A TOPOLOGICAL ISOMORPHISM INVARIANT FOR CERTAIN AF ALGEBRAS

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For certain AF algebras, a topological space is described which provides an isomorphism invariant for the algebras in this class. These AF algebras can be described in graphical terms by virtue of the existence of a certain type of Bratteli diagram, and the order-preserving automorphisms of the corresponding AF algebra's dimension group are then studied by utilizing this graph. This will also provide information about the automorphism groups of the corresponding AF algebras.

1. Introduction

In studying approximately finite-dimensional (AF) C^*-algebras, [2] introduced an infinite graph, now called a Bratteli diagram, which can be used to describe the structure of the algebra. These graphs have come to play an important role in the theory of AF algebras, and in this paper, we will be concerned with a certain class of Bratteli diagrams which have natural subgraphs that provide isomorphism invariants for the corresponding AF algebras. Not only this, but these subgraphs serve to provide information about the automorphism group of the AF algebras to which they correspond. The UHF algebras will belong to this class, as well as commutative AF algebras such as $C(X)$, where X is the Cantor set.

This paper follows a theme similar to that in [3, 4], where invariants for a certain class of AF algebras are developed by studying the corresponding dimension groups, see [1, 6]. In addition to the information it will provide about AF algebras, the invariant described here will be useful in that it will allow us to say something about the order-preserving automorphisms on these dimension groups. Therefore, these results may also be of interest to those working in the more general context of dimension group theory.

To establish some notation, for a given Bratteli diagram, we will let $V(n)$ denote the vertices at level n, $n \geq 0$ (where $V(0)$ will be a singleton set) and $E(n)$, $n \geq 1$, the set of edges connecting the vertices in $V(n-1)$ with those in $V(n)$. Furthermore, for $e \in E(n)$, let $s(e) \in V(n-1)$ be the vertex at level $n-1$ to which e is connected, and $r(e) \in V(n)$ be the vertex at level n to which e is connected. The Bratteli diagrams to which the results of this paper will then apply are those with the following property.

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Property 1.1. *For each $n \geq 1$ and each $e_i, e_j \in E(n)$, if $r(e_i) = r(e_j)$, then $s(e_i) = s(e_j)$.*

In other words, each vertex at level n is connected to only one vertex at level $n-1$. This implies that the cardinality of the sets $V(n)$ is such that $|V(n+1)| \geq |V(n)|$, and therefore we may assume, without a loss of generality, that either $|V(i)| = |V(j)|$ for all $i, j \neq 0$, or, by inserting levels of vertices if necessary, that $|V(n)| = n+1$ for all $n \geq 0$.

It should be noted that a given AF algebra, due to the fact that it has many different Bratteli diagrams, may have one Bratteli diagram which satisfies Property 1.1 and others which do not. One such example is the UHF 2^∞ algebra, which can be represented by each of the diagrams

$$\bullet \rightrightarrows \bullet \rightrightarrows \cdots \qquad\qquad \begin{tikzcd} & \bullet \arrow[r] \arrow[ddr] & \bullet \arrow[r] \arrow[ddr] & \cdots \\ \bullet \arrow[ur] \arrow[dr] & & & \\ & \bullet \arrow[r] \arrow[uur] & \bullet \arrow[r] \arrow[uur] & \cdots \end{tikzcd} \tag{1.1}$$

The former diagram satisfies Property 1.1 and the latter does not.

Given a diagram satisfying Property 1.1, there will correspond a unique subgraph of the Bratteli diagram (which is itself also a Bratteli diagram), which we will call $G_{\min}$, with the same vertex sets $V(n)$, $n \geq 0$, as the Bratteli diagram, but with edge sets $E'(n) \subset E(n)$ having the following additional property: for each $v \in V(n)$, there exists a unique edge $e \in E'(n)$ such that $r(e) = v$. For the diagram above and on the left, the subgraph $G_{\min}$ will simply be the graph with a single vertex at each level and a single edge connecting these vertices. See Example 3.6 for another such example.

