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METHOD OF SEMIDISCRETIZATION IN TIME FOR QUASILINEAR INTEGRODIFFERENTIAL EQUATIONS

D. BAHUGUNA and REETA SHUKLA

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We consider a class of quasilinear integrodifferential equations in a reflexive Banach space. We apply the method of semidiscretization in time to establish the existence, uniqueness, and continuous dependence on the initial data of strong solutions.

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1. Introduction. Let X and Y be two real reflexive Banach spaces such that the embedding $Y \hookrightarrow X$ is dense and continuous. We consider the following quasilinear evolution equation:

$$\begin{gathered} u'(t)+A(u(t))u(t)=f(t,u(t),G(u)(t)), \quad 0<t\le T, \\ u(0)=u_0, \end{gathered} \tag{1.1}$$

where $0<T<\infty$, $A(u)$ is a linear operator in X for each u in an open subset W of Y, G is a nonlinear Volterra operator defined from $C([0,T];X)$ into itself, and the nonlinear map f is defined from $[0,T]\times Y\times Y$ into Y. By a strong solution to (1.1) on $[0,T']$, $0<T'\le T$, we mean an absolutely continuous function u from $[0,T']$ into X such that $u(t)\in W$ for almost every $t\in[0,T']$ and satisfies (1.1) a.e. on $[0,T']$.

We use the ideas and techniques of Zeidler [10] and the method of semidiscretization in time to establish existence, uniqueness, and continuous dependence on initial data of strong solutions to (1.1) on $[0,T']$ for some $0<T'\le T$. For the study of particular cases of (1.1) in which $f(t,u,v)\equiv 0$ and $f(t,u,v)\equiv f(t,u)$, we refer to Crandall and Souganidis [2], Kato [6], and references cited therein. The crucial assumption in these works is that there exists an open subset W of Y such that for each $w\in W$, $A(w)$ generates a C_0-semigroup in X, $A(\cdot)$ is locally Lipschitz continuous on W from X into itself, f, defined from W into Y, is bounded and globally Lipschitz continuous from Y into itself, and there exists an isometric isomorphism $S:Y\to X$ such that

$$SA(w)S^{-1}=A(w)+B(w), \tag{1.2}$$

where $B(w)$ is in the space $B(X)$ of all bounded linear operators from X into itself. A generalization to the quasilinear evolution equation,

$$\begin{gathered} u'(t)+A(t,u(t))u(t)=f(t,u(t)), \quad 0<t\le T, \\ u(0)=u_0, \end{gathered} \tag{1.3}$$

has been considered by Katō [4] under similar conditions on $A(t,w)$ and $f(t,w)$ for $(t,w) \in [0,T] \times W$.

The method of semidiscretization in time is developed and applied to linear as well as nonlinear evolution equations by Rektorys [9], Kartsatos and Zigler [3], Nečas [7], Bahuguna and Raghavendra [1], and others. This method consists in replacing the time derivatives in an evolution equation by the corresponding difference quotients giving rise to a system of time-independent operator equations. With the help of the theory of semigroups and the theory of monotone operators, these systems are guaranteed to have unique solutions. An approximate solution to the evolution equation is defined in terms of the solutions of these time-independent systems. After proving a priori estimates for the approximate solution, the convergence of the approximate solution to the unique solution of the evolution equation is established. In these works, either global Lipschitz conditions or local Lipschitz conditions with some growth conditions on nonlinear forcing terms have been assumed.

In this paper, we assume only local Lipschitz conditions on the nonlinear maps f and G. We first prove that the discrete points lie in a ball in X of fixed radius R, where R is independent of the discretization parameters. Then using the local Lipschitz continuity, we establish a priori estimates on the difference quotients. With the help of these a priori estimates, we prove the convergence of a sequence of approximate solutions defined in terms of the discrete points to a unique solution of the problem.

2. Preliminaries. Let X and Y be as in Section 1. Let $\|x\|_Z$ denote the norm of an element x belonging to a Banach space Z. For $r > 0$, let $B_Z(x,r)$ denote the open ball $\{z \in Z : \|z-x\|_Z < r\}$ of radius r and let $\bar{B}_Z(x,r)$ be its closure. For an interval J of real numbers, we denote by $C(J;Z)$, $\mathrm{WC}(J;Z)$, $\mathrm{Lip}(J;Z)$, and $\mathrm{ABS}(J;Z)$ the spaces of all continuous, weakly continuous, Lipschitz continuous, and absolutely continuous functions from J into Z, respectively.

