ON THE EIGENVALUES WHICH EXPRESS ANTIEIGENVALUES

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We showed previously that the first antieigenvalue and the components of the first antieigenvectors of an accretive compact normal operator can be expressed either by a pair of eigenvalues or by a single eigenvalue of the operator. In this paper, we pin down the eigenvalues of T that express the first antieigenvalue and the components of the first antieigenvectors. In addition, we will prove that the expressions which state the first antieigenvalue and the components of the first antieigenvectors are unambiguous. Finally, based on these new results, we will develop an algorithm for computing higher antieigenvalues.

1. Introduction

An operator T on a Hilbert space is called accretive if $\operatorname{Re}(Tf,f) \geq 0$ and strictly accretive if $\operatorname{Re}(Tf,f) > 0$ for every vector $f \neq 0$. For an accretive operator or matrix T on a Hilbert space, the first antieigenvalue of T, denoted by $\mu_1(T)$, is defined by Gustafson to be

$$\mu_1(T) = \inf_{Tf\neq 0} \frac{\operatorname{Re}(Tf,f)}{\|Tf\|\|f\|} \tag{1.1}$$

(see [2, 3, 4, 5]). The quantity $\mu_1(T)$ is also denoted by $\cos T$ and is called the cosine of T. Definition (1.1) is equivalent to

$$\mu_1(T) = \inf_{\substack{Tf\neq 0\\ \|f\|=1}} \frac{\operatorname{Re}(Tf,f)}{\|Tf\|}. \tag{1.2}$$

$\mu_1(T)$ measures the maximum turning capability of T. A vector f for which the infimum in (1.1) is attained is called an antieigenvector of T. Higher antieigenvalues may be defined by

$$\mu_n(T) = \inf_{Tf\neq 0} \frac{\operatorname{Re}(Tf,f)}{\|Tf\|\|f\|}, \quad f \perp \left(f^{(1)},\dots,f^{(n-1)}\right), \tag{1.3}$$

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where $f^{(k)}$ denotes the kth antieigenvector. In [8, 9] (see also [7]), we found $\mu_1(T)$ for normal matrices directly, by first expressing $\operatorname{Re}(Tf,f)/\|Tf\|$ in terms of eigenvalues of T and components of vectors on eigenspaces and then minimizing it on the unit sphere $\|f\| = 1$. The result was the following theorem.

Theorem 1.1. *Let T be an n by n accretive normal matrix. Suppose $\lambda_i = \beta_i + \delta_i i$, $1 \leq i \leq m$, are the distinct eigenvalues of T. Let $E(\lambda_i)$ be the eigenspace corresponding to λ_i and $P(\lambda_i)$ the orthogonal projection on $E(\lambda_i)$. For each vector f let $z_i = P(\lambda_i)f$. If f is an antieigenvector with $\|f\| = 1$, then one of the following cases holds.*

(1) *Only one of the vectors z_i is nonzero, that is, $\|z_i\| = 1$, for some i and $\|z_j\| = 0$ for $j \neq i$. In this case it holds that*

$$\mu_1(T) = \frac{\beta_i}{|\lambda_i|}. \tag{1.4}$$

(2) *Only two of the vectors z_i and z_j are nonzero and the rest of the components of f are zero, that is, $\|z_i\| \neq 0$, $\|z_j\| \neq 0$, and $\|z_k\| = 0$ if $k \neq i$ and $k \neq j$. In this case it holds that*

$$\begin{aligned} ||z_i||^2 &= \frac{\beta_j|\lambda_j|^2 - 2\beta_i|\lambda_j|^2 + \beta_j|\lambda_i|^2}{\left(|\lambda_i|^2 - |\lambda_j|^2\right)(\beta_i - \beta_j)}, \\ ||z_j||^2 &= \frac{\beta_i|\lambda_i|^2 - 2\beta_j|\lambda_i|^2 + \beta_i|\lambda_j|^2}{\left(|\lambda_i|^2 - |\lambda_j|^2\right)(\beta_i - \beta_j)}. \end{aligned} \tag{1.5}$$

Furthermore

$$\mu_1(T) = \frac{2\sqrt{(\beta_j - \beta_i)\left(\beta_i|\lambda_j|^2 - \beta_j|\lambda_i|^2\right)}}{|\lambda_j|^2 - |\lambda_i|^2}. \tag{1.6}$$

In [12], we were able to extend the above theorem to the case of normal compact operators on an infinite-dimensional Hilbert space by modifying our techniques in [8, 9] to fit the situation in an infinite-dimensional space. However, in [12] we also took a completely different approach to compute $\mu_1(T)$ for general strictly accretive normal operators on Hilbert spaces of arbitrary dimension. In that approach, we took advantage of the fact [11] that $\mu_1^2(T) = \inf\{x^2/y : x + iy \in W(S)\}$ for strictly accretive normal operators T. Here, $S = \operatorname{Re}T + iT^*T$ and $W(S)$ denotes the numerical range of S. The result was the following.

