IJMMS 2004:72, 3989–3999
PII. S0161171204403391
http://ijmms.hindawi.com

PROPERTIES OF SOME $*$-DENSE-IN-ITSELF SUBSETS

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Received 22 March 2004

$\mathcal{I}$-open sets were introduced and studied by Janković and Hamlett (1990) to generalize the well-known Banach category theorem. Quasi-$\mathcal{I}$-openness was introduced and studied by Abd El-Monsef et al. (2000). These are $*$-dense-in-itself sets of the ideal spaces. In this note, properties of these sets are further investigated and characterizations of these sets are given. Also, their relation with $\mathcal{I}$-dense sets and $\mathcal{I}$-locally closed sets is discussed. Characterizations of completely codense ideals are given in terms of semi-preopen sets.

2000 Mathematics Subject Classification: 54A05, 54A10.

1. Introduction and preliminaries. The subject of ideals in topological spaces has been studied by Kuratowski [12] and Vaidyanathaswamy [20]. An *ideal* $\mathcal{I}$ on a topological space (X,τ) is a collection of subsets of X which satisfies that (i) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. Given a topological space (X,τ) with an ideal $\mathcal{I}$ on X and if $\wp(X)$ is the set of all subsets of X, a set operator $(\cdot)^* : \wp(X) \to \wp(X)$, called a *local function* [12] of A with respect to $\mathcal{I}$ and τ, is defined as follows: for $A \subset X$, $A^*(\mathcal{I},\tau) = \{x \in X \mid U \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau \mid x \in U\}$. We will make use of the basic facts concerning the local functions [10, Theorem 2.3] without mentioning it explicitly. A Kuratowski closure operator $\mathrm{cl}^*(\cdot)$ for a topology $\tau^*(\mathcal{I},\tau)$, called the $*$-*topology*, finer than τ, is defined by $\mathrm{cl}^*(A) = A \cup A^*(\mathcal{I},\tau)$ [19]. When there is no chance for confusion, we will simply write A^* for $A^*(\mathcal{I},\tau)$ and τ^* or $\tau^*(\mathcal{I})$ for $\tau^*(\mathcal{I},\tau)$. If $\mathcal{I}$ is an ideal on X, then $(X,\tau,\mathcal{I})$ is called an ideal space. By a space, we always mean a topological space (X,τ) with no separation properties assumed. If $A \subset X$, $\mathrm{cl}(A)$ and $\mathrm{int}(A)$ will denote the closure and interior of A in (X,τ), respectively, and $\mathrm{cl}^*(A)$ and $\mathrm{int}^*(A)$ will denote the closure and interior of A in (X,τ^*), respectively. A subset A of a space (X,τ) is *semiopen* [13] if there exists an open set G such that $G \subset A \subset \mathrm{cl}(G)$ or, equivalently, $A \subset \mathrm{cl}(\mathrm{int}(A))$. The complement of a semiopen set is semiclosed. The smallest semiclosed set containing A is called the *semiclosure* of A and is denoted by $\mathrm{scl}(A)$. Also, $\mathrm{scl}(A) = A \cup \mathrm{int}(\mathrm{cl}(A))$ [4, Theorem 1.5(a)]. The largest semiopen set contained in A is called the *semi-interior* of A and is denoted by $\mathrm{sint}(A)$. A subset A of a space (X,τ) is an α-set [15] if $A \subset \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))$. The family of all α-sets in (X,τ) is denoted by τ^α. τ^α is a topology on X which is finer than τ. The complement of an α-set is called an α-*closed* set. The closure and interior of A in (X,τ^α) are denoted by $\mathrm{cl}_\alpha(A)$ and $\mathrm{int}_\alpha(A)$, respectively. If $\mathcal{N}$ is the ideal of all nowhere dense subsets in (X,τ), then $\tau^*(\mathcal{N},\tau) = \tau^\alpha$ and $\mathrm{cl}_\alpha(A) = A \cup A^*(\mathcal{N})$ [10]. An open subset A of a space (X,τ) is said to be *regular open* if $A = \operatorname{int}(\operatorname{cl}(A))$. The complement of a regular open set is *regular closed*. A subset A of a space (X,τ) is said to be *preopen* [14] if $A \subset \operatorname{int}(\operatorname{cl}(A))$. The family of all preopen sets is denoted by $\mathrm{PO}(X,\tau)$ or simply $\mathrm{PO}(X)$. The largest preopen set contained in A is called the *preinterior* of A and is denoted by $\operatorname{pint}(A)$ and $\operatorname{pint}(A) = A \cap \operatorname{int}(\operatorname{cl}(A))$ [4]. A is preopen if and only if there is a regular open set G such that $A \subset G$ and $\operatorname{cl}(A) = \operatorname{cl}(G)$ [7, Proposition 2.1]. A subset A of a space (X,τ) is *semi-preopen* [4] if there exists a preopen set G such that $G \subset A \subset \operatorname{cl}(G)$. The family of all semi-preopen sets in (X,τ) is denoted by $\mathrm{SPO}(X,\tau)$ or simply $\mathrm{SPO}(X)$. The complement of a semi-preopen set is called *semi-preclosed*. The largest semi-preopen set contained in A is called the *semi-preinterior* of A and is denoted by $\operatorname{spint}(A)$. Also, $\operatorname{spint}(A) = A \cap \operatorname{cl}(\operatorname{int}(\operatorname{cl}(A)))$ for every A of X [4]. Given a space (X,τ) and ideals $\mathcal{I}$ and $\mathfrak{I}$ on X, the *extension* of $\mathcal{I}$ via $\mathfrak{I}$ [11], denoted by $\mathcal{I} * \mathfrak{I}$, is the ideal given by $\mathcal{I} * \mathfrak{I} = \{A \subset X \mid A^*(\mathcal{I}) \in \mathfrak{I}\}$. In particular, $\mathcal{I} * \mathcal{N} = \{A \subset X \mid \operatorname{int}(A^*(\mathcal{I})) = \phi\}$ is an ideal containing both $\mathcal{I}$ and $\mathcal{N}$ and $\mathcal{I} * \mathcal{N}$ is usually denoted by $\tilde{\mathcal{I}}$. The following lemmas will be useful in the sequel.