The primary object of interest in this paper will then be the topological space $X_{\min}$ obtained from the graph $G_{\min}$ by letting $X_{\min}$ be the set of all infinite paths in $G_{\min}$. To describe the topology on $X_{\min}$, we let $B(v)$ be the set of all infinite paths in $G_{\min}$ which pass through the vertex $v \in V(n)$. The collection of all such subsets will form a basis for the topology on $X_{\min}$. This makes $X_{\min}$ into a compact 0-dimensional (basis consisting of sets which are both open and closed) Hausdorff space. We note in passing that $X_{\min}$ is the spectrum of the commutative AF algebra with Bratteli diagram given by $G_{\min}$.

The set $X_{\min}$ is also referred to in [8] in the context of ordered Bratteli diagrams. There, $X_{\min}$ is the set of all infinite paths in the ordered diagram for which each edge is minimal. Our definition amounts to the same as in [8] for those Bratteli diagrams satisfying Property 1.1. For more general (ordered) Bratteli diagrams, the set $X_{\min}$ will depend on the order placed on the edges. That is to say, for a given ordered Bratteli diagram, different orderings of the edges will lead to different (nonhomeomorphic) possibilities for $X_{\min}$. For the Bratteli diagrams considered here, this ambiguity does not arise as any order placed on the edges will yield the same set $X_{\min}$. In a more general setting such as [8], some of the results obtained here will still be possible. However, the nonuniqueness of $X_{\min}$ in this case means that the order structure of the corresponding dimension group is significantly more complicated, see [9, Example 4.5].

To briefly preview what follows, we comment that the results of this paper will generalize the following three facts about the AF algebra $C(X)$, where X is a 0-dimensional compact Hausdorff space.

(i) The dimension group of $C(X)$ can be identified with $C(X,\mathbb{Z})$, the continuous functions from X to $\mathbb{Z}$. Such dimension groups are not uncommon in the literature, such as in [10, 11], where certain AF subalgebras of crossed product algebras are studied. For an AF algebra $\mathfrak{A}$ with a Bratteli diagram satisfying Property 1.1, it will be shown that the dimension group of $\mathfrak{A}$ can be identified with a subgroup of $C(X_{\min},\mathbb{Q})$. In this case, the set $X_{\min}$ will be in general a proper closed subspace of the spectrum of the MASA $\mathfrak{D}\subset\mathfrak{A}$ which is spanned by the diagonal matrix units in $\mathfrak{A}$.

(ii) If $C(X_1)\cong C(X_2)$, then X_1 and X_2 are homeomorphic. We show that for AF algebras $\mathfrak{A}_1\cong\mathfrak{A}_2$ which have diagrams satisfying Property 1.1, the sets $X_{\min,1}$ and $X_{\min,2}$ are homeomorphic (Theorem 2.1).

(iii) The order-preserving automorphisms ψ on the dimension group of $C(X)$ are of the form $\psi(f)=f\circ\phi$ for $\phi: X\to X$ a homeomorphism. We will show that this carries over to AF algebras with Bratteli diagrams satisfying Property 1.1 with X replaced by $X_{\min}$.

2. Dimension groups of continuous functions

To begin, let $\mathfrak{A}=\varinjlim(\mathfrak{A}_n,\phi_n)$ be a unital AF algebra with unital injective $*$-homomorphisms ϕ_n and suppose that the Bratteli diagram for $\mathfrak{A}$ satisfies Property 1.1. As mentioned earlier, we may assume, without a loss of generality, that either $|V(n)|$ is constant or $|V(n)|=n+1$ for all n. We begin by considering the latter case. One can then show that, up to unitary equivalence, the connecting homomorphisms ϕ_n correspond to multiplicity matrices (see, e.g., [5]) of the form

