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COMPLETELY MULTIPLICATIVE FUNCTIONS ARISING FROM SIMPLE OPERATIONS

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Given two multiplicative arithmetic functions, various conditions for their convolution, powers, and logarithms to be completely multiplicative, based on values at the primes, are derived together with their applications.

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1. Introduction. By an *arithmetic function* we mean a complex-valued function whose domain is the set of positive integers $\mathbb{N}$. We define the addition and convolution of two arithmetic functions f and g, respectively, by

$$(f+g)(n)=f(n)+g(n), \qquad (f*g)(n)=\sum_{ij=n} f(i)g(j). \tag{1.1}$$

It is well known (see, e.g., [2, 6, 15, 16]) that the set $(\mathcal{A},+,*)$ of all arithmetic functions is a unique factorization domain with the arithmetic function

$$I(n)=\begin{cases}1 & \text{if } n=1,\\ 0 & \text{otherwise,}\end{cases} \tag{1.2}$$

being its convolution identity.

A nonzero arithmetic function $f\in\mathcal{A}$ is called *multiplicative* if and only if

$$f(mn)=f(m)f(n) \quad \text{whenever } (m,n)=1. \tag{1.3}$$

It is called *completely multiplicative* if this equality holds for all $m,n\in\mathbb{N}$.

Denote by $\mathcal{M}$ the set of all multiplicative functions and by $\mathcal{C}$ the set of all completely multiplicative functions. It is easy to see that the elements of $\mathcal{M}$ are uniquely determined via their values at all prime powers, while those of $\mathcal{C}$ are uniquely determined only via their values at the primes. Unlike $\mathcal{M}$, which is a group with respect to convolution, $\mathcal{C}$ is closed neither with respect to addition nor convolution.

A natural problem is that of characterizing $\mathcal{M}$ and $\mathcal{C}$ by various means and there have appeared a number of such investigations, for example, by Apostol [1, 2], Carroll [5], Laohakosol et al. [10, 11], and Rearick [12, 13].

In this paper, we propose the naive problem: given $f,g\in\mathcal{M}$, find (simple) necessary and sufficient condition(s) for their convolution, powers, and logarithms to be in $\mathcal{C}$.

Though the problem seems elementary, we have not been able to locate its complete solution. As we will see, several of the techniques used to resolve certain cases of this problem are not elementary, that is, making use of logarithmic operators (see [10, 11, 13]).

The question about sum is trivial because by looking at the values at $n = 1$ we see immediately that a sum of two multiplicative functions is never multiplicative.

To facilitate later discussions, we recall some basic facts about logarithmic operators on $\mathcal{A}$. For $f \in \mathcal{A}$, its Rearick logarithm, denoted by $\operatorname{Log} f \in \mathcal{A}$, is defined by

$$\begin{aligned}(\operatorname{Log} f)(1) &= \log f(1) \quad (\text{assuming } f(1) > 0),\\ (\operatorname{Log} f)(n) &= \frac{1}{\log n}\sum_{d\mid n} f(d) f^{-1}\left(\frac{n}{d}\right)\log d\\ &= \frac{1}{\log n}(df * f^{-1})(n) \quad (n > 1),\end{aligned} \tag{1.4}$$

where $df(n) = f(n)\log n$ denotes the log-derivation of f.

In 1973, Carlitz and Subbarao [4] defined another logarithmic operator of f, denoted by $\beta f \in \mathcal{A}$, as follows:

$$\begin{gathered}(\beta f)(1) = 1 \quad (\text{assuming } f(1) = 1),\\ (\beta f)(n) = \sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{r}\sum_{\substack{a_1\cdots a_r = n\\ a_i \neq 1\,(i=1,\dots,r)}} f(a_1)\cdots f(a_r) \quad (n > 1).\end{gathered} \tag{1.5}$$

As proved in [10], these two logarithms are

(i) essentially identical, that is, both take the same values except only at $n = 1$,

(ii) both bijective,

(iii) both transform convolution into sum.

Indeed, as shown by Carlitz and Subbarao [4], the inverse (with respect to convolution) of β is the operator $\gamma : \mathcal{U}_1 \to \mathcal{U}_1$, where $\mathcal{U}_1$ is the set of all arithmetic functions f such that $f(1) = 1$, defined by

$$\begin{gathered}(\gamma g)(1) = 1,\\ (\gamma g)(n) = \sum_{r=1}^{\infty}\frac{1}{r!}\sum_{\substack{a_1\cdots a_r = n\\ a_i \neq 1\,(i=1,\dots,r)}} g(a_1)\cdots g(a_r) \quad (n > 1).\end{gathered} \tag{1.6}$$

2. Convolution.

We first look at the case of convolution.