Theorem 1.2. *Let T be a strictly accretive normal operator such that the numerical range of*

$$S = \operatorname{Re}T + iT^*T \tag{1.7}$$

is closed. Then one of the following casesholds:

(1) *$\mu_1(T) = \beta_i/|\lambda_i|$ for some $\lambda_i = \beta_i + \delta_i i$ in the spectrum of T,*

(2)

$$\mu_1(T) = \frac{2\sqrt{(\beta_j - \beta_i)\left(\beta_i |\lambda_j|^2 - \beta_j |\lambda_i|^2\right)}}{|\lambda_j|^2 - |\lambda_i|^2} \tag{1.8}$$

for a pair of distinct points

$$\lambda_i = \beta_i + \delta_i \mathrm{i}, \qquad \lambda_j = \beta_j + \delta_j \mathrm{i}, \tag{1.9}$$

in the spectrum of T.

Mirman's [11] observation that $\mu_1^2(T)$ can be obtained in terms of $S = \operatorname{Re} T + iT^*T$ is so immediate that no proof was given in [7, 11, 12], or this paper where this fact is employed. So for completeness, $S = \operatorname{Re} T + iT^*T$ and $\|z\| = 1$ implies numerical range element $(Sz,z) \equiv x + iy = \operatorname{Re}(Tz,z) + i\|Tz\|^2 \Rightarrow \mu_1^2(T) = \inf\{x^2/y : x + iy \in W(S)\}$.

It is both interesting and important to pinpoint the pair of eigenvalues of T, among all possible pairs, that actually express $\mu_1(T)$ in (1.6) in case (2) of Theorem 1.1. In the next section we will introduce the concept of the first and the second critical eigenvalues for an accretive normal operator and show that, among all possible pairs of eigenvalues of T, these two eigenvalues are the ones that express $\mu_1(T)$. This will help us further to discover which pair of eigenvalues of T express $\mu_2(T)$ and other higher antieigenvalues of T. Based on the properties of the first and the second critical eigenvalues of T, we will also show that the denominators in (1.5), and (1.6) are all nonzero for this particular pair of eigenvalues. We will also show that the radicand in the numerator of (1.6) is nonzero if (1.6) expresses $\mu_1(T)$.

It should be mentioned that Davis [1] first showed that for strictly accretive normal matrices, the antieigenvalues are determined by just two of the eigenvalues T. However, quoting Davis [1, page 174] "in general normal case I'm afraid I know no simple criterion for picking out a critical pair of eigenvalues to which attention can at once be confined." In [8, 9] we implicitly answered this question, with the ordering of the eigenvalues according to their real parts and absolute values, which more or less determines which ones led to $\mu_1(T)$ according to Theorem 1.1. Also we knew that an appearance of zero denominators and undefined numerators would not represent a problem, since the convexity arguments usually lead to the determination of μ_1 by λ_i and λ_j with $\|\lambda_i\| \neq \|\lambda_j\|$.

2. The eigenvalues expressing antieigenvalues

Assume T is a strictly accretive normal n by n matrix with distinct eigenvalues $\lambda_i = \beta_i + \delta_i i$, $1 \le i \le m$. Then as noted above, we have $\mu_1^2(T) = \inf\{x^2/y : x + iy \in W(S)\}$, where $S = \operatorname{Re} T + iT^*T$ and $W(S)$ denotes the numerical range of S. Since T is normal, so is S. By spectral mapping theorem, if $\sigma(S)$ denotes the spectrum of S, then $\sigma(S) = \{\beta_i + i|\lambda_i|^2 : \lambda_i = (\beta_i + \delta_i i) \in \sigma(T)\}$. Since S is normal, we have $W(S) = \operatorname{co}(\sigma(S))$, where $\operatorname{co}(\sigma(S))$ denotes the convex hull of $\sigma(S)$. Therefore $W(S)$ is a convex polygon contained in the first quadrant. Throughout this paper, for convenience, we consider an eigenvalue $\beta_i + i|\lambda_i|^2$ of S and the point $(\beta_i, |\lambda_i|^2)$ in the Cartesian plane to be the same. Therefore, in place of

