IJMMS 2004:72, 3959–3964
PII. S0161171204306277
http://ijmms.hindawi.com

ON A HIGHER-ORDER EVOLUTION EQUATION WITH A STEPANOV-BOUNDED SOLUTION

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Received 12 June 2003 and in revised form 5 August 2004

We study strong solutions $u : \mathbb{R} \to X$, a Banach space X, of the nth-order evolution equation $u^{(n)} - Au^{(n-1)} = f$, an infinitesimal generator of a strongly continuous group $A : D(A) \subseteq X \to X$, and a given forcing term $f : \mathbb{R} \to X$. It is shown that if X is reflexive, u and $u^{(n-1)}$ are Stepanov-bounded, and f is Stepanov almost periodic, then u and all derivatives $u', \ldots, u^{(n-1)}$ are strongly almost periodic. In the case of a general Banach space X, a corresponding result is obtained, proving weak almost periodicity of u, $u', \ldots, u^{(n-1)}$.

2000 Mathematics Subject Classification: 34G10, 34C27, 47D03.

1. Introduction. In this paper, we are concerned with an nth-order evolution equation of the form

$$u^{(n)} - Au^{(n-1)} = f. \tag{1.1}$$

Here $A : D(A) \subseteq X \to X$ is an infinitesimal generator of a strongly continuous group, $f : \mathbb{R} \to X$ a given forcing term, X a Banach space with scalar field C, n a positive integer, and $\mathbb{R}$ denotes the set of reals. We will give suitable assumptions to ensure that almost periodicity of the forcing term f carries over to the solution u and its derivatives up to order $(n-1)$.

The reason for studying this rather special evolution equation may be classified as a first pilot study of the issue of higher-order evolution equations, which probably has not been studied before.

We first recall the relevant concepts. A continuous function $f : \mathbb{R} \to X$ is said to be strongly (or Bochner) almost periodic if, for every given $\varepsilon > 0$, there is an $r > 0$ such that any interval in $\mathbb{R}$ of length r contains a point τ for which

$$\sup_{t\in\mathbb{R}} \|f(t+\tau) - f(t)\| \le \varepsilon. \tag{1.2}$$

Here $\|\cdot\|$ denotes the norm in X.

A function $f : \mathbb{R} \to X$ is called weakly almost periodic if $x^* f(\cdot) : \mathbb{R} \to C$ is continuous and almost periodic for every x^* in the dual space X^* of X.

We will call a function $f \in L^1_{\text{loc}}(\mathbb{R}, X)$ Stepanov-bounded or briefly S-bounded if

$$\|f\|_S := \sup_{t\in\mathbb{R}} \int_t^{t+1} \|f(s)\| ds < \infty. \tag{1.3}$$

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$$\sup_{t\in\mathbb{R}} \int_t^{t+1} \|f(s+\tau) - f(s)\| ds \le \varepsilon. \tag{1.4}$$

We denote by $L(X,X)$ the set of all bounded linear operators on X into itself. An operator-valued function $T : \mathbb{R} \to L(X,X)$ will be called a strongly continuous group if

$$T(t_1 + t_2) = T(t_1)T(t_2) \quad \forall t_1, t_2 \in \mathbb{R}, \tag{1.5}$$

$$T(0) = I = \text{the identity operator on } X, \tag{1.6}$$

$$T(\cdot)x : \mathbb{R} \to X \text{ is continuous for every } x \in X. \tag{1.7}$$

We recall (e.g., from Dunford and Schwartz [4]) that the infinitesimal generator $A : D(A) \subseteq X \to X$ of a strongly continuous group $T : \mathbb{R} \to L(X,X)$ is a densely defined, closed linear operator.

An operator-valued function $T : \mathbb{R} \to L(X,X)$ is said to be strongly (weakly) almost periodic if $T(\cdot)x : \mathbb{R} \to X$ is strongly (weakly) almost periodic for every $x \in X$.

Suppose $A : D(A) \subseteq X \to X$ is a densely defined, closed linear operator, and $f : \mathbb{R} \to X$ is a continuous function. Then a strong solution of the evolution equation

$$u^{(n)}(t) - Au^{(n-1)}(t) = f(t) \quad \text{a.e. for } t \in \mathbb{R} \tag{1.8}$$

is an n times strongly differentiable function $u : \mathbb{R} \to X$ with $u^{(n-1)}(t) \in D(A)$ for all $t \in \mathbb{R}$, and satisfies problem (1.8).

Our first result is as follows (see Zaidman [7, 8] for first-order evolution equations).

