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ESSENTIAL NORM OF WEIGHTED COMPOSITION OPERATOR BETWEEN α-BLOCH SPACE AND β-BLOCH SPACE IN POLYDISCS

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Let $\varphi(z) = (\varphi_1(z),\ldots,\varphi_n(z))$ be a holomorphic self-map of $\mathbb{D}^n$ and $\psi(z)$ a holomorphic function on $\mathbb{D}^n$, where $\mathbb{D}^n$ is the unit polydiscs of $\mathbb{C}^n$. Let $0 < \alpha$, $\beta < 1$, we compute the essential norm of a weighted composition operator ψC_φ between α-Bloch space $\mathcal{B}^\alpha(\mathbb{D}^n)$ and β-Bloch space $\mathcal{B}^\beta(\mathbb{D}^n)$.

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1. Introduction. Let $\mathbb{D}^n$ be the unit polydiscs of $\mathbb{C}^n$, the class of all holomorphic functions with domain $\mathbb{D}^n$ will be denoted by $H(\mathbb{D}^n)$. Let φ be a holomorphic self-map of $\mathbb{D}^n$, the composition operator C_φ induced by φ is defined by $(C_\varphi f)(z) = f(\varphi(z))$ for z in $\mathbb{D}^n$ and $f \in H(\mathbb{D}^n)$. If, in addition, ψ is a holomorphic function defined on $\mathbb{D}^n$, the weighted composition operators ψC_φ induced by ψ and φ is defined by $\psi C_\varphi(z) = \psi(z)f(\varphi(z))$ for z in $\mathbb{D}^n$ and $f \in H(\mathbb{D}^n)$.

Let $0 < \alpha < 1$, a function f holomorphic in $\mathbb{D}^n$ is said to belong to the α-Bloch space $\mathcal{B}^\alpha(\mathbb{D}^n)$ if

$$\|f\|_\alpha = |f(0)| + \sup_{z\in\mathbb{D}^n} \sum_{k=1}^{n} \left|\frac{\partial f}{\partial z_k}(z)\right| \left(1-|z_k|^2\right)^\alpha < +\infty. \tag{1.1}$$

It is easy to show that $\mathcal{B}^\alpha(\mathbb{D}^n)$ is a Banach space with the norm $\|\cdot\|_\alpha$. These spaces are called Lipschitz space $\mathrm{Lip}_\alpha(\mathbb{D}^n)$ by Zhou (see [6, 8]). It is easy to show that the usual norm on $\mathrm{Lip}_{1-\alpha}(\mathbb{D}^n)$ defined by

$$\|f\|_{\mathrm{Lip}} = |f(0)| + \sup_{z,w\in\mathbb{D}^n} \frac{|f(z)-f(w)|}{|z-w|^{1-\alpha}} \tag{1.2}$$

induces a Banach space structure on $\mathrm{Lip}_{1-\alpha}(\mathbb{D}^n)$. Clahane in [1] has shown that this norm is equivalent to $\|\cdot\|_\alpha$.

Essential norm formulas for composition operators are known in various settings. Shapiro has given a formula for $\|C_\varphi\|_e$ when C_φ acting on the Hardy space $H^2(\mathbb{D})$ in [5]; Montes-Rodríguez [4] has given the essential norm of composition operator on the Bloch space in the unit disc; Donaway [2] has given upper and lower estimates for $\|C_\varphi\|_e$ when C_φ maps the Bloch space, the Dirichlet space, or the Besov-p space to itself; MacCluer and Zhao [3] have given an exact formula of essential norm of weighted composition operator between the Bloch-type spaces in the unit disc, namely,

$$\|uC_\varphi\|_e = \lim_{s\to 1} \sup_{|\varphi(z)|>s} |u(z)||\varphi'(z)| \frac{(1-|z|^2)^\beta}{\left(1-|\varphi(z)|^2\right)^\alpha}. \tag{1.3}$$