$$\overline{A}_{n,n+1}=\begin{bmatrix} a^n_{1,1} & & & & \\ & \ddots & & & \\ & & a^n_{j(n),j(n)} & & \\ & & a^n_{j(n)+1,j(n)} & & \\ & & & a^n_{j(n)+2,j(n)+1} & \\ & & & & \ddots \\ & & & & & a^n_{n+2,n+1}\end{bmatrix}, \tag{2.1}$$

for some $1\le j(n)\le n+1$, where all unspecified entries are zero. By choosing nonzero integers b_n $(=a_{j(n)+1,n+2})$ define, for all $n\ge 0$,

$$A_{n,n+1}=\begin{bmatrix} a^n_{1,1} & & & & & \\ & \ddots & & & & \\ & & a^n_{j(n),j(n)} & & & \\ & & a^n_{j(n)+1,j(n)} & & & b_n \\ & & & a^n_{j(n)+2,j(n)+1} & & \\ & & & & \ddots & \\ & & & & & a^n_{n+2,n+1} & \end{bmatrix}, \tag{2.2}$$

in which case the inverse $A^{-1}_{n,n+1}$ exists.

Now define, for all $n \geq 1$, the linear maps $R_n : \mathbb{Q}^{n+1} \to C(X_{\min}, \mathbb{Q})$, where $C(X_{\min}, \mathbb{Q})$ are the continuous functions from $X_{\min}$ to $\mathbb{Q}$ (giving $\mathbb{Q}$ the discrete topology), by

$$R_n(\alpha_1, \ldots, \alpha_{n+1}) = \alpha_1 \chi_{B(1,0)} + \sum_{l=0}^{n-1} \alpha_{l+2} \chi_{B(j(l)+1, l+1)}, \tag{2.3}$$

where χ_A is the function which is equal to 1 on A and 0 on A^c, and as a notational convenience, we write $B(i,n)$ to denote the clopen subset of $X_{\min}$ which consists of all infinite paths passing through the vertex $v_i(n) \in V(n)$. Here we are labeling the vertices at level n by $v_1(n), \ldots, v_{n+1}(n)$, where the vertex $v_i(n)$ corresponds to the ith row/column in the corresponding multiplicity matrices. It is left to the reader to show that R_n injects.

To compute the dimension group $K_0(\mathfrak{A})$, we utilize the fact that $K_0(\mathfrak{A}) = \varinjlim K_0(\mathfrak{A}_n)$. However, to be more explicit about the nature of $\varinjlim K_0(\mathfrak{A}_n)$, we define, for all $n \geq 1$,

$$A_n = [A_{0,1}^{-1} \oplus I_{n-1}] \cdots [A_{n-2,n-1}^{-1} \oplus I_1] A_{n-1,n}^{-1}. \tag{2.4}$$

Note that A_n has rational entries. Then, let $\Phi_n : \mathbb{Z}^{n+1} \to C(X_{\min}, \mathbb{Q})$ be given by $\Phi_n = R_n \circ A_n$. To show that $K_0(\mathfrak{A})$ can be identified with a subgroup of $C(X_{\min}, \mathbb{Q})$, we would like to show that the diagram

$$\begin{tikzcd} K_0(\mathfrak{A}_n) \cong \mathbb{Z}^{n+1} \arrow[r, "\phi_{n*}"] \arrow[dr, "\Phi_n"'] & K_0(\mathfrak{A}_{n+1}) \cong \mathbb{Z}^{n+2} \arrow[d, "\Phi_{n+1}"] \\ & C(X_{\min}, \mathbb{Q}) \end{tikzcd} \tag{2.5}$$

commutes, where ϕ_{n*} is the homomorphism induced by the standard embedding $\phi_n : \mathfrak{A}_n \to \mathfrak{A}_{n+1}$.