For a real β, $N(Z,\beta)$ represents the set of all densely defined linear operators L in Z such that if $\lambda > 0$ and $\lambda\beta < 1$, then $(I+\lambda L)$ is one-to-one with a bounded inverse defined everywhere on Z and

$$||(I+\lambda L)^{-1}||_{B(Z)} \le (1+\lambda\beta)^{-1}, \tag{2.1}$$

where I is the identity operator on Z. The Hille-Yosida (cf. Pazy [8]) theorem states that $L \in N(Z,\beta)$ if and only if $-L$ is the infinitesimal generator of a strongly continuous semigroup e^{-tL}, $t \ge 0$, on Z satisfying $\|e^{-tL}\|_{B(Z)} \le e^{\beta t}$, $t \ge 0$. A linear operator L on $D(L) \subseteq Z$ into Z is said to be accretive in Z if for every $u \in D(L)$,

$$\langle Lu, u^* \rangle \ge 0 \quad \text{for some } u^* \in F(u), \tag{2.2}$$

where $\langle z, z^* \rangle$ is the value of $z^* \in Z^*$ at $z \in Z$ and $F : Z \to 2^{Z^*}$ is the duality map given by

$$F(z) = \left\{ z^* \in Z^* : \langle z, z^* \rangle = \|z\|_Z^2 = ||z^*||_{Z^*}^2 \right\}. \tag{2.3}$$

Here, 2^{Z^*} denotes the power set of Z^*. If $L \in N(Z,\beta)$, then $(L+\beta I)$ is m-accretive in Z, that is, $(L+\beta I)$ is accretive and the range $R(L+\lambda I) = Z$ for some $\lambda > \beta$. If Z^* is uniformly convex, then F is single-valued and uniformly continuous on bounded subsets of Z.

(H) We assume in addition that the embedding $Y \hookrightarrow X$ is compact and the dual X^* is uniformly convex. Furthermore, we state the following hypotheses:

(H1) there exist an open subset W of Y and $\beta \geq 0$ such that $u_0 \in W$ and

$$A : W \longrightarrow N(X,\beta); \tag{2.4}$$

(H2) there exist positive constants μ_A and γ_A such that for all $w, w_1, w_2 \in W$ and $v \in Y$,

$$\begin{gathered} Y \subseteq D(A(w)), \qquad \|(A(w_1) - A(w_2))v\|_X \leq \mu_A \|w_1 - w_2\|_X \|v\|_Y, \\ \|A(w)v\|_X \leq \gamma_A \|v\|_Y; \end{gathered} \tag{2.5}$$

(H3) there exist a linear isometric isomorphism $S : Y \to X$, a map $P : W \to B(X)$, and positive constants μ_P and γ_P such that for all $w, w_1, w_2 \in W$,

$$\begin{gathered} SA(w) = A(w)S + P(w)S, \qquad \|P(w)\|_X \leq \gamma_P, \\ \|P(w_1) - P(w_2)\|_X \leq \mu_P \|w_1 - w_2\|_Y; \end{gathered} \tag{2.6}$$

(H4) the nonlinear map $G : C([0,T];X) \to C(0,T;X)$ satisfies

(a) for all $u, v \in \bar{B}_{C([0,T];X)}(\tilde{u}_0, r)$,

$$\|G(u) - G(v)\|_{C([0,T];X)} \leq \mu_G(r) \|u - v\|_{C([0,T];X)}, \tag{2.7}$$

where $\mu_G(r)$ is a nonnegative nondecreasing function and $\tilde{u}_0 \in C([0,T];X)$ is defined by $\tilde{u}_0(t) = u_0$ for all $t \in [0,T]$,

(b) for all $t, s \in [0,T]$ and $u \in \operatorname{Lip}([0,T];X) \cap \bar{B}_{C([0,T];X)}(\tilde{u}_0, r)$,

$$\|G(u)(t) - G(u)(s)\|_X \leq \gamma_G(r) |t-s| (1 + \|du/dt\|_{L^\infty([0,T];X)}), \tag{2.8}$$

where $\gamma_G(r)$ is a nonnegative nondecreasing function,

(c) furthermore that the subspace $C([0,T];Y)$ of space $(C[0,T];X)$ is an invariant subspace of the map G, that is, $G : C([0,T];Y) \to C([0,T];Y)$ which satisfies

$$\|G(u)(t)\|_Y \leq \lambda_G(r) \quad \text{for } u \in B_Y(u_0, r), \tag{2.9}$$

where $\lambda_G(r)$ is a nonnegative nondecreasing function.