Lemma 1.1. *Let $(X,\tau,\mathcal{I})$ be an ideal space and $A \subset X$. If $A \subset A^*$, then*
(a) $A^* = \operatorname{cl}(A) = \operatorname{cl}^*(A)$,
(b) $A^*(\tilde{\mathcal{I}}) = A^*(\mathcal{N})$.

Proof. Clearly, for every subset A of X, $\operatorname{cl}^*(A) \subset \operatorname{cl}(A)$. Let $x \notin \operatorname{cl}^*(A)$. Then there exists a τ^*-open set G containing x such that $G \cap A = \phi$. There exists $V \in \tau$ and $I \in \mathcal{I}$ such that $x \in V - I \subset G$. $G \cap A = \phi \Rightarrow (V - I) \cap A = \phi \Rightarrow (V \cap A) - I = \phi \Rightarrow ((V \cap A) - I)^* = \phi \Rightarrow (V \cap A)^* - I^* = \phi \Rightarrow (V \cap A)^* = \phi \Rightarrow V \cap A^* = \phi \Rightarrow V \cap A = \phi$. Since V is an open set containing x, $x \notin \operatorname{cl}(A)$ and so $\operatorname{cl}(A) \subset \operatorname{cl}^*(A)$. Hence $\operatorname{cl}(A) = \operatorname{cl}^*(A)$. Since $A \subset A^* \subset \operatorname{cl}(A)$, $\operatorname{cl}(A) = A^*$. This proves (a).

(b) By [11, Theorem 3.2], $A^*(\tilde{\mathcal{I}}) = \operatorname{cl}(\operatorname{int}(A^*))$ and so by (a), $A^*(\tilde{\mathcal{I}}) = \operatorname{cl}(\operatorname{int}(\operatorname{cl}(A))) = A^*(\mathcal{N})$. $\square$

Lemma 1.2. *Let (X,τ) be a space and $A \subset X$. If A is semiopen, then* $\operatorname{cl}(A) = \operatorname{cl}_\alpha(A)$ *and if A is semiclosed, then* $\operatorname{int}(A) = \operatorname{int}_\alpha(A)$ [18, Lemma 2.1].

Lemma 1.3. *If $(X,\tau,\mathcal{I})$ is an ideal space, then the following are equivalent.*
(a) *For every $A \in \tau$, $A \subset A^*$.*
(b) *For every $A \in \mathrm{SO}(X,\tau)$, $A \subset A^*$.*

Proof. Since $\tau \subset \mathrm{SO}(X,\tau)$, it is enough to prove that (a)⇒(b). Suppose $A \in \mathrm{SO}(X,\tau)$. Then there exists an open set H such that $H \subset A \subset \operatorname{cl}(H)$. Since H is open, $H \subset H^*$ and so, by Lemma 1.1, $A \subset \operatorname{cl}(H) = H^* \subset A^*$. Hence $A \subset A^*$. $\square$

2. Completely codense ideal. An ideal $\mathcal{I}$ on a space (X,τ) is said to be *codense* [6] if $\tau \cap \mathcal{I} = \{\phi\}$ or, equivalently, $X = X^*$ [10]. $\mathcal{I}$ is said to be *completely codense* [6] if $\mathrm{PO}(X) \cap \mathcal{I} = \{\phi\}$ or, equivalently, $\mathcal{I} \subset \mathcal{N}$ [6, Theorem 4.13]. Every completely codense ideal is codense. The converse implication is not true, since in $\mathbb{R}$, the set of all real numbers with the usual topology, the ideal $\mathscr{C}$ of all countable subsets is codense but not completely codense [6]. The following theorem characterizes completely codense ideals.

Theorem 2.1. *Let $(X,\tau,\mathcal{I})$ be an ideal space. Then the following are equivalent.*
(a) *$\mathcal{I}$ is completely codense.*
(b) $\mathrm{SPO}(X)\cap\mathcal{I}=\{\phi\}$.
(c) $A\subset A^*$ *for every* $A\in\mathrm{SPO}(X)$.
(d) $\mathrm{spint}(A)=\phi$ *for every* $A\in\mathcal{I}$.

Proof. (a)⇒(b). Suppose $A\in\mathrm{SPO}(X)\cap\mathcal{I}$. $A\in\mathcal{I}\Rightarrow A\in\mathcal{N}$ and so $\mathrm{int}(\mathrm{cl}(A))=\phi$. $A\in\mathrm{SPO}(X)\Rightarrow A\subset\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))\Rightarrow A=\phi$. Therefore, $\mathrm{SPO}(X)\cap\mathcal{I}=\{\phi\}$.

(b)⇒(c). Suppose $A\in\mathrm{SPO}(X)$ and $x\notin A^*$. Then there exists an open set G containing x such that $G\cap A\in\mathcal{I}$. Since $A\in\mathrm{SPO}(X)$, $G\cap A\in\mathrm{SPO}(X)$, by [4, Theorem 2.7] and so by hypothesis, $G\cap A=\phi$ which implies that $x\notin A$.

(c)⇒(d). Let $A\in\mathcal{I}$ such that $\mathrm{spint}(A)\neq\phi$. Then there exists a nonempty semi-preopen set G such that $G\subset A$ and so $G^*\subset A^*=\phi$. Since $G\subset G^*$, $G=\phi$ which is a contradiction. Therefore, $\mathrm{spint}(A)=\phi$.

(d)⇒(a). Let $A\in\mathrm{PO}(X)\cap\mathcal{I}$. Then $A\in\mathrm{PO}(X)\Rightarrow A\subset\mathrm{int}(\mathrm{cl}(A))\subset\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))$. $A\in\mathcal{I}\Rightarrow\mathrm{spint}(A)=\phi\Rightarrow A\cap\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))=\phi\Rightarrow A=\phi$. □

Corollary 2.2. *Let $(X,\tau,\mathcal{I})$ be an ideal space with a completely codense ideal $\mathcal{I}$.*
(a) *If $A\in\mathrm{SPO}(X)$, then $A^*(\mathcal{I})$ is regular closed, $A^*(\mathcal{I})=A^*(\mathcal{N})$, and* $\mathrm{cl}(A)=\mathrm{cl}^*(A)=\mathrm{cl}_\alpha(A)$.
(b) *If A is semi-preclosed, then* $\mathrm{int}(A)=\mathrm{int}^*(A)=\mathrm{int}_\alpha(A)$.