Here, φ is an analytic self-map of $\mathbb{D}$ and u is a fixed analytic function on $\mathbb{D}$ and $0<\alpha<1$, $0<\beta<\infty$; Zhou and Shi [7] have given the essential norm of composition operator on the Bloch space in polydiscs, that is,

$$\begin{aligned} &\frac{1}{n^2}\lim_{\delta\to 0}\sup_{\operatorname{dist}(\varphi(z),\partial\mathbb{D}^n)<\delta}\sum_{k,l=1}^{n}\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|\frac{1-|z_k|^2}{1-|\varphi_l(z)|^2}\\ &\quad\le \|C_\varphi\|_e \le 2n^2\lim_{\delta\to 0}\sup_{\operatorname{dist}(\varphi(z),\partial\mathbb{D}^n)<\delta}\sum_{k,l=1}^{n}\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|\frac{1-|z_k|^2}{1-|\varphi_l(z)|^2}. \end{aligned} \tag{1.4}$$

Recently, Zhou [6] studied weighted composition operators between α-Bloch space and β-Bloch space in polydiscs. He proved the following theorems.

Theorem 1.1. *Let $\varphi=(\varphi_1,\dots,\varphi_n)$ be a holomorphic self-map of $\mathbb{D}^n$ and $\psi(z)$ a holomorphic function of $\mathbb{D}^n$, $0<\alpha,\beta<1$. Then, $\psi C_\varphi:\mathcal{B}^\alpha(\mathbb{D}^n)\to\mathcal{B}^\beta(\mathbb{D}^n)$ is bounded if and only if $\psi\in\mathcal{B}^\beta(\mathbb{D}^n)$ and*

$$|\psi(z)|\sum_{k,l=1}^{n}\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|\frac{\left(1-|z_k|^2\right)^\beta}{\left(1-|\varphi_l(z)|^2\right)^\alpha} = O(1) \quad (|z|\longrightarrow 1^-). \tag{1.5}$$

Theorem 1.2. *Let $\varphi=(\varphi_1,\dots,\varphi_n)$ be a holomorphic function of $\mathbb{D}^n$ and $\psi(z)$ a holomorphic function of $\mathbb{D}^n$, $0<\alpha,\beta<1$. Then, $\psi C_\varphi:\mathcal{B}^\alpha(\mathbb{D}^n)\to\mathcal{B}^\beta(\mathbb{D}^n)$ is compact if and only if*

(i) *ψC_φ is bounded,*

(ii)

$$\sum_{k=1}^{n}\left|\frac{\partial\psi}{\partial z_k}(z)\right|\left(1-|z_k|^2\right)^\beta = o(1), \quad (\varphi(z)\longrightarrow\partial\mathbb{D}^n), \tag{1.6}$$

(iii)

$$|\psi(z)|\sum_{k,l=1}^{n}\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|\frac{\left(1-|z_k|^2\right)^\beta}{\left(1-|\varphi_l(z)|^2\right)^\alpha} = o(1), \quad (\varphi(z)\longrightarrow\partial\mathbb{D}^n). \tag{1.7}$$

It is reasonable to expect that the essential norm of $\psi C_\varphi:\mathcal{B}^\alpha(\mathbb{D}^n)\to\mathcal{B}^\beta(\mathbb{D}^n)$ should be given by the related lim sup expression. The following theorem is our main result.

Theorem 1.3. *Let $\varphi = (\varphi_1,\dots,\varphi_n)$ be a holomorphic self-map of $\mathbb{D}^n$ and $\psi(z)$ a holomorphic function of $\mathbb{D}^n$, $0 < \alpha, \beta < 1$. Suppose the weighted composition operator $\psi C_\varphi : \mathcal{B}^\alpha(\mathbb{D}^n) \to \mathcal{B}^\beta(\mathbb{D}^n)$ is bounded, then,*

$$\begin{aligned}\frac{1}{n}\lim_{\delta\to 0}\sup_{\operatorname{dist}(\varphi(z),\partial\mathbb{D}^n)<\delta}|\psi(z)|\sum_{k,l=1}^{n}\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|\frac{\left(1-|z_k|^2\right)^\beta}{\left(1-|\varphi_l(z)|^2\right)^\alpha}\\ \le \|\psi C_\varphi\|_e \le 2n\lim_{\delta\to 0}\sup_{\operatorname{dist}(\varphi(z),\partial\mathbb{D}^n)<\delta}|\psi(z)|\sum_{k,l=1}^{n}\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|\frac{\left(1-|z_k|^2\right)^\beta}{\left(1-|\varphi_l(z)|^2\right)^\alpha}.\end{aligned}\tag{1.8}$$

2. The proof of Theorem 1.3. In this section, we mainly give the proof of the main theorem of this paper. We divided our proof into two parts.