In particular, we may take operator G as a Volterra operator defined by

$$G(u)(t) = \int_0^t a(t-s) k(s, u(s)) ds \tag{2.10}$$

in which a is a real-valued continuous function defined on $[0,T]$ and k is defined on $[0,T]\times Y$ into Y and $\|k(t,w)\|_Y \le C_k$ for every $(t,w)\in[0,T]\times Y$, then map G satisfies hypothesis (c);

(H5) the nonlinear map $f:[0,T]\times Y\times Y\to Y$ is a bounded function

$$||f(t,u,v)||_Y \le \lambda_f(r) \tag{2.11}$$

for all $(t,u,v)\in[0,T]\times Y\times Y$ with $\|u\|_Y+\|v\|_Y\le r$ where $\lambda_f(r)$ is a nonnegative nondecreasing function. Also, this map satisfies the Lipschitz condition

$$\begin{aligned}&||f(t_1,u_1,v_1)-f(t_2,u_2,v_2)||_X\\&\quad\le \mu_f(r)\big[\,|\phi(t_1)-\phi(t_2)|+||u_1-u_2||_X+||v_1-v_2||_X\big],\end{aligned} \tag{2.12}$$

for all $t_1,t_2\in[0,T]$ and all $u_i,v_i\in\bar B_X(u_0,r)$, $i=1,2$, where ϕ is a real-valued continuous function of bounded variation on $[0,T]$ and $\mu_f(r)$ is a nonnegative nondecreasing function.

Let $R>0$ be such that $W_R=\bar B_Y(u_0,R)\subseteq W$. We set

$$\begin{aligned}R_0&=\frac{R}{3}\big(1+e^{2\theta T}\big)^{-1},\\R_1&=\max\{R,\lambda_G(3R)+||u_0||_Y\},\\M&=\lambda_f\big(R+||u_0||+\lambda_G(3R)\big),\end{aligned} \tag{2.13}$$

where $\theta=\beta+\|P\|_{B(X)}$ and $V(\phi)$ is the total variation of ϕ on $[0,T]$.

Let $z_0\in Y$ and let T_0, $0<T_0\le T$, be such that

$$\begin{gathered}||Su_0-z_0||_X\le R_0,\\T_0\big[\gamma_A||z_0||_Y+\gamma_P||z_0||_X+M\big]\le R_0.\end{gathered} \tag{2.14}$$

We note that (2.14) implies that

$$\big(1+e^{2\theta T}\big)\big[||Su_0-z_0||+T_0\{\gamma_A||z_0||_Y+\gamma_P||z_0||_X+M\}\big]\le\frac{2R}{3}. \tag{2.15}$$

We have the following main result for the existence, uniqueness, and continuous dependence on the initial data of the strong solutions to (1.1).

Theorem 2.1. *Let (H), (H1), (H2), (H3), (H4), and (H5) hold. Then there exists a unique strong solution u to (1.1) on $[0,T_0]$ such that $u\in\mathrm{Lip}([0,T_0];X)$. Furthermore, if $v_0\in\bar B_Y(u_0,R_0)$, then there exists a strong solution v to (1.1) on $[0,T_0]$ with the initial point u_0 replaced by v_0 and*

$$||u(t)-v(t)||_X\le C||u_0-v_0||_X,\quad t\in[0,T_0], \tag{2.16}$$

where C is a positive constant depending only on T_0.

3. Basic lemmas. We will state and prove several lemmas required to prove Theorem 2.1. A proof of Theorem 2.1 will be given in the next section.

Let $w_0 = Su_0$. Let $h = T_0/n$ for all positive integers $n \geq N$ where N is a positive integer such that $\theta(T_0/N) < 1/2$. For $n \geq N$, we set $u_0^n = u_0$, $\tilde{u}_0^n = \tilde{u}_0$, and $t_j^n = jh$ for $j = 1,2,\dots,n$. We consider the scheme

$$\delta u_j^n + A(u_{j-1}^n)u_j^n = f(t_j^n, u_{j-1}^n, G(\tilde{u}_{j-1}^n)(t_j^n)), \quad j = 1,2,\dots,n, \tag{3.1}$$

where, for $j = 1,2,\dots,n$, $n \geq N$,

$$\delta u_j^n = \frac{u_j^n - u_{j-1}^n}{h},$$
$$\tilde{u}_j^n(t) = \begin{cases} u_0, & t = 0, \\ u_{i-1}^n + \left(\frac{1}{h}\right)(t - t_{i-1}^n)(u_i^n - u_{i-1}^n), & t \in [t_{i-1}^n, t_i^n],\ i = 1,2,\dots,j, \\ u_j^n, & t \in [t_j^n, T_0]. \end{cases} \tag{3.2}$$