Proof. (a) If $A\in\mathrm{SPO}(X)$, by Theorem 2.1(c), $A\subset A^*\subset\mathrm{cl}(A)$ and so $A^*=\mathrm{cl}(A)$ which implies that A^* is regular closed, since the closure of a semi-preopen set is regular closed [4, Theorem 2.4]. Therefore, $A^*=\mathrm{cl}(\mathrm{int}(A^*))=\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))=A^*(\mathcal{N})$. $\mathrm{cl}(A)=\mathrm{cl}^*(A)$ by Theorem 2.1(c) and Lemma 1.1. Also, $\mathrm{cl}^*(A)=A\cup A^*(\mathcal{I})=A\cup A^*(\mathcal{N})=\mathrm{cl}_\alpha(A)$. This proves (a).

(b) The proof follows from (a). □

3. $\mathcal{I}$-open sets.

A subset A of an ideal space $(X,\tau,\mathcal{I})$ is τ^*-*closed* [10] (resp., $*$-*dense in itself* [9], $*$-*perfect* [9]) if $A^*\subset A$ (resp., $A\subset A^*$, $A=A^*$). Clearly, A is $*$-perfect if and only if A is τ^*-closed and $*$-dense in itself. The following Theorem 3.1 is useful in the sequel.

Theorem 3.1. *Let $(X,\tau,\mathcal{I})$ be an ideal space and let U and A be subsets of X such that $A\subset U\subset A^*$. Then U is $*$-dense in itself, and U^* and A^* are $*$-perfect.*

Proof. $A\subset U\subset A^*$ implies that $U^*=A^*$ and so U is $*$-dense in itself. Since $(A^*)^*\subset A^*$, $A\subset A^*$ implies that A^* is $*$- perfect and so U^* is $*$-perfect. □

A subset A of an ideal space $(X,\tau,\mathcal{I})$ is $\mathcal{I}$-locally closed, [5] if $A=G\cap V$, where G is open and V is $*$-perfect. Clearly, every $*$-perfect set is $\mathcal{I}$-locally closed. The following theorem gives a characterization of $\mathcal{I}$-locally closed sets.

Theorem 3.2. *Let $(X,\tau,\mathcal{I})$ be an ideal space. A subset A of X is $\mathcal{I}$-locally closed if and only if $A=G\cap A^*$ for some open set G.*

Proof. Suppose A is $\mathcal{I}$-locally closed. Then $A = G\cap V$ where G is open and V is $*$-perfect. Now $A^* = (G\cap V)^* \supset G\cap V^* = G\cap V = A$. Also, $A\subset V$ implies that $A^*\subset V^* = V$. Therefore, $G\cap A^* = G\cap(A^*\cap V) = (G\cap V)\cap A^* = A\cap A^* = A$. Conversely, if $A = G\cap A^*$ where G is open, then $A\subset A^*$ and so by Theorem 3.1, A^* is $*$-perfect and so A is $\mathcal{I}$-locally closed. □

The following corollary follows from [10, Theorems 2.1 and 2.2 and Theorem 6.1(d)].

Corollary 3.3. *Let $(X,\tau,\mathcal{I})$ be an ideal space.*
(a) *Every $\mathcal{I}$-locally closed set is $*$-dense in itself.*
(b) *Every open, $*$-dense-in-itself subset of X is $\mathcal{I}$-locally closed.*
(c) *If $\mathcal{I}$ is codense, then every open set is $\mathcal{I}$-locally closed.*

A subset A of an ideal space $(X,\tau,\mathcal{I})$ is $\mathcal{I}$-open [11] if $A\subset \operatorname{int}(A^*)$. The family of all $\mathcal{I}$-open sets is denoted by $\mathrm{IO}(X,\tau,\mathcal{I})$, $\mathrm{IO}(X,\tau)$, or $\mathrm{IO}(X)$. The complement of an $\mathcal{I}$-open set is said to be $\mathcal{I}$-closed. The largest $\mathcal{I}$-open set contained in A is called the $\mathcal{I}$-interior of A and is denoted by $\operatorname{Iint}(A)$ and $\operatorname{Iint}(A) = A\cap\operatorname{int}(A^*)$ [11, Theorem 4.1(3)]. The following theorem gives some properties of $\mathcal{I}$-open sets.

Theorem 3.4. *If A is an $\mathcal{I}$-open subset of an ideal space $(X,\tau,\mathcal{I})$, then*
(a) *A is $*$-dense in itself,*
(b) *$A^* = \operatorname{cl}(A) = \operatorname{cl}^*(A)$ and $\operatorname{cl}(A)$ and A^* are regular closed,*
(c) *A^* is $*$-perfect and $\mathcal{I}$-locally closed,*
(d) *$\operatorname{int}(A^*)$ is $*$-dense in itself and $\mathcal{I}$-locally closed,*
(e) *$\operatorname{cl}(\operatorname{int}(A^*)) = A^*(\tilde{\mathcal{I}})$ is $*$-dense in itself,*
(f) *$A^* = (\operatorname{int}(A^*))^* = (\operatorname{cl}(\operatorname{int}(A^*)))^* = (A^*(\tilde{\mathcal{I}}))^*(\mathcal{I})$,*
(g) *$(\operatorname{int}(A^*))^*$ and $(\operatorname{cl}(\operatorname{int}(A^*)))^*$ are $\mathcal{I}$-locally closed,*
(h) *$\operatorname{int}(A^*)$ is $\mathcal{I}$-open.*