Theorem 2.1. *Let $f, g \in \mathcal{M}$. Then $f * g \in \mathcal{C} \Leftrightarrow$ either*

$$\begin{aligned}g(p^a) &= f^{-1}(p^a) + \big(f(p)+g(p)\big)f^{-1}(p^{a-1})\\ &\quad + \cdots + \big(f(p)+g(p)\big)^{a-1}f^{-1}(p) + \big(f(p)+g(p)\big)^a\end{aligned} \tag{2.1}$$

or

$$f(p^a) = g^{-1}(p^a) + (f(p)+g(p))g^{-1}(p^{a-1}) + \cdots + (f(p)+g(p))^{a-1}g^{-1}(p) + (f(p)+g(p))^a \tag{2.2}$$

for all primes p and all $a \in \mathbb{N}$.

Proof. Writing $g = (f*g)*f^{-1}$ and evaluating at prime powers, we get

$$g(p^a) = f^{-1}(p^a)(f*g)(1) + f^{-1}(p^{a-1})(f*g)(p) + \cdots + f^{-1}(1)(f*g)(p^a). \tag{2.3}$$

If $f*g \in \mathscr{C}$, then $(f*g)(p^i) = (f*g)(p)^i = (f(p)+g(p))^i$, and (2.1) results immediately. Further, (2.2) follows by interchanging f and g in the above arguments.

On the other hand, if (2.1) holds, then comparing (2.1) and (2.3) for successive values of $a = 1,2,3,\ldots$, we conclude that

$$(f*g)(p^a) = (f*g)(p)^a, \tag{2.4}$$

and so

$$(f*g) \in \mathscr{C}. \tag{2.5}$$

The same result holds similarly for (2.2). □

Two remarkable consequences are now in order.

Corollary 2.2. *Let $f \in \mathscr{C}$ and $g \in \mathscr{M}$. Then*

$$f*g \in \mathscr{C} \Longleftrightarrow g(p^a) = g(p)(g(p)+f(p))^{a-1} \tag{2.6}$$

for all primes p and all $a \in \mathbb{N}$.

Proof. This is immediate from (2.1) in Theorem 2.1, noting that $f^{-1}(p) = -f(p)$ and

$$f \in \mathscr{C} \Longleftrightarrow f^{-1}(p^i) = 0 \quad \forall i \geq 2. \tag{2.7}$$

□

Corollary 2.3. *Let $f, g \in \mathscr{C}$. Then*

$$f*g \in \mathscr{C} \Longleftrightarrow f(p)g(p) = 0 \tag{2.8}$$

for all primes p.

Proof. From Corollary 2.2, we have

$$\begin{aligned} f*g \in \mathscr{C} &\Longrightarrow g(p^2) = g(p)(g(p)+f(p)) \\ &\Longrightarrow f(p)g(p) = 0, \end{aligned} \tag{2.9}$$

while

$$f(p)g(p)=0 \Longrightarrow g(p^a)=g(p)\big(g(p)+f(p)\big)^{a-1} \\ \Longrightarrow f*g\in\mathscr{C}. \tag{2.10}$$

□

3. Powers. We first recall some of the results in [13]. For $h\in\mathscr{A}$, the (Rearick) exponential $\operatorname{Exp} h$ is defined as the unique element

$$f\in\mathscr{A},\quad f(1)>0,\quad \text{such that } h=\operatorname{Log} f. \tag{3.1}$$

For $f\in\mathscr{A}$, $f(1)>0$, and $r\in\mathbb{R}$, the rth power function f^r is defined as

$$f^r=\operatorname{Exp}(r\operatorname{Log} f). \tag{3.2}$$

From [12], we know that if $f\in\mathscr{M}$, $r\in\mathbb{R}$, then $f^r\in\mathscr{M}$, which also implies that if $f^r\in\mathscr{M}$, $r\in\mathbb{R}\setminus\{0\}$, is multiplicative, then f is also multiplicative.