Let $(\alpha_1, \ldots, \alpha_{n+1})^T \in \mathbb{Z}^{n+1}$. Then, $\Phi_{n+1} \circ \phi_{n*}(\alpha_1, \ldots, \alpha_{n+1})^T$ equals

$$\left(R_{n+1} \circ [A_{0,1}^{-1} \oplus I_n] \cdots [A_{n-1,n}^{-1} \oplus I_1] A_{n,n+1}^{-1}\right) \circ A_{n,n+1}(\alpha_1, \ldots, \alpha_{n+1}, 0)^T. \tag{2.6}$$

If we define $(\beta_1^{n-1}, \ldots, \beta_{n+1}^{n-1})^T = A_{n-1,n}^{-1}(\alpha_1, \ldots, \alpha_{n+1})^T$ and, in general, for $2 \leq i < n$, $(\beta_1^{n-i}, \ldots, \beta_{n-i+2}^{n-i})^T = A_{n-i,n-i+1}^{-1}(\beta_1^{n-i+1}, \ldots, \beta_{n-i+2}^{n-i+1})^T$, we see that $\Phi_{n+1} \circ \phi_{n*}(\alpha_1, \ldots, \alpha_{n+1})^T$ equals

$$\begin{aligned} R_{n+1} &\circ [A_{0,1}^{-1} \oplus I_n] \cdots [A_{n-2,n-1}^{-1} \oplus I_2] (\beta_1^{n-1}, \ldots, \beta_{n+1}^{n-1}, 0)^T \\ &= R_{n+1}(\beta_1^0, \beta_2^0, \beta_3^1, \ldots, \beta_n^{n-2}, \beta_{n+1}^{n-1}, 0) \\ &= \beta_1^0 \chi_{B(1,0)} + \sum_{l=0}^{n-1} \beta_{l+2}^l \chi_{B(j(l)+1, l+1)}. \end{aligned} \tag{2.7}$$

However, $\Phi_n(\alpha_1,\dots,\alpha_{n+1})^T = R_n(\beta_1^0,\beta_2^0,\beta_3^1,\dots,\beta_n^{n-2},\beta_{n+1}^{n-1})$, and thus $\Phi_n = \Phi_{n+1}\circ\phi_{n*}$. We conclude that the diagram commutes. By the universal property of the direct limit, it follows that there exists a homomorphism $\Psi : K_0(\mathfrak{A}) \to C(X_{\min},\mathbb{Q})$. Since each of the maps Φ_n is injective, Ψ injects.

Next, we consider the nature of the positive cone $K_0^+(\mathfrak{A})$. Letting $(\alpha_1,\alpha_2)^T \in \mathbb{Z}^2$, we see that $\Phi_1(\alpha_1,\alpha_2)^T$ equals

$$\begin{aligned} R_1\left(\frac{\alpha_1}{a_{1,1}^0},\frac{\alpha_2}{b_0}-\frac{\alpha_1 a_{2,1}^0}{a_{1,1}^0 b_0}\right) &= \frac{\alpha_1}{a_{1,1}^0}\chi_{B(1,0)} + \left(\frac{\alpha_2}{b_0}-\frac{\alpha_1 a_{2,1}^0}{a_{1,1}^0 b_0}\right)\chi_{B(2,1)} \\ &= \frac{\alpha_1}{a_{1,1}^0}\chi_{B(1,1)} + \left(\frac{\alpha_1}{a_{1,1}^0}+\frac{\alpha_2}{b_0}-\frac{\alpha_1 a_{2,1}^0}{a_{1,1}^0 b_0}\right)\chi_{B(2,1)}, \end{aligned} \tag{2.8}$$

since $B(1,0) = B(1,1)\cup B(2,1)$. This is simply $(\alpha_1/a_{1,1}^0)\chi_{B(1,1)} + (\alpha_2/a_{2,1}^0)\chi_{B(2,1)}$ if we choose $b_0 = a_{2,1}^0$. In order to obtain a convenient description of $K_0^+(\mathfrak{A})$, we would like to show that there exists a sequence $\{b_n\}_{n=0}^{\infty}$ of positive integers such that for every $n \ge 1$,

$$\Phi_n(\alpha_1,\dots,\alpha_{n+1})^T = \sum_{i=1}^{n+1}\frac{\alpha_i}{k(i,n)}\chi_{B(i,n)}, \tag{2.9}$$

where $k(i,n) \in \mathbb{Z}^+$, for all $1 \le i \le n+1$. Our above calculations prove that it is possible to choose b_0 accordingly.