Proof. (a) follows from the definition. (b) follows from (a), Lemma 1.1, and the fact that every $\mathcal{I}$-open set is preopen [1] and the closure of a preopen set is regular closed [7, Proposition 2.1(ii)]. (c) follows from Theorem 3.1 and from the fact that every $*$-perfect set is $\mathcal{I}$-locally closed. (d) follows from Theorem 3.1 and Corollary 3.3(b). (e) $\operatorname{cl}(\operatorname{int}(A^*)) = A^*(\tilde{\mathcal{I}})$ by [11, Theorem 3.2] and since $A\subset\operatorname{int}(A^*)\subset\operatorname{cl}(\operatorname{int}(A^*))\subset A^*$, by Theorem 3.1, $\operatorname{cl}(\operatorname{int}(A^*))$ is $*$-dense in itself. (f) From the inequality in the proof of (e), we have $A^* = (\operatorname{int}(A^*))^* = (\operatorname{cl}(\operatorname{int}(A^*)))^*$. Each is equal to $(A^*(\tilde{\mathcal{I}}))^*(\mathcal{I})$ by (e). (g) and (h) follow from (c) and (f), respectively. □

Theorem 3.5. *Let $(X,\tau,\mathcal{I})$ be an ideal space. If A is $\mathcal{I}$-open and V is semiopen, then*
(a) *$V\cap A$ is $*$-dense in itself,*
(b) *$(V\cap A)^*$ is $*$-perfect and $\mathcal{I}$-locally closed,*
(c) *$\operatorname{cl}(V)\cap A$ is $*$-dense in itself,*
(d) *$(\operatorname{cl}(V)\cap A)^*$ is $*$-perfect and $\mathcal{I}$-locally closed.*

Proof. Since $V\cap A\subset\operatorname{cl}(V)\cap A\subset(V\cap A)^*$ by [1, Theorem 2.10], $V\cap A$ is $*$-dense in itself and by Theorem 3.1, $\operatorname{cl}(V)\cap A$ is $*$-dense in itself and so by Theorem 3.1, $(V\cap A)^*$ and $(\operatorname{cl}(V)\cap A)^*$ are $*$-perfect and so are $\mathcal{I}$-locally closed. □

The following theorem shows that (X,τ) and (X,τ^α) have the same collection of $\mathcal{I}$-open sets.

Theorem 3.6. *If $(X,\tau,\mathcal{I})$ is an ideal space, then* $\mathrm{IO}(X,\tau,\mathcal{I}) = \mathrm{IO}(X,\tau^\alpha,\mathcal{I})$.

Proof. $A \in \mathrm{IO}(X,\tau)$ if and only if $A \subset \mathrm{int}(A^*)$ if and only if $A \subset \mathrm{int}_\alpha(A^*)$, by Lemma 1.2 if and only if $A \in \mathrm{IO}(X,\tau^\alpha)$. □

Corollary 3.7. *If $(X,\tau,\mathcal{I})$ is an ideal space where $\mathcal{I}$ is completely codense, then* $\mathrm{IO}(X,\tau) = \mathrm{IO}(X,\tau^*) = \mathrm{IO}(X,\tau^\alpha)$.

Proof. Follows from Corollary 2.2(b). □

The following theorem and corollary are generalizations of [1, Theorem 2.6(iii) and Corollary 2.1(ii)], respectively.

Theorem 3.8. *Let $(X,\tau,\mathcal{I})$ be an ideal space. If $A \in \mathrm{IO}(X)$ and $B \in \tau^\alpha$, then $A \cap B \in \mathrm{IO}(X)$.*

Proof. $A \in \mathrm{IO}(X,\tau) \Rightarrow A \in \mathrm{IO}(X,\tau^\alpha)$ and so by [1, Theorem 2.6(ii)], $A\cap B \in \mathrm{IO}(X,\tau^\alpha)$ which implies that $A \cap B \in \mathrm{IO}(X,\tau)$. □

Corollary 3.9. *Let $(X,\tau,\mathcal{I})$ be an ideal space. If A is $\mathcal{I}$-closed and B is α-closed, then $A \cup B$ is $\mathcal{I}$-closed.*

Every $\mathcal{I}$-open set is preopen but the converse need not be true [1, Example 2.3]. The following theorem characterizes $\mathcal{I}$-open sets in terms of preopen sets.

Theorem 3.10. *Let $(X,\tau,\mathcal{I})$ be an ideal space and $A \subset X$. Then the following are equivalent.*
(a) *A is $\mathcal{I}$-open.*
(b) *$A \subset A^*$ and* $\mathrm{scl}(A) = \mathrm{int}(\mathrm{cl}(A))$.
(c) *$A \subset A^*$ and A is preopen.*

Proof. $A \in \mathrm{IO}(X)$ if and only if $A \subset A^*$ and $A \subset \mathrm{int}(A^*)$ if and only if $A \subset A^*$ and $A \subset \mathrm{int}(\mathrm{cl}(A))$, since $\mathrm{cl}(A) = A^*$ if and only if $A \subset A^*$ and $A \cup \mathrm{int}(\mathrm{cl}(A)) = \mathrm{int}(\mathrm{cl}(A))$ if and only if $A \subset A^*$ and $\mathrm{scl}(A) = \mathrm{int}(\mathrm{cl}(A))$. Therefore, (a) and (b) are equivalent. It is clear that (a) and (c) are equivalent. □

Corollary 3.11. *Let $(X,\tau,\mathcal{I})$ be an ideal space and $A \subset X$.*
(a) *If A is semiclosed and $\mathcal{I}$-open, then A is regular open.*
(b) *If A is semiopen and $\mathcal{I}$-closed, then A is regular closed.*
(c) *If A is $\mathcal{I}$-open, then* $\mathrm{sint}(\mathrm{scl}(A)) = \mathrm{int}(\mathrm{scl}(A)) = \mathrm{int}(\mathrm{cl}(A))$.

For subsets of any ideal space $(X,\tau,\mathcal{I})$, openness and $\mathcal{I}$-openness are independent concepts [1, Examples 2.1 and 2.2]. The following Theorem 3.12 shows that the two concepts coincide for $*$-perfect sets. Corollary 3.13 follows from the fact that every τ^*-closed, $\mathcal{I}$-open set is $*$-perfect.

Theorem 3.12. *Let $(X,\tau,\mathcal{I})$ be an ideal space and $A \subset X$.*
(a) *If A is $*$-dense in itself, then* $\mathrm{Iint}(A^*) = \mathrm{int}(A^*)$.