Recall also the Hsu's generalized Möbius function (see [3, 8, 11])

$$\mu_\alpha(n)=\prod_{p\mid n}\binom{\alpha}{\nu_p(n)}(-1)^{\nu_p(n)}, \tag{3.3}$$

where $\alpha\in\mathbb{R}$ and $\nu_p(n)$ is the highest power of the prime p dividing n.

Theorem 3.1. *Let $f\in\mathscr{M}$ and $r\in\mathbb{R}\setminus\{0\}$. Then*

$$f^r\in\mathscr{C}\Longleftrightarrow f(p^a)=\binom{-\frac{1}{r}}{a}(-r)^a f(p)^a \tag{3.4}$$

for all primes p and all $a\in\mathbb{N}$.

Proof. Assume $f^r\in\mathscr{C}$. By Haukkanen's theorem [8] and the property of power function, we have

$$\mu_{-1/r}f^r=(f^r)^{1/r}=f, \tag{3.5}$$

so that

$$f(p^a)=\mu_{-1/r}(p^a)f^r(p^a)=\binom{-\frac{1}{r}}{a}(-1)^a f^r(p)^a. \tag{3.6}$$

By the result of Carroll [5], see also [10], for $f^r\in\mathscr{C}$, we have

$$f^r(p)=\operatorname{Log} f^r(p)=r\operatorname{Log} f(p) \\ =\frac{r}{\log p}\left[f(1)f^{-1}(p)\log 1+f(p)f^{-1}(1)\log p\right]=rf(p), \tag{3.7}$$

and (3.4) follows.

Conversely, assuming (3.4), to show $f^r \in \mathscr{C}$, we use a characterization of Apostol [1], namely,

$$f^r \in \mathscr{C} \Longleftrightarrow f^{-r}(p^a) = 0 \tag{3.8}$$

for all primes p and all integers $a \geq 2$.

Writing $g = f^{-r}$, then $\operatorname{Log} g = -r \operatorname{Log} f$. We will show that $g(p^a) = 0$ for all $a \geq 0$. Since

$$(\operatorname{Log} f)(n) = \frac{1}{\log n}(df * f^{-1})(n) \quad (n > 1), \tag{3.9}$$

then

$$f * dg = -r(g * df). \tag{3.10}$$

Thus

$$\sum_{ij=p^a} f(i)dg(j) = -r \sum_{ij=p^a} g(i)df(j). \tag{3.11}$$

Taking $a = 1$, we get $g(p) = -rf(p)$.

Taking $a = 2$ and using $g(p) = -rf(p)$, we get

$$2g(p^2) = -r[2f(p^2) - (r+1)f(p)^2] = 0. \tag{3.12}$$

By induction, using (3.4), for $a \geq 2$, we deduce that $g(p^a) = 0$. □

Remark 3.2. An alternative proof of the "only if" part of Theorem 3.1 can be done as follows. Assuming (3.4), to show $f^r \in \mathscr{C}$, we use a characterization of Carroll [5], namely, for $f \in \mathscr{M}$, $f \in \mathscr{C}$ if and only if

$$(\operatorname{Log} f)(n) = \begin{cases} \dfrac{f(p)^a}{a} & \text{if } n = p^a,\ p \text{ prime},\ a \geq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.13}$$

From the hypothesis and using induction on a, we easily verify that

$$f^{-1}(p^a) = \binom{\frac{1}{r}}{a}(-r)^a f(p)^a \tag{3.14}$$

for all primes p and all $a \in \mathbb{N}$.

Next we want to show that $r(\operatorname{Log} f)(p^a) = (rf(p))^a/a$. From the definition of Rearick logarithm $\operatorname{Log} f$, we have

$$\begin{aligned}(\operatorname{Log} f)(p^a) = \frac{1}{a}[&f(p)f^{-1}(p^{a-1}) + 2f(p^2)f^{-1}(p^{a-2}) \\ &+ \cdots + (a-1)f(p^{a-1})f^{-1}(p) + af(p^a)],\end{aligned} \tag{3.15}$$

and so, using (3.14), we get

$$(\operatorname{Log} f)(p^a) = \frac{r^a}{a} f(p)^a \left[\sum_{i=1}^{a} i \binom{-\frac{1}{r}}{i} \binom{\frac{1}{r}}{a-i} (-1)^a \right]. \tag{3.16}$$

Using Riordan [14, identity (3d), page 10], we deduce that

$$(\operatorname{Log} f)(p^a) = \frac{(r f(p))^a}{a} \cdot \frac{1}{r}, \tag{3.17}$$

and the desired result follows.