The lower estimates. Since $\psi C_\varphi : \mathcal{B}^\alpha(\mathbb{D}^n) \to \mathcal{B}^\beta(\mathbb{D}^n)$ is bounded, we have $\psi \in \mathcal{B}^\beta(\mathbb{D}^n)$ by Theorem 1.1.

Note that for $m \ge 2$,

$$\|z_1^m\|_\alpha = \sup_{z\in\mathbb{D}^n}\left(1-|z_1|^2\right)^\alpha |mz_1^{m-1}| = m\left(\frac{2\alpha}{m-1+2\alpha}\right)^\alpha\left(\frac{m-1}{m-1+2\alpha}\right)^{(m-1)/2}, \tag{2.1}$$

where the maximum is attained at any point on the circle with radius

$$r_m = \left(\frac{m-1}{m-1+2\alpha}\right)^{1/2}. \tag{2.2}$$

Hence, the sequence $\{z_1^m\}_{m\ge 2}$ is bounded in $\mathcal{B}^\alpha(\mathbb{D}^n)$. Let $f_m = z_1^m/\|z_1^m\|_\alpha$, then $\|f_m\|_\alpha = 1$ and f_m is bounded sequence and converges weakly to 0 in $\mathcal{B}^\alpha(\mathbb{D}^n)$. This follows since a bounded sequence contained in $\mathcal{B}^\alpha(\mathbb{D}^n)$ which tends to 0 uniformly on compact subsets of $\mathbb{D}^n$ converges weakly to 0 in $\mathcal{B}^\alpha(\mathbb{D}^n)$. In particular, if K is any compact operator from $\mathcal{B}^\alpha(\mathbb{D}^n)$ to $\mathcal{B}^\beta(\mathbb{D}^n)$, then $\lim_{m\to\infty}\|Kf_m\|_\beta = 0$.

For $m \ge 2$, let

$$A_m = \{z = (z_1,z_2,\dots,z_n) \in \mathbb{D}^n,\ r_m \le |z_1| \le r_{m+1}\}, \tag{2.3}$$

here, $r_m = ((m-1)/(m-1+2\alpha))^{1/2}$. Let $g(x) = m(1-x^2)^\alpha x^{m-1}$, then

$$g'(x) = -mx^{m-2}(1-x^2)^{\alpha-1}(2\alpha x^2-(m-1)(1-x^2)) \le 0 \tag{2.4}$$

for $x \in [((m-1)/(m-1+2\alpha))^{1/2},1)$, that is, $g(x)$ is a decreasing function for $x \in [((m-1)/(m-1+2\alpha))^{1/2},1)$. Therefore,

$$\begin{aligned}\min_{A_m}\left|\frac{\partial f_m}{\partial z_1}\right|\left(1-|z_1|^2\right)^\alpha &= \frac{(1-r_{m+1}^2)|mr_{m+1}^{m-1}|}{\|z_1^m\|_\alpha}\\ &= \left(\frac{m-1+2\alpha}{m+2\alpha}\right)^\alpha\left(\frac{m^2+(2\alpha-1)m}{m^2+(2\alpha-1)m-2\alpha}\right)^{(m-1)/2} = C_m.\end{aligned}\tag{2.5}$$

Note that C_m tends to 1 as $m \to \infty$. Take any compact operator K from $\mathcal{B}^{\alpha}(\mathbb{D}^n)$ to $\mathcal{B}^{\beta}(\mathbb{D}^n)$, we have