Theorem 1.1. *Let X be reflexive, $f : \mathbb{R} \to X$ continuous, S-almost periodic, A infinitesimal generator of a strongly almost periodic group $T : \mathbb{R} \to L(X,X)$. In this case, if, for the strong solution $u : \mathbb{R} \to X$ of problem (1.8), both u and $u^{(n-1)}$ are S-bounded on $\mathbb{R}$, then $u, u', \ldots, u^{(n-1)}$ are all strongly almost periodic.*

Our second result refers to a weak variant of our first theorem in the case of a general—not necessarily reflexive—Banach space X.

Theorem 1.2. *Suppose $f : \mathbb{R} \to X$ is an S-almost periodic (or a weakly almost periodic) continuous function, A an infinitesimal generator of a strongly continuous group $T : \mathbb{R} \to L(X,X)$ such that the conjugate operator group $T^* : \mathbb{R} \to L(X^*, X^*)$ is strongly almost periodic. If, for the strong solution $u : \mathbb{R} \to X$ of problem (1.8), both u and $u^{(n-1)}$ are S-bounded on $\mathbb{R}$, then $u, u', \ldots, u^{(n-1)}$ are all weakly almost periodic.*

Remark 1.3. For some examples of first-order and higher-order evolution equations with strongly almost periodic solutions, the reader may wish to consult Cooke [3] and Zaidman [9].

2. Lemmas

Lemma 2.1. *If A is the infinitesimal generator of a strongly continuous group $G : \mathbb{R} \to L(X,X)$, then the $(n-1)$th derivative of any solution of (1.8) has the representation*

$$u^{(n-1)}(t) = G(t)u^{(n-1)}(0) + \int_0^t G(t-s)f(s)ds \quad \text{for } t \in \mathbb{R}. \tag{2.1}$$

Proof. For an arbitrary but fixed $t \in \mathbb{R}$, we have

$$\begin{aligned} \frac{d}{ds}\left[G(t-s)u^{(n-1)}(s)\right] &= G(t-s)\left[u^{(n)}(s) - Au^{(n-1)}(s)\right] \\ &= G(t-s)f(s) \quad \text{a.e. for } s \in \mathbb{R}, \text{ by (1.8)}. \end{aligned} \tag{2.2}$$

Now, integrating (2.2) from 0 to t, we obtain

$$\int_0^t \frac{d}{ds}\left[G(t-s)u^{(n-1)}(s)\right]ds = \int_0^t G(t-s)f(s)ds, \tag{2.3}$$

which gives the desired representation, by (1.6). □

Lemma 2.2. *If $g : \mathbb{R} \to X$ is a strongly almost periodic function, and $G : \mathbb{R} \to L(X,X)$ is a strongly (weakly) almost periodic operator-valued function, then $G(\cdot)g(\cdot) : \mathbb{R} \to X$ is a strongly (weakly) almost periodic function.*

For the proof of Lemma 2.2, see [6, Theorem 1] for weak almost periodicity.

Lemma 2.3. *If $g : \mathbb{R} \to X$ is an S-almost periodic continuous function, and $G : \mathbb{R} \to L(X,X)$ is a weakly almost periodic operator-valued function, then $x^*G(\cdot)g(\cdot) : \mathbb{R} \to C$ is an S-almost periodic continuous function for every $x^* \in X^*$.*

Proof. By our assumption, for an arbitrary but fixed $x^* \in X^*$, the function $x^*G(\cdot)x : \mathbb{R} \to C$ is almost periodic, and so is bounded on $\mathbb{R}$, for every $x \in X$. Hence, by the uniform-boundedness principle,

$$\sup_{t\in\mathbb{R}} \left\|x^*G(t)\right\| = K < \infty. \tag{2.4}$$

We note that the function $x^*G(\cdot)g(\cdot)$ is continuous on $\mathbb{R}$ (see [6, proof of Theorem 1]).

Consider the functions g_η given by

$$g_\eta(t) = \frac{1}{\eta}\int_0^\eta g(t+s)ds \quad \text{for } \eta > 0,\ t \in \mathbb{R}. \tag{2.5}$$

Since g is S-almost periodic from $\mathbb{R}$ to X, g_η is strongly almost periodic from $\mathbb{R}$ to X for every fixed $\eta > 0$. Further, as shown for C-valued functions in [2, pages 80-81], we can prove that $g_\eta \to g$ as $\eta \to 0+$ in the S-sense, that is,

$$\sup_{t\in\mathbb{R}} \int_t^{t+1} \left\|g(s) - g_\eta(s)\right\|ds \longrightarrow 0 \quad \text{as } \eta \longrightarrow 0+. \tag{2.6}$$

Now we have

$$x^*G(s)g(s) = x^*G(s)\left[g(s) - g_\eta(s)\right] + x^*G(s)g_\eta(s) \quad \text{for } s \in \mathbb{R}, \tag{2.7}$$

and, by (2.4) and (2.6),

$$\begin{aligned}\sup_{t\in\mathbb{R}}\int_t^{t+1}&|x^*G(s)[g(s)-g_\eta(s)]|\,ds\\&\le K\sup_{t\in\mathbb{R}}\int_t^{t+1}\|g(s)-g_\eta(s)\|ds\longrightarrow 0\quad\text{as }\eta\longrightarrow 0+.\end{aligned}\tag{2.8}$$