Corollary 3.3. *Let $f \in \mathscr{C}$ and $r \in \mathbb{R} \setminus \{0,1\}$. Then*

$$f^r \in \mathscr{C} \Longleftrightarrow f = I. \tag{3.18}$$

Proof. For all primes p and all $a \in \mathbb{N}$, by Haukkanen's theorem [8] and Theorem 3.1, we have

$$\begin{aligned} f^r \in \mathscr{C} &\Longrightarrow (\mu_{-r} f)(p^a) = f^r(p^a) = (f^r(p))^a \\ &\Longrightarrow \binom{r+a-1}{a} f(p^a) = (r f(p))^a \\ &\Longrightarrow \left[\binom{r+a-1}{a} - r^a \right] f(p)^a = 0. \end{aligned} \tag{3.19}$$

Taking $a = 2$, since $r \neq 0, 1$, we get $f(p) = 0$ for all primes p. Being in $\mathscr{C}$, this yields $f = I$.

If $f = I \in \mathscr{C}$, by Haukkanen's theorem [8], we get

$$I^r(n) = (\mu_{-r} I)(n) = \mu_{-r}(n) I(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{otherwise}, \end{cases} \tag{3.20}$$

and so $I^r \in \mathscr{C}$. □

Remark 3.4. Corollary 3.3 is false if the assumption $f \in \mathscr{C}$ is dropped as seen from the following two examples.

(1) Take f as the Möbius function $\mu \neq I$ and $r = -1$. We know that (see [2]) μ^{-1} is the unit function u, $u(n) = 1$ $(n \geq 1)$, that is, $\mu^{-1} \in \mathscr{C}$.

(2) Take $f = |\mu| = \mu^2 \neq I$ and $r = -1$. Hence, $|\mu|^{-1}$ is the Liouville function $\lambda \in \mathscr{C}$ (see [2]).

4. Logarithms.

Since any two of the three known logarithms are the same except at $n = 1$, then we may use any of them whichever appropriate. We start by observing a trivial fact that if f is multiplicative, then its Rearick logarithm $\operatorname{Log} f$ is not multiplicative because $(\operatorname{Log} f)(1) = \log f(1) = 0$.

A natural question to consider here is that of finding a necessary and sufficient condition for the Carlitz-Subbarao logarithm βf to be (completely) multiplicative for a given $f \in \mathscr{A}$. This proves to be a difficult task, and we have the following partial result.

Theorem 4.1. *Let $f \in \mathcal{A}$ with $f(1) = 1$. Then its Carlitz-Subbarao logarithm $\beta f \in \mathcal{C} \Leftrightarrow$ for distinct primes $p_1,\ldots,p_k$ and positive integers $n_1,\ldots,n_k$, there holds*

$$f(p_1^{n_1}\cdots p_k^{n_k}) = \alpha(n_1,\ldots,n_k) f(p_1)^{n_1}\cdots f(p_k)^{n_k}, \tag{4.1}$$

where

$$\alpha(n_1,\ldots,n_k) = \sum_{r=1}^{\infty}\frac{1}{r!}I_r(n_1,\ldots,n_k), \tag{4.2}$$

$I_r(n_1,\ldots,n_k) = \sum^* 1$; $\sum^*$ denotes the sum taken over nonnegative integers $n_{11},\ldots,n_{r1}$; $\ldots; n_{1k},\ldots,n_{rk}$ such that

$$\begin{gathered} n_{11}+\cdots+n_{r1} = n_1,\ldots,n_{1k}+\cdots+n_{rk} = n_k, \\ n_{11}+\cdots+n_{1k} > 0,\ldots,n_{r1}+\cdots+n_{rk} > 0. \end{gathered} \tag{4.3}$$

Proof. Put $g = \beta f$. Then $g(1) = \beta f(1) = 1$. Assuming $g = \beta f \in \mathcal{C}$, by [4, Theorem 4.28 and equation (1.2)] of Carlitz and Subbarao, we see that for $m > 1$,

$$f(m) = \sum_{r=1}^{\infty}\frac{1}{r!}\sum_{\substack{a_1\cdots a_r = m \\ a_i\neq 1\,(i=1,\ldots,r)}} g(a_1)\cdots g(a_r) = g(m)c(m), \tag{4.4}$$

where

$$c(m) = \sum_{r=1}^{\infty}\frac{1}{r!}\sum_{\substack{a_1\cdots a_r = m \\ a_i\neq 1\,(i=1,\ldots,r)}} 1. \tag{4.5}$$