$$\begin{aligned}
&\|\psi C_\varphi - K\| \\
&\quad \geq \limsup_{m\to\infty} \|(\psi C_\varphi - K) f_m\|_\beta \\
&\quad \geq \limsup_{m\to\infty} (\|\psi C_\varphi f_m\|_\beta - \|K f_m\|_\beta) \\
&\quad = \limsup_{m\to\infty} \|\psi C_\varphi f_m\|_\beta \\
&\quad \geq \limsup_{m\to\infty} \sup_{z\in D^n} \sum_{k=1}^n \left| \frac{\partial(\psi f_m\circ\varphi)}{\partial z_k}(z)\right| \left(1-|z_k|^2\right)^\beta \\
&\quad \geq \limsup_{m\to\infty} \sup_{\varphi(z)\in A_m} \sum_{k=1}^n \left| \frac{\partial(\psi f_m\circ\varphi)}{\partial z_k}(z)\right| \left(1-|z_k|^2\right)^\beta \\
&\quad \geq \limsup_{m\to\infty} \sup_{\varphi(z)\in A_m} \sum_{k=1}^n \left| \frac{\partial\psi(z)}{\partial z_k} f_m(\varphi(z)) + \psi(z) \sum_{l=1}^n \frac{\partial f_m}{\partial w_l}(\varphi(z)) \frac{\partial \varphi_l}{\partial z_k}(z)\right| \left(1-|z_k|^2\right)^\beta \\
&\quad = \limsup_{m\to\infty} \sup_{\varphi(z)\in A_m} \sum_{k=1}^n \left| \frac{\partial\psi(z)}{\partial z_k} f_m(\varphi(z)) + \psi(z) \frac{\partial f_m}{\partial w_1}(\varphi(z)) \frac{\partial \varphi_1}{\partial z_k}(z)\right| \left(1-|z_k|^2\right)^\beta \\
&\quad \geq \limsup_{m\to\infty} \sup_{\varphi(z)\in A_m} \left| \sum_{k=1}^n |\psi(z)| \left| \frac{\partial f_m}{\partial w_1}(\varphi(z)) \frac{\partial \varphi_1}{\partial z_k}(z)\right| \right. \\
&\qquad\qquad \times \frac{\left(1-|z_k|^2\right)^\beta}{\left(1-|\varphi_1(z)|^2\right)^\alpha} \left(1-|\varphi_1(z)|^2\right)^\alpha \\
&\qquad\qquad \left. - \sum_{k=1}^n \left|\frac{\partial \psi}{\partial z_k}(z)\right| \left(1-|z_k|^2\right)^\beta |f_m(\varphi(z))| \right|.
\end{aligned} \tag{2.6}$$

When $0<\alpha<1$, we know that $\psi \in \mathcal{B}^{\beta}(\mathbb{D}^n)$, so that

$$\begin{aligned}
&\limsup_{m\to\infty} \sup_{\varphi(z)\in A_m} \sum_{k=1}^n \left|\frac{\partial \psi}{\partial z_k}(z)\right| \left(1-|z_k|^2\right)^\beta |f_m(\varphi(z))| \\
&\quad \leq \|\psi\|_\beta \limsup_{m\to\infty} \sup_{\varphi(z)\in A_m} |f_m(\varphi(z))| \\
&\quad = \|\psi\|_\beta \limsup_{m\to\infty} \frac{(m/(m+2\alpha))^{m/2}}{\|z_1^m\|_\alpha} \\
&\quad = \|\psi\|_\beta \limsup_{m\to\infty} \frac{(m/(m+2\alpha))^{m/2}}{m(2\alpha/(m-1+2\alpha))^\alpha((m-1)/(m-1+2\alpha))^{(m-1)/2}} \\
&\quad = 0.
\end{aligned} \tag{2.7}$$

Therefore,

$$
\begin{aligned}
&\|\psi C_\varphi\|_e \\
&\geq \limsup_{m\to\infty} \sup_{\varphi(z)\in A_m} \sum_{k=1}^{n} |\psi(z)| \left| \frac{\partial f_m}{\partial w_1}(\varphi(z)) \frac{\partial \varphi_1}{\partial z_k}(z) \right| \\
&\qquad\qquad \times \frac{\left(1-|z_k|^2\right)^\beta}{\left(1-|\varphi_1(z)|^2\right)^\alpha}\left(1-|\varphi_1(z)|^2\right)^\alpha \\
&\geq \limsup_{m\to\infty} \sup_{\varphi(z)\in A_m} \sum_{k=1}^{n} |\psi(z)| \left| \frac{\partial \varphi_1}{\partial z_k}(z) \right| \frac{\left(1-|z_k|^2\right)^\beta}{\left(1-|\varphi_1(z)|^2\right)^\alpha} \\
&\quad \times \liminf_{m} \min_{\varphi(z)\in A_m} \left| \frac{\partial f_m}{\partial w_1}(\varphi(z)) \right| \left(1-|\varphi_1(z)|^2\right)^\alpha \\
&\geq \limsup_{m\to\infty} \sup_{\varphi(z)\in A_m} \sum_{k=1}^{n} |\psi(z)| \left| \frac{\partial \varphi_1}{\partial z_k}(z) \right| \frac{\left(1-|z_k|^2\right)^\beta}{\left(1-|\varphi_1(z)|^2\right)^\alpha} \liminf_{m} C_m \\
&\geq \limsup_{m\to\infty} \sup_{\varphi(z)\in A_m} \sum_{k=1}^{n} |\psi(z)| \left| \frac{\partial \varphi_1}{\partial z_k}(z) \right| \frac{\left(1-|z_k|^2\right)^\beta}{\left(1-|\varphi_1(z)|^2\right)^\alpha}.
\end{aligned}
\tag{2.8}
$$