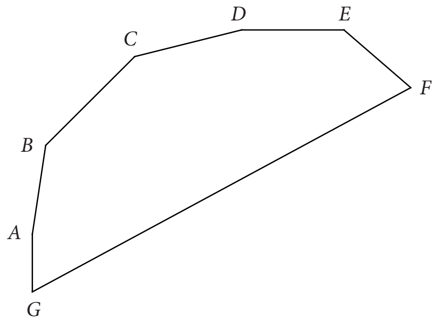

Figure 2.1

$\beta_i + i|\lambda_i|^2$, we may refer to $(\beta_i, |\lambda_i|^2)$ as an eigenvalue of S. The convexity of the function $f(x,y) = x^2/y$ implies that the minimum of this function on $W(S)$ is equal to the smallest value of k such that a member of the family of convex functions $y = x^2/k$ touches just one point of the polygon representing $W(S)$. Obviously, if any parabola from the family $y = x^2/k$ touches only one point of $W(S)$, that point should be on $\partial W(S)$, the boundary of $W(S)$. Therefore to find $\mu_1^2(T)$, first we need to identify those values of k for which $y = x^2/k$ touches only one point of $\partial W(S)$ and then select the smallest such value. The trivial case is when a member of the family of convex functions $y = x^2/k$ touches $\partial W(S)$ at a corner point such as $(\beta_i, |\lambda_i|^2)$. If $y = x^2/k$ is the parabola that is passing through $(\beta_i, |\lambda_i|^2)$, then the components of this point should satisfy $y = x^2/k$. Hence we must have $|\lambda_i|^2 = \beta_i^2/k$, which implies $k = \beta_i^2/|\lambda_i|^2$. Next consider the more interesting case when a member of the family $y = x^2/k$ touches $\partial W(S)$ at an interior point of an edge. In this case the parabola $y = x^2/k$ must be tangent to that edge at the point of contact. It is clear that such parabolas cannot be tangent to an edge of $\partial W(S)$ if that edge has a slope which is negative, zero, or undefined because the slopes of tangent lines to the right half of parabolas $y = x^2/k$ are always positive for positive values of k. For example, in Figure 2.1 no member of the family $y = x^2/k$ can be tangent to edges AG, DE, and EF. It is also clear that no member of the family of parabolas $y = x^2/k$ can be tangent to an edge with positive slope if $W(S)$ is above the line of support of $W(S)$ which contains that edge without having other points in common with $W(S)$. For instance, in Figure 2.1 no parabola of the form $y = x^2/k$ can be tangent to the edge GF at an interior point of that edge without actually entering into the interior of $W(S)$. A member of the family $y = x^2/k$ can however be tangent to an edge at an interior point of that edge, without having any other common point with $W(S)$, if the slope of that edge is positive and $W(S)$ falls below the line of support which contains that edge. For example, in Figure 2.1 members of the family $y = x^2/k$ can be tangent to edges AB, BC, and CD, without having any other common points with $W(S)$.

Definition 2.1. An edge of the polygon representing $W(S)$ is called an upper positive edge if the slope of that edge is positive and $W(S)$ falls below the line of support of $W(S)$ which contains that edge.

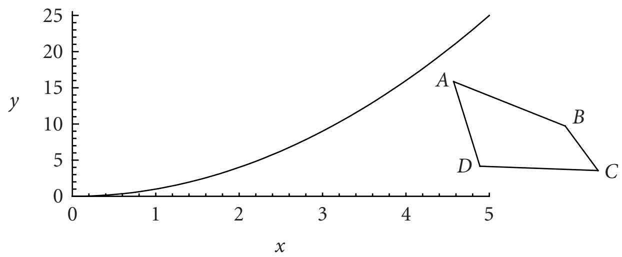

Figure 2.2

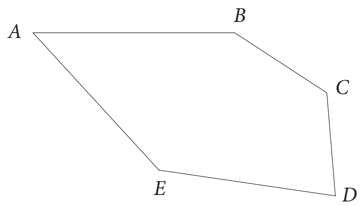

Figure 2.3

Definition 2.2. Let $d = \inf\{\beta_i : (\beta_i + i|\lambda_i|^2) \in \sigma(S)\}$. Define D to be

$$D = \sup\left\{ |\lambda_i|^2 : \left(\beta_i + i|\lambda_i|^2\right) \in \sigma(S),\ d = \beta_i \right\}. \tag{2.1}$$

Let $\beta_p + i|\lambda_p|^2$ be that eigenvalue of S for which $\beta_p = d$ and $|\lambda_p|^2 = D$. $\beta_p + i|\lambda_p|^2$ is the first critical eigenvalue γ_p of S. The corresponding eigenvalue $\lambda_p = \beta_p + \delta_p i$ of T is called the first critical eigenvalue of T.

The point A labelling $(\beta_p, |\lambda_p|^2)$ is shown in Figures 2.2, 2.3, 2.4, 2.5, and 2.6. It represents that eigenvalue of S which has the highest imaginary component among all eigenvalues of S which have the smallest real component.

Definition 2.3. If $A(\beta_p, |\lambda_p|^2)$, the first critical eigenvalue of S, is on the upper positive edge AB and $B(\beta_q, |\lambda_q|^2)$ corresponds to the eigenvalue $\gamma_q = \beta_q + i|\lambda_q|^2$ of S, then $\gamma_q = \beta_q + i|\lambda_q|^2$ is called the second critical eigenvalue of S and the corresponding eigenvalue $\lambda_q = \beta_q + \delta_q i$ of T is called the second critical eigenvalue of T. The second critical eigenvalue of S (of T) is not defined if A is not on an upper positive edge.