Assume that $b_0,\dots,b_{k-1} \in \mathbb{Z}^+$ have been chosen in order to guarantee that Φ_k is of the appropriate form for $1 \le k \le n$. Let $(\alpha_1,\dots,\alpha_{n+2})^T \in \mathbb{Z}^{n+2}$. Then,

$$\Phi_{n+1}(\alpha_1,\dots,\alpha_{n+2})^T = R_{n+1}\begin{bmatrix} A_n & 0 \\ 0 & 1\end{bmatrix} A_{n,n+1}^{-1}(\alpha_1,\dots,\alpha_{n+2})^T, \tag{2.10}$$

from which it follows that $\Phi_{n+1}(\alpha_1,\dots,\alpha_{n+2})^T$ is equal to

$$\Phi_n\left(\frac{\alpha_1}{a_{1,1}^n},\dots,\frac{\alpha_{j(n)}}{a_{j(n),j(n)}^n},\frac{\alpha_{j(n)+2}}{a_{j(n)+2,j(n)+1}^n},\dots,\frac{\alpha_{n+2}}{a_{n+2,n+1}^n}\right) + \left(\frac{\alpha_{j(n)+1}}{b_n}-\frac{\alpha_{j(n)}a_{j(n)+1,j(n)}^n}{b_n a_{j(n),j(n)}^n}\right)\chi_{B(j(n)+1,n+1)}. \tag{2.11}$$

Hence, by the induction hypothesis, $\Phi_{n+1}(\alpha_1,\dots,\alpha_{n+2})^T$ equals

$$\sum_{i=1}^{j(n)}\frac{\alpha_i}{a_{i,i}^n k(i,n)}\chi_{B(i,n)} + \sum_{i=j(n)+1}^{n+1}\frac{\alpha_{i+1}}{a_{i+1,i}^n k(i,n)}\chi_{B(i,n)} + \left(\frac{\alpha_{j(n)+1}}{b_n}-\frac{\alpha_{j(n)}a_{j(n)+1,j(n)}^n}{b_n a_{j(n),j(n)}^n}\right)\chi_{B(j(n)+1,n+1)}. \tag{2.12}$$

Now, we note that for $1 \le i \le j(n)-1$, $B(i,n) = B(i,n+1)$, and for $j(n)+1 \le i \le n+1$, $B(i,n) = B(i+1,n+1)$. Furthermore, $B(j(n),n) = B(j(n),n+1)\cup B(j(n)+1,n+1)$.

Thus, $\Phi_{n+1}(\alpha_1,\dots,\alpha_{n+2})^T$ becomes

$$\sum_{i=1}^{j(n)}\frac{\alpha_i}{a_{i,i}^n k(i,n)}\chi_{B(i,n+1)}+\sum_{i=j(n)+1}^{n+1}\frac{\alpha_{i+1}}{a_{i+1,i}^n k(i,n)}\chi_{B(i+1,n+1)}$$
$$+\left[\frac{\alpha_{j(n)}}{a_{j(n),j(n)}^n k(j(n),n)}\left(1-\frac{a_{j(n)+1,j(n)}^n k(j(n),n)}{b_n}\right)\right]\chi_{B(j(n)+1,n+1)}+\frac{\alpha_{j(n)+1}}{b_n}\chi_{B(j(n)+1,n+1)}. \tag{2.13}$$

If we let $b_n = a_{j(n)+1,j(n)}^n k(j(n),n)$, then we have the desired outcome.