So,

$$
\|\psi C_\varphi\|_e \geq \limsup_{m\to\infty} \sup_{\varphi(z)\in A_m} \sum_{k=1}^{n} |\psi(z)| \left| \frac{\partial \varphi_1}{\partial z_k}(z) \right| \frac{\left(1-|z_k|^2\right)^\beta}{\left(1-|\varphi_1(z)|^2\right)^\alpha}. \tag{2.9}
$$

For $l = 1, 2, \ldots, n$, define

$$
a_l = \lim_{\delta\to 0} \sup_{\operatorname{dist}(\varphi(z),\partial\mathbb{D}^n)<\delta} \sum_{k=1}^{n} |\psi(z)| \left| \frac{\partial \varphi_l}{\partial z_k}(z) \right| \frac{\left(1-|z_k|^2\right)^\beta}{\left(1-|\varphi_l(z)|^2\right)^\alpha}. \tag{2.10}
$$

For any $\epsilon > 0$, (2.10) shows that there exists δ_0, $0 < \delta_0 < 1$, if $\operatorname{dist}(\varphi(z), \partial\mathbb{D}^n) < \delta_0$, then

$$
\sum_{k=1}^{n} |\psi(z)| \left| \frac{\partial \varphi_1}{\partial z_k}(z) \right| \frac{\left(1-|z_k|^2\right)^\beta}{\left(1-|\varphi_1(z)|^2\right)^\alpha} > a_1 - \epsilon. \tag{2.11}
$$

Because $r_m \to 1$ $(m \to \infty)$, $r_m > 1 - \delta_0$ for m being large enough. If $\varphi(z) \in A_m$, then $r_m \leq |\varphi_1(z)| \leq r_{m+1}$, that is, $1 - r_{m+1} \leq 1 - |\varphi_1(z)| \leq 1 - r_m$, that is,

$$
\operatorname{dist}(\varphi(z), \partial\mathbb{D}^n) \leq \operatorname{dist}(\varphi_1(z), \partial\mathbb{D}) < \delta_0. \tag{2.12}
$$

By (2.9) and (2.11), we have

$$\|\psi C_\varphi\|_e \ge a_1 - \epsilon. \tag{2.13}$$

If we choose $g_m(z) = z_l^m/\|z_l^m\|$, repeating similar argument as in the case $l = 1$, we have

$$\|\psi C_\varphi\|_e \ge a_l - \epsilon \tag{2.14}$$

for $l = 2,\dots,n$. Hence,

$$\begin{aligned}\|\psi C_\varphi\|_e &\ge \frac{1}{n}\sum_{l=1}^{n}(a_l-\epsilon)\\ &= \frac{1}{n}\sum_{l=1}^{n}\left(\lim_{\delta\to 0}\sup_{\operatorname{dist}(\varphi(z),\partial\mathbb{D}^n)<\delta}\sum_{k=1}^{n}|\psi(z)|\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|\frac{\left(1-|z_k|^2\right)^\beta}{\left(1-|\varphi_l(z)|^2\right)^\alpha}-\epsilon\right)\\ &= \frac{1}{n}\lim_{\delta\to 0}\sup_{\operatorname{dist}(\varphi(z),\partial\mathbb{D}^n)<\delta}\sum_{k,l=1}^{n}|\psi(z)|\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|\frac{\left(1-|z_k|^2\right)^\beta}{\left(1-|\varphi_l(z)|^2\right)^\alpha}-\epsilon.\end{aligned} \tag{2.15}$$

Let $\epsilon \to 0$, we obtain the result.