(b) *If A is $*$-perfect, then* $\text{Iint}(A) = \text{int}(A)$ *and so, for $*$-perfect sets, the concepts open and $\mathcal{I}$-open coincide.*

Proof. Since A is $*$-dense in itself, A^* is $*$-perfect, by Theorem 3.1. Now $\text{Iint}(A^*) = A^* \cap \text{int}((A^*)^*) = A^* \cap \text{int}(A^*) = \text{int}(A^*)$. This proves (a). (b) follows from (a). □

Corollary 3.13. *Let $(X,\tau,\mathcal{I})$ be an ideal space and $A \subset X$. If A is τ^*-closed and $\mathcal{I}$-open, then A is open.*

In [17, Remark 4], it was stated that $\mathcal{I}$ is codense if and only if $\tau \subset \text{IO}(X)$. The following Theorem 3.14(a) follows from the above result. Theorem 3.14(b) follows from Theorem 3.6 and the fact that $\text{SO}(X) \cap \mathcal{I} = \{\phi\}$ if and only if $\tau \cap \mathcal{I} = \{\phi\}$. Theorem 3.15 is a characterization of completely codense ideals.

Theorem 3.14. *Let $(X,\tau,\mathcal{I})$ be an ideal space.*
(a) *If* $\text{SO}(X) \subset \text{IO}(X)$, *then $\mathcal{I}$ is codense.*
(b) *$\mathcal{I}$ is codense if and only if $\tau^\alpha \subset \text{IO}(X)$.*

Theorem 3.15. *Let $(X,\tau,\mathcal{I})$ be an ideal space. Then $\mathcal{I}$ is completely codense if and only if* $\text{PO}(X) = \text{IO}(X)$.

Proof. Suppose $\mathcal{I}$ is completely codense and $G \in \text{PO}(X)$. Then $G \subset G^*$, by Theorem 2.1(c) and so $\text{cl}(G) = G^*$. $G \in \text{PO}(X)$ implies $G \subset \text{int}(\text{cl}(G)) = \text{int}(G^*)$ and so $G \in \text{IO}(X)$. Therefore, $\text{PO}(X) \subset \text{IO}(X)$. Clearly, $\text{IO}(X) \subset \text{PO}(X)$. Conversely, if $G \in \text{SPO}(X)$, then there exists $V \in \text{PO}(X)$ such that $V \subset G \subset \text{cl}(V)$ and by hypothesis, $V \subset V^*$ and so by Lemma 1.1, $\text{cl}(V) = V^*$. Hence by Theorem 3.1, G is $*$-dense in itself and so by Theorem 2.1, $\mathcal{I}$ is completely codense. □

In the following Theorem 3.16, we show that if A is $\mathcal{I}$-open, then $\text{sint}(A^*)$ is regular closed.

Theorem 3.16. *Let $(X,\tau,\mathcal{I})$ be an ideal space and $A \subset X$.*
(a) *For every subset A of X,* $\text{cl}(\text{Iint}(A)) = \text{cl}(\text{int}(A^*)) = \text{sint}(A^*)$.
(b) *If A is $\mathcal{I}$-open, then* $A^* = \text{cl}(A) = \text{cl}(\text{int}(A^*)) = \text{sint}(A^*)$ *and so* $\text{sint}(A^*)$ *is regular closed.*

Proof. If A is a subset of X, then $\text{sint}(A^*) = A^* \cap \text{cl}(\text{int}(A^*)) = \text{cl}(\text{int}(A^*))$. To prove the other equality, since $\text{Iint}(A) = A \cap \text{int}(A^*)$, $\text{cl}(\text{Iint}(A)) = \text{cl}(A \cap \text{int}(A^*)) \supset \text{cl}(A) \cap \text{int}(A^*) = \text{int}(A^*)$ and so $\text{cl}(\text{Iint}(A)) \supset \text{cl}(\text{int}(A^*))$. To prove the reverse direction, note that $\text{Iint}(A) \subset \text{int}(A^*)$ and so $\text{cl}(\text{Iint}(A)) \subset \text{cl}(\text{int}(A^*))$. This completes the proof of (a). (b) follows from (a) and Theorem 3.4(b). □

A subset A of an ideal space $(X,\tau,\mathcal{I})$ is $\mathcal{I}$-dense [6] if $A^* = X$. Clearly, every $\mathcal{I}$-dense set is dense but the converse is not true. If G is any proper dense subset of an ideal space $(X,\tau,\mathcal{I})$ where $\mathcal{I}$ is the maximal ideal $\wp(X)$, then G is not $\mathcal{I}$-dense. In particular, if $\mathcal{I}$ is not codense, then X is not $\mathcal{I}$-dense and hence no subset of X is $\mathcal{I}$-dense [6]. Therefore, the existence of an $\mathcal{I}$-dense set implies that the ideal is codense. The following theorem characterizes $\mathcal{I}$-open sets in terms of $\mathcal{I}$-dense sets.

Theorem 3.17. *Let $(X,\tau,\mathcal{I})$ be an ideal space with a codense ideal $\mathcal{I}$ and $A \subset X$. Then the following are equivalent.*

(a) *A is $\mathcal{I}$-open.*

(b) *There is a regular open set G such that $A \subset G$ and $A^* = G^*$.*

(c) *$A = G \cap D$ where G is regular open and D is $\mathcal{I}$-dense.*

(d) *$A = G \cap D$ where G is open and D is $\mathcal{I}$-dense.*

Proof. (a)⇒(b). That A is $\mathcal{I}$-open implies $A \subset \operatorname{int}(A^*) \subset A^*$. Let $G = \operatorname{int}(A^*)$. Then $A \subset G$ and $\operatorname{int}(\operatorname{cl}(G)) = \operatorname{int}(\operatorname{cl}(\operatorname{int}(A^*))) = \operatorname{int}(\operatorname{cl}(\operatorname{int}(\operatorname{cl}(A)))) = \operatorname{int}(\operatorname{cl}(A)) = \operatorname{int}(A^*) = G$ and so G is regular open. $G^* = (\operatorname{int}(A^*))^* = A^*$, by Theorem 3.4(f).