Writing m as a unique prime factorization, $m = p_1^{n_1}\cdots p_k^{n_k}$, we have

$$\sum_{\substack{a_1\cdots a_r = m \\ a_i\neq 1\,(i=1,\ldots,r)}} 1 = \sum^* 1. \tag{4.6}$$

The sum $\sum^*$ depends only on r, $n_1,\ldots,n_k$, but is independent of $p_1,\ldots,p_k$; we call it $I_r(n_1,\ldots,n_k)$. Thus the function

$$c(p_1^{n_1}\cdots p_k^{n_k}) = \sum_{r=1}^{\infty}\frac{1}{r!}I_r(n_1,\ldots,n_k) \tag{4.7}$$

depends only on r and $n_1,\ldots,n_k$, but is independent of $p_1,\ldots,p_k$, so that we may rewrite it as $\alpha(n_1,\ldots,n_k)$. Such a function $c(m)$ is known as prime-independent arithmetic function (see [9, page 33]). Since $f(p) = g(p)c(p) = g(p)$, then

$$\begin{aligned} f(p_1^{n_1}\cdots p_k^{n_k}) &= \alpha(n_1,\ldots,n_k) g(p_1)^{n_1}\cdots g(p_k)^{n_k} \\ &= \alpha(n_1,\ldots,n_k) f(p_1)^{n_1}\cdots f(p_k)^{n_k}, \end{aligned} \tag{4.8}$$

as to be proved.

Conversely, from $g = \beta f$, we get (see [4]) $f = \gamma g$; that is,

$$\sum_{r=1}^{\infty} \frac{1}{r!} \sum_{\substack{a_1 \cdots a_r = p_1^{n_1} \cdots p_k^{n_k} \\ a_i \neq 1\,(i=1,\ldots,r)}} g(a_1)\cdots g(a_r) = f(p_1^{n_1}\cdots p_k^{n_k}) = \alpha(n_1,\ldots,n_k) f(p_1)^{n_1}\cdots f(p_k)^{n_k}. \tag{4.9}$$

Specializing $k = 1 = n_1$, we get $g(p) = f(p)$.

Thus

$$\sum_{r=1}^{\infty} \frac{1}{r!} \sum_{\substack{a_1 \cdots a_r = p_1^{n_1} \cdots p_k^{n_k} \\ a_i \neq 1\,(i=1,\ldots,r)}} g(a_1)\cdots g(a_r) = \alpha(n_1,\ldots,n_k) g(p_1)^{n_1}\cdots g(p_k)^{n_k}. \tag{4.10}$$

We now show that $g(p^i) = g(p)^i$ for each prime p and $i \in \mathbb{N}$.

Taking $k = 1$ and $n_1 = 2$ in (4.10), observing that $\alpha(2) = \sum_{r=1}^{\infty}(1/r!)I_r(2)$, we have

$$\frac{1}{1!}g(p^2) + \frac{1}{2!}g(p)^2 = \alpha(2)g(p)^2 = \left(\frac{1}{1!} + \frac{1}{2!}\right)g(p)^2, \tag{4.11}$$

that is, $g(p^2) = g(p)^2$.

Assuming $g(p^j) = g(p)^j$ for all $j < i$, and using induction in (4.10), we obtain

$$\frac{1}{1!}g(p^i) + g(p)^i \sum_{r=2}^{\infty} \frac{1}{r!} I_r(i) = \alpha(i)g(p)^i. \tag{4.12}$$

As observed earlier, $\alpha(i)$ is just the sum of the coefficients of the g's on the left-hand side and this yields $g(p^i) = g(p)^i$ as desired.