The upper estimates. For this purpose, we define operator K_m $(m \ge 2)$ as follows:

$$K_m f(z) = f\left(\frac{m-1}{m}z\right). \tag{2.16}$$

Using the method of [7], we can show that K_m has the following properties:

(a) K_m is compact operator from $\mathcal{B}^\alpha(\mathbb{D}^n)$ to $\mathcal{B}^\alpha(\mathbb{D}^n)$,

(b) $\lim_{m\to\infty}\sup_{\|f\|_\alpha\le 1}\sup_{z\in D^n}|(I-K_m)f(z)| = 0$,

(c) for any $f \in \mathcal{B}^\alpha(\mathbb{D}^n)$, $(I-K_m)f \to 0$ uniformly on compact subsets of $\mathbb{D}^n$, hence, for $l = 1,2,\dots,n$, $\partial(I-K_m)f/\partial w_l \to 0$ uniformly on compact subsets of $\mathbb{D}^n$,

(d) $\|I-K_m\| \le 2$.

The details of parts (b) and (c) are left to the reader, we will show the details for part (a) and (d).

First, we show the details of part (a). In fact, let $E = \{((m-1)/m)z,\ z \in \mathbb{D}^n\}$, then E is a compact subset of $\mathbb{D}^n$. For any sequence $\{f_j\} \subset \mathcal{B}^\alpha(\mathbb{D}^n)$, there exists a subsequence $\{f_{j_s}\}$ of $\{f_j\}$ converging uniformly to $f \in \mathcal{B}^\alpha(\mathbb{D}^n)$ on compact subsets of $\mathbb{D}^n$ and $\|f\|_\alpha \le M$. It is obvious that $\{\partial f_{j_s}/\partial z_i\}$, $i = 1,2,\dots,n$, converges uniformly to $\{\partial f/\partial z_i\}$ on compact subsets of $\mathbb{D}^n$. So, for large enough $s, w \in E$ and $l = 1,2,\dots,n$,

$$\left|\frac{\partial(f_{j_s}-f)}{\partial w_l}(w)\right| < \epsilon. \tag{2.17}$$

Hence,

$$
\begin{aligned}
&\|K_m f_{j_s} - K_m f\|_\alpha \\
&\quad = \left\| f_{j_s}\left(\frac{m-1}{m}z\right) - f\left(\frac{m-1}{m}z\right)\right\|_\alpha \\
&\quad \le \sup_{z\in\mathbb{D}^n} \sum_{k=1}^{n} \left| \frac{\partial[(f_{j_s}-f)(((m-1)/m)z)]}{\partial z_k}\right| \left(1-|z_k|^2\right)^\alpha \\
&\quad \le \sup_{z\in\mathbb{D}^n} \sum_{k=1}^{n}\sum_{l=1}^{n} \left| \frac{\partial[(f_{j_s}-f)(((m-1)/m)z)]}{\partial w_l}\right| \frac{m-1}{m}\left(1-|z_k|^2\right)^\alpha \\
&\quad \le \sup_{z\in\mathbb{D}^n} \sum_{k=1}^{n}\sum_{l=1}^{n} \left| \frac{\partial[(f_{j_s}-f)(((m-1)/m)z)]}{\partial w_l}\right| \frac{m-1}{m} \\
&\quad \le \sup_{w\in E} \frac{m-1}{m} \sum_{k=1}^{n}\sum_{l=1}^{n} \left| \frac{\partial(f_{j_s}-f)}{\partial w_l}(w)\right| \longrightarrow 0
\end{aligned}
\tag{2.18}
$$

for $s\to\infty$. These show that K_m is a compact operator.