The following result establishes the fact that $u_j^n \in W_R$, $j = 1,2,\dots,n$, $n \geq N$.

Lemma 3.1. *For each $n \geq N$, there exist unique $u_j^n \in W_R$, $j = 1,2,\dots,n$, satisfying (3.1).*

Proof. It follows (cf. [2, Lemma 2]) that there exists a unique $u_1^n \in Y$ such that

$$u_1^n + hA(u_0)u_1^n = u_0 + hf(t_1^n, u_0, G(\tilde{u}_0)(t_1^n)). \tag{3.3}$$

Applying S on both sides in (3.3) using (H3) and putting $w_1^n = Su_1^n$, we have

$$\begin{aligned}(w_1^n - z_0) &+ hA(u_0)(w_1^n - z_0) + hP(u_0)(w_1^n - z_0) \\ &= (w_0 - z_0) - hA(u_0)z_0 + hP(u_0)z_0 + hSf(t_1^n, u_0, G(\tilde{u}_0)(t_1^n)).\end{aligned} \tag{3.4}$$

The estimates in [2, Lemma 2] imply that

$$\|w_1^n - z_0\|_X \leq (1 - h\theta)^{-1}\left[\|w_0 - z_0\| + h(\gamma_A\|z_0\|_Y + \gamma_P\|z_0\|_X + M)\right]. \tag{3.5}$$

Since $h\theta < 1/2$, we have

$$\|w_1^n - z_0\|_X \leq e^{2\theta h}\left[\|w_0 - z_0\| + h(\gamma_A\|z_0\|_Y + \gamma_P\|z_0\|_X + M)\right]. \tag{3.6}$$

Therefore,

$$\|w_1^n - w_0\|_X \leq (1 + he^{2\theta h})\left[\|w_0 - z_0\| + h(\gamma_A\|z_0\|_Y + \gamma_P\|z_0\|_X + M)\right] \leq R \tag{3.7}$$

in view of the estimate (2.15). Hence, $u_1^n \in W_R$. Now suppose that $u_i^n \in W_R$ for $i = 1,2,\dots,j-1$. Again, [2, Lemma 2] implies that for $2 \le j \le n$ there exist unique $u_j^n \in Y$ such that

$$u_j^n + hA(u_{j-1}^n)u_j^n = u_{j-1}^n + hf(t_j^n, u_{j-1}^n, G(\tilde{u}_{j-1}^n)(t_j^n)). \tag{3.8}$$

Proceeding as before and putting $w_j^n = Su_j^n$, we get the estimate

$$||w_j^n - z_0||_X \le e^{2\theta h}[||w_{j-1} - z_0||_X + h(\gamma_A||z_0||_Y + \gamma_P||z_0||_X + M)]. \tag{3.9}$$

Reiterating the above inequality, we get

$$||w_j^n - z_0||_X \le e^{2\theta jh}[||w_0 - z_0||_X + jh(\gamma_A||z_0||_Y + \gamma_P||z_0||_X + M)]. \tag{3.10}$$

Using the fact that $jh \le T_0$, we arrive at

$$||w_j^n - w_0||_X \le (1 + e^{2\theta T})[||w_0 - z_0||_X + T_0(\gamma_A||z_0||_Y + \gamma_P||z_0||_X + M)] \le R. \tag{3.11}$$

The above estimate and (3.3) and (3.8) give the required result. This completes the proof. □

Lemma 3.2. *There exists a positive constant C independent of j, h, and n such that*

$$||\delta u_j||_X \le C, \quad j = 1,2,\dots,n;\ n \ge N. \tag{3.12}$$

Proof. Putting $j = 1$ in (3.1), we get

$$\delta u_1^n + hA(u_0)(\delta u_1^n) = -A(u_0)u_0 - f(t_1^n, u_0, G(\tilde{u}_0)(t_1^n)). \tag{3.13}$$