The significance of this is that when determining the positive cone $K_0^+(\mathfrak{A})$, we see that since the image of $K_0^+(\mathfrak{A})$ inside of $C(X_{\min},\mathbb{Q})$ is $\bigcup_{n\geq 1}\Phi_n(\mathbb{Z}_+^{n+1})$, clearly $K_0^+(\mathfrak{A})$ can be identified with a subset of $C(X_{\min},\mathbb{Q}^+)$. Furthermore, for all $U\subset X_{\min}$ clopen, each function χ_U will be an element of the positive cone. Therefore, for such AF algebras, the hypotheses of the following theorem are satisfied.

Theorem 2.1. *Suppose that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are AF algebras such that there exist injective homomorphisms $\Psi_i : K_0^+(\mathfrak{A}_i)\to C(X_i,\mathbb{Q}^+)$, for $i=1,2$, where the sets X_i are 0-dimensional compact Hausdorff spaces and where each Ψ_i maps the order unit of $K_0(\mathfrak{A}_i)$ to χ_{X_i}. Assume further that for all $U\subset X_i$ clopen, $\chi_U\in\Psi_i(K_0^+(\mathfrak{A}_i))$. If $\psi : K_0(\mathfrak{A}_1)\to K_0(\mathfrak{A}_2)$ is an order-preserving isomorphism taking order unit to order unit, then X_1 and X_2 are homeomorphic.*

Proof. To begin, let $x\in X_1$ and take $\{U_n\}_{n=1}^\infty$ to be a decreasing sequence of clopen subsets in X_1 with $\{x\}=\bigcap_{n=1}^\infty U_n$. Then, suppressing explicit mention of the maps Ψ_i, we have $\chi_{X_2}=\psi(\chi_{X_1})=\psi(\chi_{U_n})+\psi(\chi_{U_n^c})$, for all $n\geq 1$, where U_n^c is the complement of U_n in X_1. If we suppose that the supports of the functions $\psi(\chi_{U_n})$ and $\psi(\chi_{U_n^c})$ are not disjoint, then there exists a nonempty clopen set $V\subset X_2$ such that $\alpha\chi_V\leq\psi(\chi_{U_n}),\ \psi(\chi_{U_n^c})$, where $\alpha\in\mathbb{Q}^+$, $\alpha\neq 0$. Because ψ^{-1} is order preserving, it follows that $\psi^{-1}(\alpha\chi_V)\leq\chi_{U_n},\chi_{U_n^c}$, implying that $\psi^{-1}(\alpha\chi_V)=0$ since $U_n\cap U_n^c=\varnothing$. Thus, $\alpha\chi_V=0$, a contradiction, and we conclude that $\psi(\chi_{U_n})$ and $\psi(\chi_{U_n^c})$ have disjoint supports. Therefore, by continuity, there exists a nonempty clopen subset $V_n\subset X_2$ such that $\psi(\chi_{U_n})=\chi_{V_n}$, for all $n\geq 1$. Because $\{\chi_{U_n}\}_{n=1}^\infty$ is a decreasing sequence in $C(X_1,\mathbb{Q})$, $\{\chi_{V_n}\}_{n=1}^\infty$ decreases in $C(X_2,\mathbb{Q})$, and $\{V_n\}_{n=1}^\infty$ is a decreasing sequence of clopen sets. It is relatively straightforward to show that $\bigcap_{n=1}^\infty V_n$ is a singleton, and so we let y be the lone element in this intersection.

Define $\theta : X_1\to X_2$ by letting $\theta(x)=y$, for all $x\in X_1$. Clearly, this map is well defined since any two sequences $\{U_n\}$, $\{U_n'\}$ which decrease down to $\{x\}$ can be intertwined, resulting in the intertwining of the corresponding sequences $\{V_n\}$ and $\{V_n'\}$. It is also clear that θ is a bijection. Finally, for U clopen in X_1, one sees that if $\psi(\chi_U)=\chi_V$, then $\theta(U)=V$. Therefore, θ is an open map, and similarly so is its inverse. Hence, X_1 is homeomorphic to X_2. $\square$

3. Applications and further questions

Considering those AF algebras which have a Bratteli diagram satisfying Property 1.1, we have the following corollaries of Theorem 2.1.