Next, we show the details of part (d). In fact, for any $f\in\mathcal{B}^\alpha(\mathbb{D}^n)$, we have

$$
\begin{aligned}
\|(I-K_m)f\|_\alpha &\le \sup_{z\in\mathbb{D}^n} \sum_{k=1}^{n} \left| \frac{\partial(I-K_m)f}{\partial z_k}(z)\right| \left(1-|z_k|^2\right)^\alpha \\
&= \sup_{z\in\mathbb{D}^n} \sum_{k=1}^{n} \left| \frac{\partial f}{\partial z_k}(z) - \left(1-\frac{1}{m}\right)\frac{\partial f}{\partial z_k}\left(\left(1-\frac{1}{m}\right)z\right)\right| \left(1-|z_k|^2\right)^\alpha \\
&\le \sup_{z\in\mathbb{D}^n} \sum_{k=1}^{n} \left| \frac{\partial f}{\partial z_k}(z)\right| \left(1-|z_k|^2\right)^\alpha \\
&\quad + \left(1-\frac{1}{m}\right) \sup_{z\in\mathbb{D}^n} \sum_{k=1}^{n} \left| \frac{\partial f}{\partial z_k}\left(\left(1-\frac{1}{m}\right)z\right)\right| \left(1-\left|\left(1-\frac{1}{m}\right)z_k\right|^2\right)^\alpha \\
&\le \|f\|_\alpha + \|f\|_\alpha = 2\|f\|_\alpha.
\end{aligned}
\tag{2.19}
$$

Since each K_m is compact operator from $\mathcal{B}^\alpha(\mathbb{D}^n)$ to $\mathcal{B}^\alpha(\mathbb{D}^n)$, so is $\psi C_\varphi K_m$. Hence,

$$
\|\psi C_\varphi\|_e \le \|\psi C_\varphi - \psi C_\varphi K_m\| = \|\psi C_\varphi (I-K_m)\| = \sup_{\|f\|_\alpha\le 1} \|\psi C_\varphi (I-K_m) f\|_\beta. \tag{2.20}
$$

We bound the last expression from above by

$$
\sup_{\|f\|_\alpha\le 1} |\psi(0)|\,|(I-K_m)f(\varphi(0))| \tag{2.21}
$$

$$
+ \sup_{\|f\|_\alpha\le 1} \sup_{z\in\mathbb{D}^n} \sum_{k,l=1}^{n} |\psi(z)| \left| \frac{\partial(I-K_m)f}{\partial w_l}(\varphi(z)) \frac{\partial\varphi_l}{\partial z_k}(z)\right| \left(1-|z_k|^2\right)^\beta \tag{2.22}
$$

$$
+ \sup_{\|f\|_\alpha\le 1} \sup_{z\in\mathbb{D}^n} \sum_{k=1}^{n} \left| \frac{\partial\psi}{\partial z_k}(z)\right| |(I-K_m)f(\varphi(z))| \left(1-|z_k|^2\right)^\beta. \tag{2.23}
$$

By the property (b), we know that the supremum in (2.21) can be made arbitrarily small as $m \to \infty$. Since $\psi(z) \in \mathcal{B}^{\beta}(\mathbb{D}^n)$,

$$\sup_{z\in\mathbb{D}^n} \sum_{k=1}^{n} \left| \frac{\partial \psi}{\partial z_k}(z) \right| \left(1-|z_k|^2\right)^{\beta} < \infty. \tag{2.24}$$

Property (b) also ensures that the supremum in (2.23) tends to 0 as $m \to \infty$. Now, we need only consider the term

$$\sup_{\|f\|_{\alpha}\le 1} \sup_{z\in\mathbb{D}^n} \sum_{k,l=1}^{n} |\psi(z)| \left| \frac{\partial (I-K_m)f}{\partial w_l}(\varphi(z)) \frac{\partial \varphi_l}{\partial z_k}(z) \right| \left(1-|z_k|^2\right)^{\beta}. \tag{2.25}$$

For arbitrary $0<\delta<1$, we define

$$\begin{aligned} G_1 &= \{z \in D^n : \operatorname{dist}(\varphi(z), \partial\mathbb{D}^n) < \delta\}, \\ G_2 &= \{z \in D^n : \operatorname{dist}(\varphi(z), \partial\mathbb{D}^n) \ge \delta\}. \end{aligned} \tag{2.26}$$