(b)⇒(c). Let G be a regular open set such that $A \subset G$ and $A^* = G^*$. Let $D = A \cup (X - G)$. Then $A = G \cap D$ where G is regular open. Now $D^* = (A \cup (X - G))^* = A^* \cup (X - G)^* = G^* \cup (X - G)^* = (G \cup (X - G))^* = X^* = X$, since $\mathcal{I}$ is codense. Therefore, D is $\mathcal{I}$-dense which proves (c).

(c)⇒(d) is clear.

(d)⇒(a). Suppose $A = G \cap D$ where G is open and D is $\mathcal{I}$-dense. Now $G = G \cap X = G \cap D^* \subset (G \cap D)^*$ and so $G \subset \operatorname{int}((G \cap D)^*) = \operatorname{int}(A^*)$. Therefore, $A \subset G \subset \operatorname{int}(A^*)$ which implies that A is $\mathcal{I}$-open. □

The following theorem is a generalization of [1, Theorem 2.14(ii)].

Theorem 3.18. *Let $(X,\tau,\mathcal{I})$ be an ideal space and $A \subset X$. If A is $\mathcal{I}$-closed and α-open, then $A = \operatorname{cl}(A) = \operatorname{int}(\operatorname{cl}(A)) = \operatorname{cl}(\operatorname{int}(A))$ and so A is both regular open and regular closed.*

Proof. A is $\mathcal{I}$-closed ⇒ $X - A$ is $\mathcal{I}$-open ⇒ $X - A \subset \operatorname{int}(X - A)^*$ ⇒ $X - A \subset \operatorname{int}(\operatorname{cl}(X - A))$ ⇒ $X - A \subset X - \operatorname{cl}(\operatorname{int}(A))$ ⇒ $\operatorname{cl}(\operatorname{int}(A)) \subset A$. A is α-open ⇒ A is semiopen and preopen [16] ⇒ $\operatorname{cl}(A) = \operatorname{cl}(\operatorname{int}(A))$ and $A \subset \operatorname{int}(\operatorname{cl}(A))$. Therefore, $\operatorname{int}(\operatorname{cl}(A)) \subset \operatorname{cl}(A) = \operatorname{cl}(\operatorname{int}(A)) \subset A \subset \operatorname{int}(\operatorname{cl}(A))$ and so $A = \operatorname{cl}(A) = \operatorname{cl}(\operatorname{int}(A)) = \operatorname{int}(\operatorname{cl}(A))$. □

4. Quasi-$\mathcal{I}$-open sets. A subset A of an ideal space $(X,\tau,\mathcal{I})$ is quasi-$\mathcal{I}$-open [2] if $A \subset \operatorname{cl}(\operatorname{int}(A^*))$. Every $\mathcal{I}$-open set is quasi-$\mathcal{I}$-open and every quasi-$\mathcal{I}$-open set is semi-preopen but the converse implications need not be true [2, Examples 1 and 2]. Also, quasi-$\mathcal{I}$-openness and semiopenness (resp., preopenness) are independent concepts [2, Examples 1 and 2]. The family of all quasi-$\mathcal{I}$-open sets is denoted by $Q\mathcal{I}O(X,\tau)$. The following theorem gives some of the properties of quasi-$\mathcal{I}$-open sets, the proof of which is similar to the proof of Theorem 3.4.

Theorem 4.1. *Let $(X,\tau,\mathcal{I})$ be an ideal space and A a quasi-$\mathcal{I}$-open subset of X. Then*

(a) *A is $*$-dense in itself,*

(b) *$A^* = \operatorname{cl}(A) = \operatorname{cl}^*(A)$,*

(c) *A^* is $*$-perfect, regular closed, and $\mathcal{I}$-locally closed,*

(d) *$\operatorname{cl}(\operatorname{int}(A^*)) = A^*(\tilde{\mathcal{I}})$ is $*$-dense in itself,*

(e) *$A^* = (\operatorname{cl}(\operatorname{int}(A^*)))^* = (A^*(\tilde{\mathcal{I}}))^*(\mathcal{I})$,*

(f) *$(\operatorname{cl}(\operatorname{int}(A^*)))^*$ is $*$-perfect and $\mathcal{I}$-locally closed.*

Corollary 4.2. *Let $(X,\tau,\mathcal{I})$ be an ideal space. A subset A of X is quasi-$\mathcal{I}$-open if and only if $A\subset A^*(\tilde{\mathcal{I}})$* [2, Theorem 3].

Theorem 4.3. *Let $(X,\tau,\mathcal{I})$ be an ideal space and let U and A be subsets of X such that $A\subset U\subset A^*$. Then U^* is $*$-perfect, and if A is quasi-$\mathcal{I}$-open, then U is quasi-$\mathcal{I}$-open and so $\operatorname{cl}(\operatorname{int}(A^*))$ is quasi-$\mathcal{I}$-open.*

Proof. By Theorem 3.1, $U^*=A^*$ and U^* is $*$-perfect. A is quasi-$\mathcal{I}$-open $\Rightarrow A\subset \operatorname{cl}(\operatorname{int}(A^*))=\operatorname{cl}(\operatorname{int}(U^*))$. Now $U\subset A^*\Rightarrow U\subset(\operatorname{cl}(\operatorname{int}(U^*)))^*\Rightarrow U\subset \operatorname{cl}(\operatorname{cl}(\operatorname{int}(U^*)))=\operatorname{cl}(\operatorname{int}(U^*))$. Therefore, U is quasi-$\mathcal{I}$-open. Since $A\subset\operatorname{cl}(\operatorname{int}(A^*))\subset A^*$, $\operatorname{cl}(\operatorname{int}(A^*))$ is quasi-$\mathcal{I}$-open. □

Every quasi-$\mathcal{I}$-open set is semi-preopen but the converse is not true [2]. [2, Proposition 3(iii)] says that every semiopen set which is $*$-dense in itself is quasi-$\mathcal{I}$-open. The following Theorem 4.4 is a generalization of this result and shows that for $*$-dense in itself, the concepts quasi-$\mathcal{I}$-open and semi-preopen are equivalent. Theorem 4.5(a) gives a characterization of codense ideals and Theorem 4.5(b) gives a characterization of completely codense ideals.