Theorem 2.4. *If $A(\beta_p, |\lambda_p|^2)$, the first critical eigenvalue of S, is not on any upper positive edge, then the minimum of the function $f(x,y) = x^2/y$ on $W(S)$ is attained at $A(\beta_p, |\lambda_p|^2)$. If $A(\beta_p, |\lambda_p|^2)$ belongs to an upper positive edge, then the minimum of the function $f(x,y) = x^2/y$ on $W(S)$ is attained at a corner point belonging to an upper positive edge or at a point in the interior of the line segment joining the first and second critical eigenvalues $A(\beta_p, |\lambda_p|^2)$ and $B(\beta_q, |\lambda_q|^2)$.*

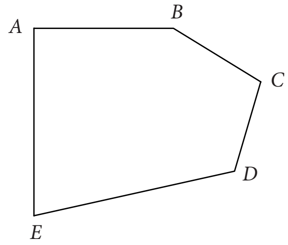

Figure 2.4

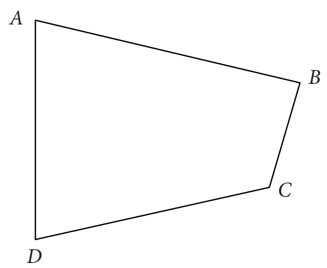

Figure 2.5

Proof. Assume AB (AA_1) has a positive slope. The convexity of the polygon representing $W(S)$ implies that if there is any set of consecutive upper positive edges $A_{i-1}A_i$, $2 \le i \le r$ following AA_1, then their slopes should decrease as we move from left to right. For example, in Figure 2.1 the edge AB is followed by the edge BC whose slope is positive but less than the slope of AB. Also BC is followed by CD whose slope is positive but less than the slope of BC. Suppose the slope of AA_1 is m_1 and $y = x^2/k_1$ is tangent to AA_1 at a point with x-component x_1. Then we have $m_1 = 2x_1/k_1$ which implies $k_1 = 2x_1/m_1$. Now suppose the slope of the segment $A_{i-1}A_i$, $2 \le i \le r$ is m_i and $y = x^2/k_i$ is tangent to $A_{i-1}A_i$ at an interior point with x-component x_i,then we have $k_i = 2x_i/m_i$. Since $m_1 > m_i$ and $x_i > x_1$, we have $k_1 < k_i$. □

The first critical eigenvalue $A(\beta_p, |\lambda_p|^2)$ and the upper positive edge that contains A (if it exists) are important in computing $\mu_1^2(T)$. For example, suppose in Figure 2.2 the polygon $ABCD$ represents $\partial W(S)$. It is obvious that the only point of this polygon that can be touched by a member of the family of functions $y = x^2/k$ is point A. Depending on the signs of the slopes of the two edges of the polygon that meet at $A(\beta_p, |\lambda_p|^2)$, we have two different cases that will be analyzed below.

(1) $A(\beta_p, |\lambda_p|^2)$ does not belong to an upper positive edge. Figures 2.2–2.5 show the situations when this occurs. In this case the only parabola from the family $y = x^2/k$ that can touch $W(S)$ at only one point is the one which touches $W(S)$ at A.

(2) $A(\beta_p, |\lambda_p|^2)$ belongs to an upper positive edge AB. By Theorem 2.4 the convex function $y = x^2/k$ that touches $W(S)$ at one point with minimum value of k should either

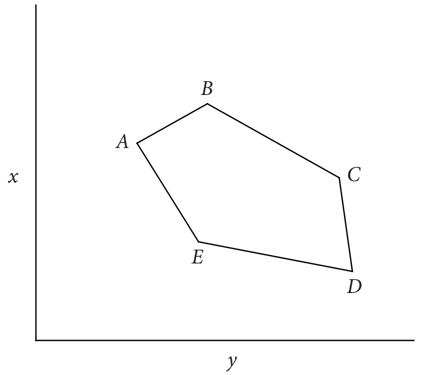

Figure 2.6

be tangent to AB or pass through a corner point of an upper positive edge (see Figures 2.1 and 2.6).

Assume that $B(\beta_q, |\lambda_q|^2)$ is the higher end of the upper positive edge AB in case (2) above.

Note that since the polygon representing $W(S)$ is the convex hull of all eigenvalues of S, there might be other eigenvalues of S located on the edge AB. However, point B is the end point of that edge and thus has the maximum distance from point A among all other points on that edge. Also note that besides eigenvalues which are located at the corners of $W(S)$, the matrix S may have other eigenvalues which are in the interior of $W(S)$. However, given one such eigenvalue $\beta_i + \delta_i i$ there exists points $x + yi \in W(S)$ such that $x < \beta_i$ and $y < \beta_i$, and hence these eigenvalues do not play any role in the computation of $\mu_1(T)$. The first and second critical eigenvalues can be found algebraically and in practice one does not have to construct the polygon representing $W(S)$ to find them. The procedure for finding $\mu_1(T)$ is outlined in the following theorem.