Here, G_2 is compact subset of $\mathbb{C}^n$. We consider

$$\begin{aligned} &\sup_{\|f\|_{\alpha}\le 1} \sup_{z\in D^n} \sum_{k,l=1}^{n} |\psi(z)| \left| \frac{\partial (I-K_m)f}{\partial w_l}(\varphi(z)) \frac{\partial \varphi_l}{\partial z_k}(z) \right| \left(1-|z_k|^2\right)^{\beta} \\ &\quad \le \sup_{\|f\|_{\alpha}\le 1} \sup_{z\in G_1} \sum_{k,l=1}^{n} |\psi(z)| \left| \frac{\partial (I-K_m)f}{\partial w_l}(\varphi(z)) \frac{\partial \varphi_l}{\partial z_k}(z) \right| \left(1-|z_k|^2\right)^{\beta} \\ &\qquad + \sup_{\|f\|_{\alpha}\le 1} \sup_{z\in G_2} \sum_{k,l=1}^{n} |\psi(z)| \left| \frac{\partial (I-K_m)f}{\partial w_l}(\varphi(z)) \frac{\partial \varphi_l}{\partial z_k}(z) \right| \left(1-|z_k|^2\right)^{\beta} \\ &\quad = I + II. \end{aligned} \tag{2.27}$$

First, we write I as follows:

$$\begin{aligned} I = \sup_{\|f\|_{\alpha}\le 1} \sup_{z\in G_1} \sum_{k,l=1}^{n} |\psi(z)| \left| \frac{\partial \varphi_l}{\partial z_k}(z) \right| \left| \frac{\left(1-|z_k|^2\right)^{\beta}}{\left(1-|\varphi_l(z)|^2\right)^{\alpha}} \right| \\ \times \left| \frac{\partial (I-K_m)f}{\partial w_l}(\varphi(z)) \right| \left(1-|\varphi_l(z)|^2\right)^{\alpha} \end{aligned} \tag{2.28}$$

and observe that this is bounded above by

$$n\|I-K_m\| \sup_{z\in G_1} \sum_{k,l=1}^{n} |\psi(z)| \left| \frac{\partial \varphi_l}{\partial z_k}(z) \right| \frac{\left(1-|z_k|^2\right)^{\beta}}{\left(1-|\varphi_l(z)|^2\right)^{\alpha}}. \tag{2.29}$$

Then, by property (d), we know that

$$I \le 2n \sup_{z\in G_1} \sum_{k,l=1}^{n} |\psi(z)| \left|\frac{\partial \varphi_l}{\partial z_k}(z)\right| \frac{\left(1-|z_k|^2\right)^{\beta}}{\left(1-|\varphi_l(z)|^2\right)^{\alpha}}. \tag{2.30}$$

Next, we prove that $\lim_{m\to\infty} II = 0$. Since ψC_φ is bounded from $\mathcal{B}^{\alpha}(\mathbb{D}^n)$ to $\mathcal{B}^{\beta}(\mathbb{D}^n)$, by Theorem 1.1 we have

$$|\psi(z)| \sum_{k,l=1}^{n} \left|\frac{\partial \varphi_l}{\partial z_k}(z)\right| \frac{\left(1-|z_k|^2\right)^{\beta}}{\left(1-|\varphi_l(z)|^2\right)^{\alpha}} < \infty \quad (|z| \longrightarrow 1^-). \tag{2.31}$$

Thus,

$$\sup_{z\in G_2} |\psi(z)| \sum_{k,l=1}^{n} \left|\frac{\partial \varphi_l}{\partial z_k}(z)\right| \left(1-|z_k|^2\right)^{\beta} < \infty. \tag{2.32}$$

Then, using property (c), we have

$$\lim_{m\to\infty} II = \lim_{m\to\infty} \sup_{\|f\|_\alpha\le 1} \sup_{z\in G_2} \sum_{k,l=1}^{n} |\psi(z)| \left|\frac{\partial (I-K_m)f}{\partial w_l}(\varphi(z))\frac{\partial \varphi_l}{\partial z_k}(z)\right| \left(1-|z_k|^2\right)^{\beta} = 0. \tag{2.33}$$

Combining the estimates for (2.21), (2.22), and (2.23) as $m\to\infty$, we get

$$\|\psi C_\varphi\|_e \le 2n \lim_{\delta\to 0} \sup_{\operatorname{dist}(\varphi(z),\partial D^n)<\delta} \sum_{k,l=1}^{n} |\psi(z)| \left|\frac{\partial \varphi_l}{\partial z_k}(z)\right| \frac{\left(1-|z_k|^2\right)^{\beta}}{\left(1-|\varphi_l(z)|^2\right)^{\alpha}}. \tag{2.34}$$

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