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POWERS OF A PRODUCT OF COMMUTATORS AS PRODUCTS OF SQUARES

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We prove that for any odd integer N and any integer $n > 0$, the Nth power of a product of n commutators in a nonabelian free group of countable infinite rank can be expressed as a product of squares of $2n+1$ elements and, for all such odd N and integers n, there are commutators for which the number $2n+1$ of squares is the minimum number such that the Nth power of its product can be written as a product of squares. This generalizes a recent result of Akhavan-Malayeri.

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1. Introduction. Lyndon et al. [2] have shown that the product of n commutators in a nonabelian free group can be written as a product of $2n+1$ squares of elements and there are commutators for which the number $2n+1$ of squares is the minimum number such that the product of these commutators can be written as a product of squares. Recently, Akhavan-Malayeri [1] proved, for an odd integer n, that $[x,y]^n$ of two distinct elements of a free generating set of a nonabelian free group is not a product of two squares but it is the product of three squares. We generalize these results in the following theorem.

Theorem 1.1. *Let F be a free group with a basis of distinct elements $x_1,\dots,x_{2n}$, and N any odd integer. Then there exist elements $u_1,\dots,u_m$ in F such that*

$$([x_1,x_2]\cdots[x_{2n-1},x_{2n}])^N = u_1^2\cdots u_m^2 \tag{1.1}$$

if and only if $m \geq 2n+1$.

Note that the theorem for even N is not true since the element in the left-hand side of the above equation is actually a square. The proof of this theorem is almost *mutatis mutandis* as the proof of the main result of [2]. Throughout this note, $[x,y] = x^{-1}y^{-1}xy$ and $[x,y,z] = [[x,y],z]$ for all elements x, y, z of a group G, and G' denotes the derived subgroup of G.

2. Proof of the main result

Proof of Theorem 1.1. We show first that this equation has a solution for $m = 2n+1$, hence trivially for $m \geq 2n+1$. Since N is odd, there is an integer k such that $N = 2k+1$. Thus it is enough to show that, for any element v of F, we can express the element $v^2[x_1,x_2]\cdots[x_{2n-1},x_{2n}]$ as a product of $2n+1$ squares. We argue by induction on n. If $n=1$, then by the following well-known identity this case is proved:

$$A^2[B,C]=\left(A^2B^{-1}A^{-1}\right)^2\left(ABA^{-1}C^{-1}A^{-1}\right)^2(AC)^2. \tag{2.1}$$

Assume $n>1$ and suppose inductively that

$$v^2[x_1,x_2]\cdots[x_{2n-3},x_{2n-2}]=u_1^2\cdots u_{2n-1}^2 \tag{2.2}$$

for some elements $u_1,\dots,u_{2n-1}$ in F. Now by the identity (2.1) we can write

$$u_{2n-1}^2[x_{2n-1},x_{2n}]=U^2V^2W^2 \tag{2.3}$$

for some elements U, V, and W in F, and so

$$v^2[x_1,x_2]\cdots[x_{2n-1},x_{2n}]=u_1^2\cdots u_{2n-2}^2U^2V^2W^2, \tag{2.4}$$

which completes the induction. This first part of the proof is essentially well known in a topological context: the nonorientable surface formed by attaching one cross-cap and n handles to a sphere (the connected sum of 1 projective plane and n tori) is homeomorphic to the surface obtained by attaching $2n+1$ cross-caps (the connected sum of $2n+1$ projective planes). In this context, the identity (2.1) is just the handle calculus that says cross-cap + handle = 3 cross-caps.

For the converse, we suppose that the equation holds. Let G be the group with the following presentation:

$$\langle y_i \mid y_i^2=[y_i,y_j,y_k]=1\ \forall i,j,k\in\{1,2,\dots,2n\}\rangle. \tag{2.5}$$

The equation would also hold in G since G is a quotient of F. So we have

$$\left([y_1,y_2]\cdots[y_{2n-1},y_{2n}]\right)^N=v_1^2\cdots v_m^2 \tag{2.6}$$

for some elements $v_1,\dots,v_m$ in G. Since N is odd, $N=2t+1$ for some integer t. Since G is nilpotent of class 2 and $y_i^2=1$ for each i, we have $[y_i,y_j]^2=1$ and all the commutators are in the center of G, so the latter equation can be rewritten as

$$[y_1,y_2]\cdots[y_{2n-1},y_{2n}]=v_1^2\cdots v_m^2. \tag{2.7}$$

Let $c_{ij}=[y_i,y_j]$. Then each element v of G has a unique expression

$$v=y_1^{a_1}\cdots y_{2n}^{a_{2n}}\prod_{i<j}c_{ij}^{d_{ij}}\quad \text{for } a_i,d_{ij}\in\mathbb{Z}_2. \tag{2.8}$$

Let

$$v_k=y_1^{a_{1k}}\cdots y_{2n}^{a_{2nk}}z_k, \tag{2.9}$$

where $a_{ik}\in\mathbb{Z}_2$ and $z_k\in G'$ for all $i\in\{1,\dots,2n\}$ and all $k\in\{1,\dots,m\}$. Since $z_k^2=1$ for all k, we have

$$v_1^2\cdots v_m^2=\prod_{i<j}c_{ij}^{\sum_{k=1}^m a_{ik}a_{jk}}. \tag{2.10}$$

If A is the matrix $A = (a_{ij})$ over $\mathbb{Z}_2$, and $A_i = (a_{i1},\ldots,a_{im})$ is the ith row of A, then from the relation $v_1^2\cdots v_m^2 = [y_1,y_2]\cdots[y_{2n-1},y_{2n}]$ we conclude that, taking inner products,

$$A_i\cdot A_j = \begin{cases} 1 & \text{if } \{i,j\} = \{2h-1,2h\} \text{ for } 1 \le h \le n, \\ 0 & \text{otherwise.} \end{cases} \tag{2.11}$$

We conclude that $A\cdot A^T = B$, where A^T is the transpose of A, and B is the direct sum of n matrices of the form $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, and hence has rank $2n$. It follows that $\operatorname{rank}(A) \ge 2n$. But the equation $A_i\cdot A_i = \sum_{j=1}^m a_{ij}a_{ij} = 0$ for each i implies that the sum of the columns of A is 0, whence $\operatorname{rank}(A) \le m-1$. Therefore $m-1 \ge 2n$. □

References

[1] M. Akhavan-Malayeri, *Powers of commutators as products of squares*, Int. J. Math. Math. Sci. **31** (2002), no. 10, 635–637.

[2] R. Lyndon, T. McDonough, and M. Newman, *On products of powers in groups*, Proc. Amer. Math. Soc. **40** (1973), 419–420.

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