Theorem 2.5. *Let T be a strictly accretive normal matrix and $\gamma_p = \beta_p + i|\lambda_p|^2$ the first critical eigenvalue of $S = \operatorname{Re} T + iT^*T$. Let $\beta_i + i|\lambda_i|^2$ represent any other eigenvalue of S for which $\beta_i > \beta_p$. Let $m_i = |\lambda_i|^2 - |\lambda_p|^2/\beta_i - \beta_p$ be the slopes of line segments connecting the point $(\beta_p, |\lambda_p|^2)$ to points $(\beta_i, |\lambda_i|^2)$. Define $m = \max\{m_i\}$. Then the following two cases hold:*

(1) *if $m \leq 0$, the second critical eigenvalue of S does not exist and $\mu_1(T) = \beta_p/|\lambda_p|$,*
(2) *if $m > 0$, let $R = \{(\beta_j, |\lambda_j|^2) : (\beta_j, |\lambda_j|^2) \epsilon \sigma(S)$ and $m_j = m\}$, and let*

$$t = \sup\left\{(\beta_j - \beta_p)^2 + \left(|\lambda_j|^2 - |\lambda_p|^2\right)^2 : \left(\beta_j, |\lambda_j|^2\right) \epsilon R\right\}. \tag{2.2}$$

If $(\beta_q, |\lambda_q|^2)$ is that element of R for which $t = (\beta_q - \beta_p)^2 + (|\lambda_q|^2 - |\lambda_q|^2)^2$, then $(\beta_q, |\lambda_q|^2)$ is the second critical eigenvalue of S. In this case $\mu_1(T)$ is equal to $2\sqrt{(\beta_q - \beta_p)(\beta_p|\lambda_q|^2 - \beta_q|\lambda_p|^2)/|\lambda_q|^2 - |\lambda_p|^2}$ or $\beta_i/|\lambda_i|$ for an eigenvalue $\lambda_i = \beta_i + \delta_i i$ of T such that $(\beta_i, |\lambda_i|^2)$ is a corner point on an upper positive edge.

Table 2.1

Point	(2,11)	(3,25)	(4,50)	(5,60)
Slope	4	9	14.333	13.25

Proof. Based on the arguments that preceded this theorem, we know that in case (1) the infimum of the function $f(x,y) = x^2/y$ on $W(S)$ is attained at $(\beta_p, |\lambda_p|^2)$. Therefore the minimum value is $f(\beta_p, |\lambda_p|^2) = \beta_p^2/|\lambda_p|^2$. Hence $\mu_1^2(T) = \beta_p^2/|\lambda_p|^2$, which implies $\mu_1(T) = \beta_p/|\lambda_p|$. In case (2), Theorem 2.4 shows that the minimum of the function $f(x,y) = x^2/y$ on $W(S)$ is attained at a corner point belonging to an upper positive edge or at a point in the interior of the line segment joining the first and second critical eigenvalues $(\beta_p, |\lambda_p|^2)$ and $(\beta_q, |\lambda_q|^2)$. As we just showed if the minimum of $f(x,y) = x^2/y$ on $W(S)$ is attained at $(\beta_i, |\lambda_i|^2)$, we have $\mu_1(T) = \beta_i/|\lambda_i|$. If the minimum of the function $f(x,y) = x^2/y$ on $W(S)$ is attained at a point in the interior of the line segment joining $(\beta_p, |\lambda_p|^2)$ and $(\beta_q, |\lambda_q|^2)$, one can use Lagrange multiplier method (see [12] for details) to show that the point of contact is at (x_1, y_1), where

$$\begin{aligned} x_1 &= 2\frac{\beta_p |\lambda_q|^2 - \beta_q |\lambda_p|^2}{|\lambda_q|^2 - |\lambda_p|^2}, \\ y_1 &= \frac{\beta_p |\lambda_q|^2 - \beta_q |\lambda_p|^2}{\beta_q - \beta_p}. \end{aligned} \tag{2.3}$$

Therefore, in this case the minimum of the function $f(x,y) = x^2/y$ on $W(S)$ is

$$f(x_1, y_1) = \frac{x_1^2}{y_1} = \frac{4\left(\beta_p |\lambda_q|^2 - \beta_q |\lambda_p|^2\right)(\beta_q - \beta_p)}{\left(|\lambda_q|^2 - |\lambda_p|^2\right)^2}. \tag{2.4}$$

Thus $\mu_1^2(T) = 4(\beta_p|\lambda_q|^2 - \beta_q|\lambda_p|^2)(\beta_q - \beta_p)/(|\lambda_q|^2 - |\lambda_p|^2)^2$, which implies

$$\mu_1(T) = \frac{2\sqrt{(\beta_q - \beta_p)\left(\beta_p |\lambda_q|^2 - \beta_q |\lambda_p|^2\right)}}{|\lambda_q|^2 - |\lambda_p|^2}. \tag{2.5}$$