Next, we show that $g(p^i q^j) = g(p)^i g(q)^j$ for distinct primes p, q and $i, j \in \mathbb{N}$. Taking $k = 2$, $p_1 = p$, $p_2 = q$, and $n_1 = n_2 = 1$ in (4.10), we get

$$\alpha(1,1)g(p)g(q) = \frac{1}{1!}g(pg) + g(p)g(q). \tag{4.13}$$

Again since $\alpha(1,1)$ is the sum of the coefficients of the g's on the right-hand side, we deduce that $g(p)g(q) = g(pq)$. Next, using $g(p^i) = g(p)^i$ and induction on $i + j$, we deduce that $g(p^i q^j) = g(p)^i g(q)^j$. Finally, $g(p_1^{i_1}\cdots p_k^{i_k}) = g(p_1)^{i_1}\cdots g(p_k)^{i_k}$ follows by induction on k. □

5. Applications. Our first application is related to one of our earlier results in [11] which states that for $f \in \mathcal{M}$, $\alpha \in \mathbb{R}\setminus\{0,1\}$, if $f^\alpha = \mu_{-\alpha} f$ and f satisfies condition (NE), then $f \in \mathscr{C}$. Here condition (NE) reads: if α is a negative even integer, then

$$f(p^{-\alpha-1}) = f(p)^{-\alpha-1} \quad \text{for each prime } p. \tag{5.1}$$

A similar result to the one above but without condition (NE) is the following proposition easily deduced from [11, Theorem 1.2].

Proposition 5.1. *Let $f \in \mathcal{M}$ and $\alpha \in \mathbb{R} \setminus \{0\}$. Then*

$$f \in \mathscr{C} \Longleftrightarrow f^{\alpha}(p^a) = \mu_{-\alpha}(p^a)\left(\frac{1}{\alpha} f^{\alpha}(p)\right)^a \tag{5.2}$$

for all primes p and all $a \in \mathbb{N}$, where μ_α denotes the Hsu's generalized Möbius function.

Our second application is related to the notion of totients (see [7]).

Recall that $f \in \mathcal{A}$ is a totient if it is of the form

$$f = f_t * f_v^{-1}, \tag{5.3}$$

where f_t (integral component) and f_v (inverse component) are both in $\mathscr{C}$. Those totients belonging to $\mathscr{C}$ are characterized in the next proposition which is an easy consequence of Corollaries 2.2 and 2.3.

Proposition 5.2. (i) *Let $f = f_t * f_v^{-1}$ be a totient. Then*

$$f \in \mathscr{C} \Longleftrightarrow f_v = I \text{ or } f_v^{-1}(p) + f_t(p) = 0 \tag{5.4}$$

for all primes p.

(ii) *Let $f = f_t * f_v^{-1}$ and $g = g_t * g_v^{-1}$ be two totients. Then*

$$f * g \text{ is a totient } \Longleftrightarrow f_t(p) g_t(p) = 0 = f_v(p) g_v(p) \tag{5.5}$$

for all primes p.

Another related concept is the following: $f \in \mathcal{A}$ is said to be a *rational arithmetical function of degree* (r,s) (see, e.g., [7]) if it is of the form

$$f = g_1 * g_2 * \cdots * g_r * h_1^{-1} * h_2^{-1} * \cdots * h_s^{-1}, \quad g_i, h_i \in \mathscr{C}. \tag{5.6}$$

Again by Corollaries 2.2 and 2.3, we have, for example, the following characterization which can be evidently generalized.

Proposition 5.3. (i) *Let $f = g_1 * g_2 * h_1^{-1}$ be a rational arithmetical function of degree $(2,1)$. Then*

$$f \text{ is a totient } \Longleftrightarrow g_1(p) g_2(p) = 0 \tag{5.7}$$

for each prime p.

(ii) *Let $f = g_1 * h_1^{-1} * h_2^{-1}$ be a rational arithmetical function of degree $(1,2)$. Then*

$$f \text{ is a totient } \Longleftrightarrow h_1(p) h_2(p) = 0 \tag{5.8}$$

for each prime p.

Our last application is related to the concept of specially multiplicative functions. Recall that an arithmetic function F is said to be *specially multiplicative* (see [16]) if there is a completely multiplicative function f_A such that

$$F(m)F(n) = \sum_{d|(m,n)} F\left(\frac{mn}{d^2}\right) f_A(d). \tag{5.9}$$

It is also known that each specially multiplicative function is a product of two completely multiplicative functions.

A straightforward consequence of Corollary 2.3 is the following result.

Proposition 5.4. *Let $F = f * g$ be a specially multiplicative function with $f, g \in \mathscr{C}$. Then*

$$F \in \mathscr{C} \Longleftrightarrow f(p)g(p) = 0 \tag{5.10}$$

for all primes p.

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