Theorem 4.4. *Let $(X,\tau,\mathcal{I})$ be an ideal space. If A is semi-preopen and $*$-dense in itself, then A is quasi-$\mathcal{I}$-open.*

Proof. $A\subset A^*\Rightarrow \operatorname{cl}(A)=A^*$, by Lemma 1.1. A is semi-preopen $\Rightarrow A\subset\operatorname{cl}(\operatorname{int}(\operatorname{cl}(A)))=\operatorname{cl}(\operatorname{int}(A^*))$ and so A is quasi-$\mathcal{I}$-open. □

Theorem 4.5. *Let $(X,\tau,\mathcal{I})$ be an ideal space. Then*
(a) *$\mathcal{I}$ is codense if and only if $\mathrm{SO}(X)\subset Q\mathcal{I}O(X)$,*
(b) *$\mathcal{I}$ is completely codense if and only if $\mathrm{SPO}(X)=Q\mathcal{I}O(X)$.*

Proof. (a) Suppose $\mathcal{I}$ is codense. Let $G\in\mathrm{SO}(X)$. By [10, Theorem 6.1] and Lemma 1.3, G is $*$-dense in itself and so by [2, Proposition 3(iii)], $G\in Q\mathcal{I}O(X)$. Conversely, suppose that $\mathrm{SO}(X)\subset Q\mathcal{I}O(X)$. If $G\in\mathrm{SO}(X)$, then $G\in Q\mathcal{I}O(X)$ and so $G\subset G^*$. Therefore, $\mathcal{I}$ is codense by [10, Theorem 6.1] and Lemma 1.3.

(b) Suppose $\mathcal{I}$ is completely codense and $G\in\mathrm{SPO}(X)$. Then $G\subset G^*$, by Theorem 2.1(c) and so $\operatorname{cl}(G)=G^*$. $G\in\mathrm{SPO}(X)\Rightarrow G\subset\operatorname{cl}(\operatorname{int}(\operatorname{cl}(G)))=\operatorname{cl}(\operatorname{int}(G^*))$ and so $G\in Q\mathcal{I}O(X)$. Therefore, $\mathrm{SPO}(X)\subset Q\mathcal{I}O(X)$. Clearly, $Q\mathcal{I}O(X)\subset\mathrm{SPO}(X)$. Conversely, if $G\in\mathrm{SPO}(X)$, then $G\in Q\mathcal{I}O(X)$, by hypothesis, and so $G\subset G^*$, and so by Theorem 2.1(c), $\mathcal{I}$ is completely codense. □

In [2], it was established that the intersection of a quasi-$\mathcal{I}$-open set with an α-set is semi-preopen. The following theorem is a generalization of the above result.

Theorem 4.6. *Let $(X,\tau,\mathcal{I})$ be an ideal space. Then (a) $Q\mathcal{I}O(X,\tau)=Q\mathcal{I}O(X,\tau^\alpha)$ and (b) $A\in Q\mathcal{I}O(X,\tau)$ and $B\in\tau^\alpha$ implies $A\cap B\in Q\mathcal{I}O(X,\tau)$.*

Proof. $A\in Q\mathcal{I}O(X,\tau)$ if and only if $A\subset\operatorname{cl}(\operatorname{int}(A^*))$ if and only if $A\subset\operatorname{cl}_\alpha(\operatorname{int}_\alpha(A^*))$ [3] if and only if $A\in Q\mathcal{I}O(X,\tau^\alpha)$ which proves (a). $A\in Q\mathcal{I}O(X,\tau)$ and $B\in\tau^\alpha\Rightarrow A\in Q\mathcal{I}O(X,\tau^\alpha)$ and $B\in\tau^\alpha\Rightarrow A\cap B\in Q\mathcal{I}O(X,\tau^\alpha)$; by [2, Proposition 2] implies $A\cap B\in Q\mathcal{I}O(X,\tau)$. □

[2, Lemma 2] states that $W^*(\mathcal{N}) \subset W$ for every subset W of X in the ideal space $(X,\tau,\mathcal{N})$. That is, every subset of X is τ^*-closed and so τ^* is the discrete topology. This is not always the case. For example, if we consider $\mathbb{R}$ with the usual topology τ and the ideal $\mathcal{N}$ of nowhere dense subsets of $\mathbb{R}$, then $Q^* = \mathbb{R}$ and so Q is not τ^*-closed. Therefore, [2, Proposition 4] is no longer valid. Also, it was established that every τ^*-closed, quasi-$\mathcal{I}$-open set is semiopen [2, Proposition 3(iii)]. The following Theorem 4.7(a) is a generalization of the above result and also shows that the condition *preclosed* is not necessary in [2, Proposition 5(i)], and Theorem 4.7(b) shows that [2, Proposition 3(iii)] is also true if we replace the condition τ^*-closed by semiclosed.

Theorem 4.7. *Let $(X,\tau,\mathcal{I})$ be an ideal space and $A \subset X$.*
(a) *If A is τ^*-closed and quasi-$\mathcal{I}$-open, then A is regular closed.*
(b) *If A is semiclosed and quasi-$\mathcal{I}$-open, then A is semiopen and $A^* = A^*(\mathcal{N})$.*

Proof. (a) That A is τ^*-closed and quasi-$\mathcal{I}$-open implies $A = A^*$. Also, $A \in Q\mathcal{I}O(X)$ $\Rightarrow A \subset \mathrm{cl}(\mathrm{int}(A^*)) \Rightarrow \mathrm{int}(A^*) \subset A^* \subset \mathrm{cl}(\mathrm{int}(A^*)) \Rightarrow \mathrm{cl}(\mathrm{int}(A^*)) \subset A^* \subset \mathrm{cl}(\mathrm{int}(A^*))$. Therefore, $A = A^* = \mathrm{cl}(\mathrm{int}(A^*)) = \mathrm{cl}(\mathrm{int}(A))$ and so A and A^* are regular closed. (b) A is semiclosed $\Rightarrow \mathrm{int}(A) = \mathrm{int}(\mathrm{cl}(A))$ by [8, Proposition 1]. That A is quasi-$\mathcal{I}$-open implies $A \subset \mathrm{cl}(\mathrm{int}(A^*)) = \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A))) = \mathrm{cl}(\mathrm{int}(A))$ and so A is semiopen. By Theorem 4.1(b), $\mathrm{cl}(A) = A^*$. Since $\mathrm{int}(\mathrm{cl}(A)) \subset A \subset \mathrm{cl}(\mathrm{int}(A^*)) = \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))$, $\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A))) \subset \mathrm{cl}(A)$ $\subset \mathrm{cl}(\mathrm{int}(A^*)) = \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))$ and so $A^* = \mathrm{cl}(A) = \mathrm{cl}(\mathrm{int}(A^*)) = A^*(\mathcal{N})$. $\square$

The following theorem gives a characterization of quasi-$\mathcal{I}$-open sets.