□

Example 2.6. Find $\mu_1(T)$ if T is a normal matrix with eigenvalues $1+\sqrt{6}i$, $2+\sqrt{7}i$, $3+4i$, $4+\sqrt{34}i$, and $5+\sqrt{35}i$. First, we need to compute the corresponding eigenvalues of $S = \operatorname{Re} T + iT^*T$. These eigenvalues are $1+7i$, $2+11i$, $3+25i$, $4+50i$, and $5+60i$. The first critical eigenvalue of S is $\gamma_p = 1+7i$. Thus $\lambda_p = 1+\sqrt{6}i$ is the first critical eigenvalue of T. Table 2.1 shows the slopes (or approximate values for slopes) of the line segments between the point (1,7) and points (2,11), (3,25), (4,50), and (5,60). Since the largest slope obtained is 14.333, the second critical eigenvalue for S is $\gamma_q = 4+50i$. The corresponding second critical eigenvalue for T is therefore $4+\sqrt{34}i$. To find out exactly what $\mu_1(T)$ is, we need to compare the values $1/\sqrt{7}$, $2/\sqrt{11}$, $3/\sqrt{25}$, $4/\sqrt{50}$, $5/\sqrt{60}$, and $2\sqrt{(\beta_q - \beta_p)(\beta_p|\lambda_q|^2 - \beta_q|\lambda_p|^2)}/|\lambda_q|^2 - |\lambda_p|^2 = 2\sqrt{(4-1)(50-28)}/50 - 7 = 2\sqrt{66}/43$. The smallest of these numbers is $2\sqrt{66}/43$. Hence we have $\mu_1(T) = 2\sqrt{66}/43$.

We can indeed develop an algorithm to compute all higher antieigenvalues of a strictly accretive normal matrix T. Notice that if T is the direct sum of two operators T_1 and T_2, $T = T_1 \oplus T_2$, and $S = \operatorname{Re} T + iT^*T$; then $S = S_1 \oplus S_2$ where $S_1 = \operatorname{Re} T_1 + iT_1^*T_1$ and $S_2 = \operatorname{Re} T_2 + iT_2^*T_2$. Hence, by Halmos [10, page 116], we have $W(S) = \operatorname{co}(W(S_1), W(S_2))$, where $\operatorname{co}(W(S_1), W(S_2))$ denotes the convex hull of the numerical ranges of S_1 and S_2. To compute $\mu_2(T)$, strike out those eigenvalues of S that express $\mu_1(T)$. Let E_1 be the direct sum of the eigenspaces that correspond to eigenvalues which are stricken out and let E_2 be the direct sum of the eigenspaces corresponding to the remaining eigenvalues. We have $T = T_1 \oplus T_2$ where T_1 is the restriction of T on E_1 and T_2 is the restriction of T on E_2. Therefore,

$$\mu_2(T) = \mu_1(T_2) = \inf\left\{\frac{x^2}{y} : x + iy \in W(S_2)\right\}. \tag{2.6}$$

Thus, to compute $\mu_2(T)$, we can replace T with T_2 and compute $\mu_1(T_2)$ as discussed above.

Example 2.7. Compute all antieigenvalues of a normal matrix T whose eigenvalues are $1+\sqrt{6}i$, $2+\sqrt{7}i$, $3+4i$, $4+\sqrt{34}i$, and $5+\sqrt{35}i$. When computing $\mu_1(T)$ in the previous example we found out that the first and the second critical eigenvalues of S are $1+7i$ and $4+50i$, respectively, and they express $\mu_1(T)$. Hence the corresponding first and second eigenvalues of T are $1+\sqrt{6}i$ and $4+\sqrt{34}i$, respectively. If we strike out the first and the second eigenvalues of S, the remaining eigenvalues of S are $2+11i$, $3+25i$, and $5+60i$. The first critical eigenvalue for S_2 is $(2,11)$. The slope of the line segment connecting $(2,11)$ to $(3,25)$ is 14, and the slope of the line segment connecting $(2,11)$ to $(5,60)$ is 16.33. Therefore, the second critical eigenvalue of S_2 is $(5,60)$. Hence $\mu_2(T)$ is the minimum of the numbers $2/\sqrt{11}$, $5/\sqrt{60}$, $3/\sqrt{25}$, and $2\sqrt{(5-2)((2)(60)-(5)(11))}/50-11 = 2\sqrt{195}/49$. The minimum of these numbers is $2\sqrt{195}/49$. Thus $\mu_2(T) = 2\sqrt{195}/49$. After striking out the first and second critical eigenvalues of S_2, which express $\mu_1(T_2)$, the only eigenvalue left is $(3,25)$ and hence $\mu_3(T) = 3/\sqrt{25}$.

If T is a positive matrix with n distinct eigenvalues $r_1 < r_2 < \cdots < r_n$, it was proved by Gustafson (see [2, 3]) that

$$\mu_1(T) = \frac{2\sqrt{r_1 r_2}}{r_1 + r_2}. \tag{2.7}$$

In [6] Gustafson extended the notion of first antieigenvalue μ_1 to arbitrary A, with polar decomposition $A = U\|A\|$. According to [6], the first antieigenvalue of A is defined to be the first antieigenvalue of $\|A\|$. In that case r_1 and r_2 in (2.7) are the smallest and largest singular values σ_n and σ_1 of A.