Using [2, Lemma 2], we have

$$||\delta u_1^n||_X \le e^{2\theta T}[\gamma_A||u_0||_Y + M] := C_0. \tag{3.14}$$

From (3.1), for $2 \le j \le n$, we have

$$\begin{aligned} &\delta u_j^n + hA(u_{j-1}^n)(\delta u_j^n) \\ &\quad = \delta u_{j-1}^n - (A(u_{j-1}^n) - A(u_{j-2}^n))u_{j-1}^n \\ &\qquad + f(t_j^n, u_{j-1}^n, G(\tilde{u}_{j-1}^n)(t_j^n)) - f(t_{j-1}^n, u_{j-2}^n, G(\tilde{u}_{j-2}^n)(t_{j-1}^n)). \end{aligned} \tag{3.15}$$

Using (H2) and [2, Lemma 2], for $2 \le j \le n$, we get

$$\begin{aligned} ||\delta u_j^n||_X \le e^{2\theta h}\Big[&(1 + C_1 h)||\delta u_{j-1}^n||_X \\ &+ ||f(t_j^n, u_{j-1}^n, G(\tilde{u}_{j-1}^n)(t_j^n)) - f(t_{j-1}^n, u_{j-2}^n, G(\tilde{u}_{j-2}^n)(t_{j-1}^n))||_X\Big] \end{aligned} \tag{3.16}$$

for some positive constant C_1 independent of j, h, and n. We note that

$$\begin{aligned}
&\|G(\tilde{u}^n_{j-1})(t^n_j)-G(\tilde{u}^n_{j-2})(t^n_{j-1})\|_X \\
&\quad\le \mu_G(3R)h\|\delta u^n_{j-1}\|_X+\gamma_G(3R)h\Big(1+\max_{1\le i\le j-1}\|\delta u^n_i\|_X\Big), \\
&\|f(t^n_j,u^n_{j-1},G(\tilde{u}^n_{j-1})(t^n_j))-f(t^n_{j-1},u^n_{j-2},G(\tilde{u}^n_{j-2})(t^n_{j-1}))\|_X \\
&\quad\le \mu_f(R_1)\Big[\,|\phi(t^n_j)-\phi(t^n_{j-1})|+h\|\delta u^n_{j-1}\|_X+\mu_G(3R)h\|\delta u^n_{j-1}\|_X \\
&\qquad\qquad +\gamma_G(3R)h\Big(1+\max_{1\le i\le j-1}\|\delta u^n_i\|_X\Big)\Big].
\end{aligned} \tag{3.17}$$

Using (3.17) in (3.16), we obtain

$$\max_{1\le i\le j}\|\delta u^n_i\|_X\le e^{2\theta h}(1+C_2h)\Big[\max_{1\le i\le j-1}\|\delta u^n_i\|_X+C_2|\phi(t^n_j)-\phi(t^n_{j-1})|+C_2h\Big], \tag{3.18}$$

where C_2 is another positive constant independent of j, h, and n. Reiterating inequality (3.18) and using (3.14), we get

$$\max_{1\le i\le j}\|\delta u^n_i\|_X\le e^{2\theta jh}(1+C_1h)^j[\|\delta u^n_1\|_X+C_2V(\phi)+C_2T_0], \tag{3.19}$$

where $V(\phi)$ is the total variation of ϕ. Hence,

$$\|\delta u^n_i\|_X\le e^{2(\theta+c_2)T}[c_0+c_2V(\phi)+C_2T_0]:=C. \tag{3.20}$$

This completes the proof. □

Now we define a sequence of functions $\{U^n\}$ from J_0 into Y by

$$U^n(t)=u^n_{j-1}+\Big(\frac{1}{h}\Big)(t-t_{j-1})(u^n_j-u^n_{j-1}),\quad t\in[t^n_{j-1},t^n_j],\ j=1,2,\dots,n. \tag{3.21}$$

Furthermore, we define a sequence of step functions $\{X^n\}$ from $(-h,T_0]$ into Y given by

$$X^n(t)=\begin{cases}u_0, & t\in(-h,0],\\ u_j, & t\in(t^n_{j-1},t^n_j],\ j=1,2,\dots,n.\end{cases} \tag{3.22}$$

Remark 3.3. We observe that $X^n(t)\in W_R$ for all $t\in(-h,T_0]$ and $n\ge N$. Also, $X^n(t)-U^n(t)\to 0$ in X uniformly on J_0 as $n\to\infty$ and $\{U^n\}$ are in $\mathrm{Lip}(J_0,X)$ with uniform Lipschitz constant C.