By Lemma 2.2, the functions $x^*G(\cdot)g_\eta(\cdot)$ are almost periodic from $\mathbb{R}$ to C. Therefore, it follows from (2.7)-(2.8) that $x^*G(\cdot)g(\cdot)$ is S-almost periodic from $\mathbb{R}$ to C. □

Lemma 2.4. *If $g:\mathbb{R}\to X$ is an S-almost periodic continuous function, and $G:\mathbb{R}\to L(X,X)$ is a strongly almost periodic operator-valued function, then $G(\cdot)g(\cdot):\mathbb{R}\to X$ is an S-almost periodic continuous function.*

The proof of this lemma parallels that of Lemma 2.3 and may therefore be safely omitted.

Lemma 2.5. *In a reflexive space X, assume $h:\mathbb{R}\to X$ is an S-almost periodic continuous function, and*

$$H(t)=\int_0^t h(s)ds\quad\text{for } t\in\mathbb{R}.\tag{2.9}$$

If H is S-bounded, then it is strongly almost periodic from $\mathbb{R}$ to X.

For the proof of Lemma 2.5, see [5, Notes (ii)].

Lemma 2.6. *For an operator-valued function $G:\mathbb{R}\to L(X,X)$, suppose $G^*(t)$ is the conjugate (adjoint) of the operator $G(t)$ for $t\in\mathbb{R}$. If $G^*:\mathbb{R}\to L(X^*,X^*)$ is strongly almost periodic, and $g:\mathbb{R}\to X$ is weakly almost periodic, then $G(\cdot)g(\cdot):\mathbb{R}\to X$ is weakly almost periodic.*

For the proof of Lemma 2.6, see [6, Remarks (iii)].

3. Proof of Theorem 1.1.

By (2.1), we have

$$T(-t)u^{(n-1)}(t)=u^{(n-1)}(0)+\int_0^t T(-s)f(s)ds\quad\text{for } t\in\mathbb{R}.\tag{3.1}$$

Evidently, $T(-\cdot):\mathbb{R}\to L(X,X)$ is a strongly almost periodic group. Therefore, $T(-\cdot)x:\mathbb{R}\to X$ is strongly almost periodic, and so is bounded on $\mathbb{R}$, for every $x\in X$. Hence, by the uniform-boundedness principle,

$$\sup_{t\in\mathbb{R}}\|T(-t)\|<\infty.\tag{3.2}$$

Consequently, $T(-\cdot)u^{(n-1)}(\cdot)$ is S-bounded on $\mathbb{R}$ (by our assumption, $u^{(n-1)}$ is S-bounded on $\mathbb{R}$).

Moreover, by Lemma 2.4, $T(-\cdot)f(\cdot):\mathbb{R}\to X$ is an S-almost periodic continuous function. So, by Lemma 2.5, $T(-\cdot)u^{(n-1)}(\cdot)$ is strongly almost periodic from $\mathbb{R}$ to X. Hence, by Lemma 2.2, $u^{(n-1)}(\cdot)=T(\cdot)[T(-\cdot)u^{(n-1)}(\cdot)]$ is strongly almost periodic from $\mathbb{R}$ to X.

Now consider a sequence $(\alpha_k)_{k=1,2,\dots}$ of infinitely differentiable nonnegative functions on $\mathbb{R}$ such that

$$\alpha_k(t) = 0 \quad \text{for } |t| \geq \frac{1}{k}, \qquad \int_{-1/k}^{1/k} \alpha_k(t)\,dt = 1. \tag{3.3}$$

The convolution of u and α_k is defined by

$$(u^*\alpha_k)(t) = \int_{\mathbb{R}} u(t-s)\alpha_k(s)\,ds = \int_{\mathbb{R}} u(s)\alpha_k(t-s)\,ds \quad \text{for } t \in \mathbb{R}. \tag{3.4}$$

We set

$$C_{\alpha_k} = \max_{|t|\leq 1/k} \alpha_k(t). \tag{3.5}$$

Then we have

$$\begin{aligned} ||(u^*\alpha_k)(t)|| &= \left|\left|\int_{-1}^{1} u(t-s)\alpha_k(s)\,ds\right|\right| \leq C_{\alpha_k}\int_{t-1}^{t+1} ||u(\rho)||\,d\rho \\ &\leq 2C_{\alpha_k}\|u\|_S \quad \text{for } t \in \mathbb{R}, \text{ by (1.3)}. \end{aligned} \tag{3.6}$$

That is, $u^*\alpha_k$ is bounded on $\mathbb{R}$.