Theorem 4.8. *Let $(X,\tau,\mathcal{I})$ be an ideal space and $A \subset X$. A is quasi-$\mathcal{I}$-open if and only if $A \subset A^*$ and $\mathrm{cl}_\alpha(A) = \mathrm{cl}(\mathrm{int}(A^*))$.*

Proof. Suppose $A \in Q\mathcal{I}O(X)$. Then $A \subset A^*$ and $\mathrm{cl}(A) = A^*$. Also $A \subset \mathrm{cl}(\mathrm{int}(A^*)) \Rightarrow$ $A \subset \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A))) \Rightarrow A \cup \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A))) = \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A))) \Rightarrow \mathrm{cl}_\alpha(A) = \mathrm{cl}(\mathrm{int}(A^*))$, since $\mathrm{cl}_\alpha(A) = A \cup \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))$ [3]. Conversely, suppose the conditions hold. Then $\mathrm{cl}_\alpha(A) =$ $\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))$ and so $A \subset \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A))) = \mathrm{cl}(\mathrm{int}(A^*))$. Therefore, A is quasi-$\mathcal{I}$-open. $\square$

The quasi-$\mathcal{I}$-interior of a subset A in an ideal space $(X,\tau,\mathcal{I})$ is the largest quasi-$\mathcal{I}$-open set contained in A and is denoted by $\mathrm{qIint}(A)$. The following theorem deals with the properties of the quasi-$\mathcal{I}$-interior of subsets of ideal spaces. In [11], it was established that $\mathrm{Iint}(A) = \phi$ if and only if $A \in \tilde{\mathcal{I}}$. Theorem 4.9(c) is a partial generalization of this result.

Theorem 4.9. *Let $(X,\tau,\mathcal{I})$ be an ideal space and $A \subset X$. Then*
(a) $\mathrm{qIint}(A) = A \cap \mathrm{cl}(\mathrm{int}(A^*))$ *for every subset A of X,*
(b) *if A is α-closed, then* $\mathrm{qIint}(A) = \mathrm{cl}(\mathrm{int}(A^*))$ *and the converse holds if $A \subset A^*$,*
(c) $\mathrm{qIint}(A) = \phi$ *if and only if $A \in \tilde{\mathcal{I}}$.*

Proof. (a) $A \cap \mathrm{cl}(\mathrm{int}(A^*)) \subset \mathrm{cl}(\mathrm{int}(A^*)) = \mathrm{cl}(\mathrm{int}(\mathrm{int}(A^*))) = \mathrm{cl}(\mathrm{int}(A^* \cap (\mathrm{int}\,A^*))) \subset$ $\mathrm{cl}(\mathrm{int}((A \cap \mathrm{int}(A^*))^*)) \subset \mathrm{cl}(\mathrm{int}((A \cap \mathrm{cl}(\mathrm{int}(A^*)))^*))$. Therefore, $A \cap \mathrm{cl}(\mathrm{int}(A^*))$ is a quasi-$\mathcal{I}$-open set contained in A and so $A \cap \mathrm{cl}(\mathrm{int}(A^*)) \subset \mathrm{qIint}(A)$. Since $\mathrm{qIint}(A)$ is quasi-$\mathcal{I}$-open, $\mathrm{qIint}(A) \subset \mathrm{cl}(\mathrm{int}(\mathrm{qIint}(A))^*) \subset \mathrm{cl}(\mathrm{int}(A^*))$ and so $A \cap \mathrm{qIint}(A) \subset A \cap \mathrm{cl}(\mathrm{int}(A^*))$ which implies that $\mathrm{qIint}(A) \subset A \cap \mathrm{cl}(\mathrm{int}(A^*))$. Hence $\mathrm{qIint}(A) = A \cap \mathrm{cl}(\mathrm{int}(A^*))$.

(b) A is α-closed $\Rightarrow \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A))) \subset A \Rightarrow \mathrm{cl}(\mathrm{int}(A^*)) \subset A \Rightarrow \mathrm{qIint}(A) = \mathrm{cl}(\mathrm{int}(A^*))$. Conversely, if $A \subset A^*$, then $A^* = \mathrm{cl}(A)$. $\mathrm{qIint}(A) = \mathrm{cl}(\mathrm{int}(A^*)) \Rightarrow \mathrm{cl}(\mathrm{int}(A^*)) \subset A$ and so $\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A))) \subset A$ and so A is α-closed.

(c) $\mathrm{qIint}(A) = \phi \Rightarrow A \cap \mathrm{cl}(\mathrm{int}(A^*)) = \phi \Rightarrow A \cap \mathrm{int}(A^*) = \phi \Rightarrow \mathrm{Iint}(A) = \phi \Rightarrow A \in \tilde{\mathcal{I}}$. Conversely, $A \in \tilde{\mathcal{I}} \Rightarrow \mathrm{int}(A^*) = \phi \Rightarrow \mathrm{cl}(\mathrm{int}(A^*)) = \phi \Rightarrow A \cap \mathrm{cl}(\mathrm{int}(A^*)) = \phi \Rightarrow \mathrm{qIint}(A) = \phi$. $\square$

Acknowledgments. The authors sincerely thank Professors T. R. Hamlett and D. A. Rose for sending some of their reprints which were helpful in the preparation of this note, and also thank the referees for their valuable suggestions.

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