y) = -\sum_i (-1)^i \dim \pi_i(Y) \otimes \mathbb{Q} = \dim V_Y^{\mathrm{odd}} - \dim V_Y^{\mathrm{even}} = 1. \tag{3.19}$$

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