We note that, for $m = 1,2,\dots,n-1$ and $k = 1,2,\dots$,

$$(u^*\alpha_k)^{(m)}(t) = (u^{(m)*}\alpha_k)(t) \quad \text{for } t \in \mathbb{R}. \tag{3.7}$$

Further, since $u^{(n-1)}$ is strongly almost periodic from $\mathbb{R}$ to X, $(u^*\alpha_k)^{(n-1)} = (u^{(n-1)*}\alpha_k)$ is strongly almost periodic from $\mathbb{R}$ to X. Consequently, by [3, corollary to Lemma 5], $u^*\alpha_k, u'^*\alpha_k,\dots,u^{(n-2)*}\alpha_k$ are all strongly almost periodic from $\mathbb{R}$ to X.

With $u^{(n-1)}$ being bounded on $\mathbb{R}$, $u^{(n-2)}$ is uniformly continuous on $\mathbb{R}$. Therefore, the sequence of convolutions $(u^{(n-2)*}\alpha_k)(t) \to u^{(n-2)}(t)$ as $k \to \infty$, uniformly for $t \in \mathbb{R}$. Hence $u^{(n-2)}$ is strongly almost periodic from $\mathbb{R}$ to X. We thus conclude successively that $u^{(n-2)},\dots,u',u$ are all strongly almost periodic from $\mathbb{R}$ to X, completing the proof of the theorem.

4. Proof of Theorem 1.2. By our assumption, for an arbitrary but fixed $x^* \in X^*$, $x^*T(\cdot) = T^*(\cdot)x^* : \mathbb{R} \to X^*$ is strongly almost periodic, and so $x^*T(\cdot)x : \mathbb{R} \to C$ is almost periodic for every $x \in X$. Therefore, it follows that $T : \mathbb{R} \to L(X,X)$ is a weakly almost periodic group.

By (3.1), we have

$$x^*T(-t)u^{(n-1)}(t) = x^*u^{(n-1)}(0) + \int_0^t x^*T(-s)f(s)\,ds \quad \text{for } t \in \mathbb{R}. \tag{4.1}$$

By Lemma 2.3, $x^*T(-\cdot)f(\cdot) : \mathbb{R} \to C$ is an S-almost periodic continuous function. By (2.4), $x^*T(-\cdot)u^{(n-1)}(\cdot)$ is S-bounded on $\mathbb{R}$, and so, by Lemma 2.5, is almost periodic from $\mathbb{R}$ to C. That is, $T(-\cdot)u^{(n-1)}(\cdot)$ is weakly almost periodic from $\mathbb{R}$ to X. Consequently, by Lemma 2.6, $u^{(n-1)}(\cdot) = T(\cdot)[T(-\cdot)u^{(n-1)}(\cdot)]$ is weakly almost periodic from $\mathbb{R}$ to X.

For the sequence $(\alpha_k)_{k=1,2,\dots}$ defined by (3.3), $(x^* u^* \alpha_k) = x^*(u^* \alpha_k)$ is bounded on $\mathbb{R}$ (by (3.6)). Further, for $m = 1, 2, \dots, n-1$ and $k = 1, 2, \dots$, we have

$$(x^* u^* \alpha_k)^{(m)}(t) = (x^* u^{(m)*} \alpha_k)(t) \quad \text{for } t \in \mathbb{R}. \tag{4.2}$$

Now the rest of the proof is obvious.

If $f : \mathbb{R} \to X$ is weakly almost periodic, then by Lemma 2.6, $T(-\cdot)f(\cdot) : \mathbb{R} \to X$ is weakly almost periodic.

Remark 4.1. If $T(t) \equiv I$ for $t \in \mathbb{R}$, and so $A = 0$, then problem (1.8) reduces to

$$u^{(n)}(t) = f(t) \quad \text{a.e. for } t \in \mathbb{R}. \tag{4.3}$$

(i) In a reflexive space X, suppose f is defined as in Theorem 1.1. If $u : \mathbb{R} \to X$ is an S-bounded strong solution of problem (4.3), then $u, u', \dots, u^{(n-1)}$ are all strongly almost periodic from $\mathbb{R}$ to X.

(ii) Assume $f : \mathbb{R} \to X$ is a weakly almost periodic continuous function. If $u : \mathbb{R} \to X$ is an S-bounded strong solution of problem (4.3), then $u, u', \dots, u^{(n-1)}$ are all weakly almost periodic from $\mathbb{R}$ to X.

These statements are clearly special cases of Theorems 1.1 and 1.2 if we take into account that the assumption $u^{(n-1)}$ S-bounded can be omitted, since, by (4.3), $u^{(n)}$ is S-almost periodic, and so $u^{(n-1)}$ is strongly (weakly) uniformly continuous on $\mathbb{R}$ (by Amerio and Prouse [1, Theorem 8, page 79]).

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