A new proof for (2.7) may be obtained within the context of this paper by clarifying that r_1 and r_n are the first and the second critical eigenvalues of T, respectively.

Theorem 2.8. *Let T be a positive matrix with n distinct eigenvalues $r_1 < r_2 < \cdots < r_n$. Then*

$$\mu_i(T) = \frac{2\sqrt{r_i r_{n-i+1}}}{r_i + r_{n-i+1}}, \quad 1 \le i \le n, \tag{2.8}$$

correspond to the critical eigenvalues criteria of this paper.

Proof. Eigenvalues of S are $r_1 + r_1^2 i$, $r_2 + r_2^2 i, \dots, r_n + r_n^2 i$. By definition $r_1 + r_1^2 i$ is the first critical eigenvalue of S. To find the second critical eigenvalue of S, we look at the slopes of line segments joining (r_1, r_1^2) to points (r_2, r_2^2), $(r_3, r_3^2), \dots, (r_n, r_n^2)$. These slopes are $(r_2^2 - r_1^2)/(r_2 - r_1) = r_1 + r_2$, $(r_3^2 - r_1^2)/(r_3 - r_1) = r_1 + r_3, \dots, (r_n^2 - r_1^2)/(r_n - r_1) = r_1 + r_n$. The largest among these slopes is $r_1 + r_n$ which shows that (r_n, r_n^2) is the second critical eigenvalue of T. Therefore $\mu_1(T)$ is the minimum of the three numbers $r_1/\sqrt{r_1^2} = 1$, $r_n/\sqrt{r_n^2} = 1$, and $2\sqrt{(r_n - r_1)(r_1 r_n^2 - r_n r_1^2)}/r_n^2 - r_1^2 = 2\sqrt{r_1 r_n}/r_1 + r_n$. This minimum is obviously $2\sqrt{r_1 r_n}/r_1 + r_n$. The expression for $\mu_2(T)$ is obtained by striking out the first and second critical eigenvalues of S we just obtained and by looking at matrix S_2 whose eigenvalues are (r_2, r_2^2), (r_3, r_3^2), (r_{n-1}, r_{n-1}^2). Higher antieigenvalues are obtained similarly. □

3. Antieigenvalues and antieigenvectors are well defined

Now that we have pinned down which pair of eigenvalues express the first antieigenvalue $\mu_1(T)$, we can restate our previous results as follows.

Theorem 3.1. *Let T be an n by n accretive normal matrix. Suppose $\lambda_i = \beta_i + \delta_i i$, $1 \le i \le m$, are eigenvalues of T. Let $E(\lambda_i)$ be the eigenspace corresponding to λ_i and $P(\lambda_i)$ the orthogonal projection on $E(\lambda_i)$. For each vector f let $z_i = P(\lambda_i) f$. Let $\lambda_p = \beta_p + \delta_p i$ be the first critical eigenvalue of T, then one of the following cases holds:*

(1) *if the second critical eigenvalue of T does not exist, then $\mu_1(T) = \beta_p / |\lambda_p|$. In this case antieigenvectors of norm 1 satisfy $\|z_p\| = 1$ and $\|z_i\| = 0$ if $i \ne p$,*

(2) *if $\lambda_q = \beta_q + \delta_q i$, the second critical eigenvalue of T, exists then one of the following cases holds:*

(a) *$\mu_1(T) = \beta_i / \lambda_i$ for some eigenvalue $\lambda_i = \beta_i + \delta_i i$ of T and antieigenvectors of norm 1 satisfy $\|z_i\| = 1$ and $\|z_j\| = 0$ if $i \ne j$,*

(b)

$$\mu_1(T) = \frac{2\sqrt{(\beta_q - \beta_p)\left(\beta_p |\lambda_q|^2 - \beta_q |\lambda_p|^2\right)}}{|\lambda_q|^2 - |\lambda_p|^2}, \tag{3.1}$$

and antieigenvectors of norm 1 satisfy

$$\|z_p\|^2 = \frac{\beta_q |\lambda_q|^2 - 2\beta_p |\lambda_q|^2 + \beta_q |\lambda_p|^2}{\left(|\lambda_q|^2 - |\lambda_p|^2\right)(\beta_q - \beta_p)}, \tag{3.2}$$

$$\|z_q\|^2 = \frac{\beta_p |\lambda_p|^2 - 2\beta_q |\lambda_p|^2 + \beta_p |\lambda_q|^2}{\left(|\lambda_q|^2 - |\lambda_p|^2\right)(\beta_q - \beta_p)}, \tag{3.3}$$

and $\|z_i\| = 0$ if $i \ne p$ and $i \ne q$.

In particular, the antieigenvalues and antieigenvectors are well defined.