For notational convenience, we will denote

$$f^n(t)=f(t^n_j,u^n_{j-1},G(\tilde{u}^n_{j-1})(t^n_j)),\quad t\in(t^n_{j-1},t^n_j),\ j=1,2,\dots,n. \tag{3.23}$$

We note that

$$\int_0^t A(X^n(s-h))X^n(s)ds = u_0 - U^n(t) + \int_o^t f^n(s)ds, \tag{3.24}$$

$$\frac{d^-}{dt}U^n(t) + A(X^n(t-h))X^n(t) = f^n(t). \tag{3.25}$$

Lemma 3.4. *There exist a subsequence* $\{U^m\}$ *of* $\{U^n\}$ *and a function* u *in* $\mathrm{Lip}(J_0,X)$ *such that* $U^m \to u$ *in* $C(J_0,X)$ *(with the supremum norm) as* $m \to \infty$.

Proof. Since $\{X^n\}$ is uniformly bounded in Y, the compact embedding of Y implies that there exist a subsequence $\{X^m\}$ of $\{X^n\}$ and a function $u : J_0 \to X$ such that $X^m(t) \to u(t)$ in X as $m \to \infty$. The reflexivity of Y implies that $u(t)$ is the weak limit of X^m in Y, hence $u(t)$ lies in Y (in fact in W_R since $X^m(t)$ is in W_R). Now, $X^m(t) - U^m(t) \to 0$ in X so $U^m(t) \to u(t)$ as $m \to \infty$. The uniform continuity of $\{U^n\}$ on J_0 implies that $\{X^m\}$ is an equicontinuous family in $C(J_0,X)$ and the strong convergence of $U^m(t)$ to $u(t)$ in X implies that $\{U^m\}$ is relatively compact in X. We use the Ascoli-Arzelá theorem to conclude that $U^m \to u$ in $C(J_0,X)$ as $m \to \infty$. Since U^m are in $\mathrm{Lip}(J_0,X)$ with uniform Lipschitz constant, $u \in \mathrm{Lip}(J_0,X)$. This completes the proof of the lemma. □

4. Proof of Theorem 2.1. First we show that $A(X^m(t-h))X^m(t) \rightharpoonup A(u(t))u(t)$ in X as $m \to \infty$, where "$\rightharpoonup$" denotes the weak convergence in X,

$$\begin{aligned} &A(X^m(t-h))X^m(t) - A(u(t))u(t) \\ &\quad = (A(X^m(t-h)) - A(u(t)))X^m(t) + A(u(t))(X^m(t) - u(t)). \end{aligned} \tag{4.1}$$

Now,

$$\begin{aligned} &||(A(X^m(t-h)) - A(u(t)))X^m||_X \\ &\quad \le \mu_A(R + ||u_0||_Y)||X^m(t-h) - u(t)||_X \longrightarrow 0 \end{aligned} \tag{4.2}$$

as $m \to \infty$, since $X^m(t) \to u(t)$ in X uniformly on J_0. Since $A(u(t)) \in N(X,\beta)$, $\beta I + A(u)$ is m-accretive in X. We use [5, Lemma 2.5] and the fact that

$$||A(u(t))(X^m(t) - u(t))||_X \le 2\gamma_A R \tag{4.3}$$

to assert that $A(u(t))X^m(t) \rightharpoonup A(u(t))u(t)$ in X, and hence $A(X^m(t-h))X^m(t) \rightharpoonup A(u(t))u(t)$ in X as $m \to \infty$. To show that $A(u(t))u(t)$ is weakly continuous on J_0, let $\{t_k\} \subset J_0$ be a sequence such that $t_k \to t$ as $k \to \infty$. Then $u(t_k) \to u(t)$ in X as $k \to \infty$ and we may follow the same arguments as above to prove that $A(u(t_k))u(t_k) \rightharpoonup A(u(t))u(t)$ in X as $k \to \infty$. The Bochner integrability of $A(u(t))u(t)$ can be established in the similar way as in Kato [5, Lemma 4.6]. Now from (3.24), for each $x^* \in X^*$, we have

$$\langle U^m(t), x^*\rangle = \langle u_0, x^*\rangle + \int_0^t \langle -A(X^m(s-h))X^m(s) + f^m(s), x^*\rangle ds. \tag{4.4}$$