Proof. We must clarify that the denominators of the expressions (3.1), (3.2), and (3.3) are nonzero for the critical eigenvalues selected for computing the antieigenvalues. We also clarify that the radicand in the expression (3.1) is nonnegative for these critical eigenvalues. The first and second critical eigenvalues are so defined that we always have $\beta_q > \beta_p$. Also, by the definition of second critical eigenvalue, the slope $m_q = (|\lambda_q|^2 - |\lambda_p|^2)/(\beta_q - \beta_p)$ of the line segment between the two points $(\beta_q, |\lambda_q|^2)$ and $(\beta_q, |\lambda_q|^2)$ is always positive. Hence both of the terms $\beta_q - \beta_p$ and $|\lambda_q|^2 - |\lambda_p|^2$ are positive. This implies that the denominators in expressions (3.1), (3.2), and (3.3) are positive. Also note that the radicand in the numerator of expression (3.1) is negative if $\beta_p|\lambda_q|^2 - \beta_q|\lambda_p|^2 < 0$. In this case $\mu_1(T)$ is not defined by (3.1). In fact, if this happens, no member of the family of the convex functions $y = x^2/k$ can touch the interior of the line segment between $(\beta_p, |\lambda_p|^2)$ and $(\beta_q, |\lambda_q|^2)$ since the components of such a point of contact, which are given by expressions (2.3), cannot be negative (recall that $W(S)$ is a subset of the first quadrant). We also need to show that the quantities on the right side of (3.2) and (3.3) are positive numbers between 0 and 1. We have already shown that the denominators of those expressions are positive for the first critical eigenvalue $\lambda_p = \beta_p + \delta_p i$ and the second critical eigenvalue $\lambda_q = \beta_q + \delta_q i$. We now prove that the numerators of these expressions are also positive for the first and second critical eigenvalues. Recall that Theorem 3.1(2b) occurs only when a member of the family of parabolas $y = x^2/k$ intersects the line segment with end points at $(\beta_p, |\lambda_p|^2)$ and $(\beta_q, |\lambda_q|^2)$ at an interior point of this segment. Therefore, $x_1 = 2(\beta_p|\lambda_q|^2 - \beta_q|\lambda_p|^2)/(|\lambda_q|^2 - |\lambda_p|^2)$, which is the x component of the point of contact, must be between β_p and β_q. In other words

$$\beta_p < 2\frac{\beta_p|\lambda_q|^2 - \beta_q|\lambda_p|^2}{|\lambda_q|^2 - |\lambda_p|^2} < \beta_q, \tag{3.4}$$

which is equivalent to the following two inequalities:

$$\beta_p|\lambda_q|^2 - \beta_p|\lambda_p|^2 < 2\beta_p|\lambda_q|^2 - 2\beta_q|\lambda_p|^2, \tag{3.5}$$

$$2\beta_p|\lambda_q|^2 - 2\beta_q|\lambda_p|^2 < \beta_q|\lambda_q|^2 - \beta_q|\lambda_p|^2. \tag{3.6}$$

The inequality (3.5) is equivalent to the inequality $\beta_p|\lambda_p|^2 - 2\beta_q|\lambda_p|^2 + \beta_p|\lambda_q|^2 > 0$. Notice that $\beta_p|\lambda_p|^2 - 2\beta_q|\lambda_p|^2 + \beta_p|\lambda_q|^2$ is the numerator of the expression on the right side of (3.3). Similarly, the inequality (3.6) is equivalent to $\beta_q|\lambda_q|^2 - 2\beta_p|\lambda_q|^2 + \beta_q|\lambda_p|^2 > 0$. Notice that $\beta_q|\lambda_q|^2 - 2\beta_p|\lambda_q|^2 + \beta_q|\lambda_p|^2$ is the numerator of the expression on the right side of (3.2). Hence the expressions on the right-hand sides of (3.2) and (3.3) are both positive. Since the sum of these two expressions is 1, each of these expressions is a number between 0 and 1. □

Since higher antieigenvalues of T are in fact first antieigenvalues of restrictions of T on certain reducing subspaces of T, the higher antieigenvalues and antieigenvectors are also well defined.

To conclude, we mention that Davis's [1, pages 173–174] theorem may not cover one instance. Thus, for clarity, we mention it here. It is possible that (using his notations) $\rho = \max(|\lambda_1|/|\lambda_2|, |\lambda_2|/|\lambda_1|) = 1$. An example is $\lambda_1 = |\lambda_1|e^{i\theta} = e^{i\theta} = \cos\theta + i\sin\theta$ and $\lambda_2 = |\lambda_2|e^{-i\theta} = e^{-i\theta} = \cos\theta - i\sin\theta$. This pair of eigenvalues λ_1 and λ_2 cannot represent a pair of critical eigenvalues. Recall that, by definition, if $\lambda_p = \beta_p + \delta_p i$ and $\lambda_q = \beta_q + \delta_q i$ are the first and second critical eigenvalues, respectively, we must have $\beta_q > \beta_p$. Thus we are in case (1) of Theorem 3.1.

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