Letting $m \to \infty$ using bounded convergence theorem and Lemma 3.4, we get

$$\langle u(t), x^* \rangle = \langle u_0, x^* \rangle + \int_0^t \langle -A(u(s))u(s) + f(s, u(s), G(u)(s)), x^* \rangle ds. \tag{4.5}$$

The continuity of the integrand implies that $\langle u(t), x^* \rangle$ is continuously differentiable on J_0. The Bochner integrability of $A(u(t))u(t)$ implies that the strong derivative of $u(t)$ exists a.e. on J_0 and

$$u'(t) + A(u(t))u(t) = f(t, u(t), G(u)(t)), \quad \text{a.e. on } J_0. \tag{4.6}$$

Since $u(0) = u_0$, u is a strong solution to (1.1).

Now, we establish the uniqueness and the continuous dependence on the initial data of a strong solution to (1.1).

Uniqueness. Let v be another strong solution to (1.1) on J_0. Let $U = u - v$. Then for a.e. $t \in J_0$,

$$\begin{aligned} &\left\langle \frac{dU}{dt}(t), F(U(t)) \right\rangle + \langle (\beta I + A(u(t)))U(t), F(U(t)) \rangle \\ &\quad = \beta ||U(t)||_X^2 + \langle (A(u(t)) - A(v(t)))v(t), F(U(t)) \rangle \\ &\qquad + \langle f(t, u(t), G(u)(t)) - f(t, v(t), G(v)(t)), F(U(t)) \rangle. \end{aligned} \tag{4.7}$$

Using m-accretivity of $\beta I + A(w)$ and the assumptions on $A(w)$ for $w \in W$ and $f(t, u, v)$, we obtain

$$\frac{1}{2} \frac{d}{dt} ||U(t)||^2 \le C_R \|U\|_{C(J_t, X)}^2, \tag{4.8}$$

where $J_t = [0, t]$ and $C_T = \beta + \mu_A R + \mu_f(R)[1 + \mu_G(R)]$.

Integrating the above inequality on $(0, t)$ and taking the supremum, we get

$$\frac{1}{2} \|U\|_{C(J_t, X)}^2 \le C_R \int_0^t \|U\|_{C(J_s, X)}^2 ds. \tag{4.9}$$

Applying Gronwall's inequality, we get $U \equiv 0$ on J_0.

Continuous dependence. Let $v_0 \in B_Y(u_0, R_0)$. Then

$$||Sv_0 - z_0||_X \le ||Sv_0 - Su_0||_X + ||Su_0 - z_0||_X \le 2R_0. \tag{4.10}$$

Hence,

$$\begin{aligned} &(1 + e^{2\theta T})[||Sv_0 - z_0|| + T_0\{\gamma_A ||z_0||_Y + \gamma_P ||z_0||_X + M\}] \\ &\quad \le 3(1 + e^{2\theta T})R_0 = R. \end{aligned} \tag{4.11}$$

We may proceed as before to prove the existence of $v_j^n \in W_R$ satisfying scheme (1.1) with u_j^n and u_0 replaced by v_j^n and v_0, respectively. Convergence of v_j^n to $v(t)$ can be proved in a similar manner. Let $U = u - v$. Then following the steps used to prove the uniqueness, we have for a.e. $t \in J_0$,

$$\frac{1}{2}\frac{d}{dt}\left|\left|U(t)\right|\right|_X^2 \le C_R \|U\|^2_{C(J_t,X)}. \tag{4.12}$$

Integrating the above inequality on $(0,t)$ and taking the supremum, we get

$$\frac{1}{2}\|U\|^2_{C(J_t,X)} \le \frac{1}{2}\left|\left|U(0)\right|\right|_X^2 + C_R\int_0^t \|U\|^2_{C(J_s,X)}ds. \tag{4.13}$$

Applying Gronwall's inequality, we get

$$\|U\|_{C(J_t,X)} \le C\left|\left|U(0)\right|\right|_X, \tag{4.14}$$

where C is a positive constant. This proves the required result. This completes the proof of Theorem 2.1.

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D. Bahuguna: Department of Mathematics, Indian Institute of Technology Kanpur, Kanpur 208 016, India

E-mail address: dhiren@iitk.ac.in

Reeta Shukla: Department of Mathematics, Indian Institute of Technology Kanpur, Kanpur 208 016, India

E-mail address: reetas@lycos.com