Corollary 3.1. *Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be AF algebras, each with Bratteli diagrams satisfying Property 1.1. If $\mathfrak{A}_1 \cong \mathfrak{A}_2$, then $X^1_{\min}$ is homeomorphic to $X^2_{\min}$.*

Corollary 3.2. *Suppose that $\mathfrak{A}$ is an AF algebra with a Bratteli diagram satisfying Property 1.1. Then, under the identification of $K_0(\mathfrak{A})$ with a subgroup of $C(X_{\min}, \mathbb{Q})$ as above, every order-preserving automorphism ψ of $K_0(\mathfrak{A})$ which takes the order unit onto itself must be of the form $\psi(f) = f \circ \theta^{-1}$ for some homeomorphism θ of $X_{\min}$.*

Remark 3.3. When a Bratteli diagram satisfies Property 1.1 and $|V(n)|$ remains constant for all $n \geq 1$, the multiplicity matrices for the corresponding AF algebra are permutation similar to diagonal matrices. Thus, the above results generalize to this situation as well.

Remark 3.4. One can use Corollary 3.2 to provide a new proof of the well-known result that identifies automorphisms on $C(X)$, X a 0-dimensional compact metric space, with homeomorphisms on X, see [7, Theorem 3.4.3].

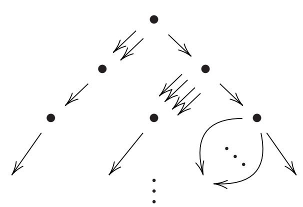
We now mention some applications of Corollary 3.2 which provide information about the automorphism groups of certain AF algebras.

Corollary 3.5. *Suppose that $\mathfrak{A}$ is an AF algebra with a Bratteli diagram satisfying Property 1.1. If the only homeomorphism of $X_{\min}$ is the identity map, then* $\mathrm{Aut}(\mathfrak{A}) = \overline{\mathrm{Inn}(\mathfrak{A})}$.

Proof. Each automorphism $\alpha \in \mathrm{Aut}(\mathfrak{A})$ induces an automorphism $\alpha_* \in \mathrm{Aut}(K_0(\mathfrak{A}))$. By Corollary 3.2, $\alpha_*(g) = g \circ \theta^{-1}$ for some homeomorphism θ of $X_{\min}$, where we are identifying $K_0(\mathfrak{A})$ with a subgroup of $C(X_{\min}, \mathbb{Q})$. By hypothesis, it follows that $\alpha_* = \mathrm{id}_*$, and therefore, by [5, Theorem IV.5.7], $\alpha \in \overline{\mathrm{Inn}(\mathfrak{A})}$. Hence, $\mathrm{Aut}(\mathfrak{A}) = \overline{\mathrm{Inn}(\mathfrak{A})}$. In particular, this applies to the UHF algebras, and should be compared with [5, Corollary IV.5.8]. □

The converse of Corollary 3.5 is in general not true. The following example illustrates this and demonstrates an application of Corollary 3.2 which utilizes the structure of $X_{\min}$ to obtain complete information about the automorphism groups of $K_0(\mathfrak{A})$ and $\mathfrak{A}$.

Example 3.6. Consider the AF algebra $\mathfrak{A}$ with Bratteli diagram



(3.1)

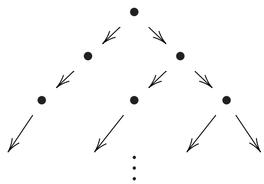
where the multiplicity matrices are such that

$$\overline{A}_{n,n+1} = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 2^{n+1} \\ & & & 1 \end{bmatrix} \in M_{n+2,n+1}, \tag{3.2}$$

for all $n \geq 0$. In defining the matrix $A_{n,n+1}$, if we choose the sequence $\{b_n\}$ from Section 2 so that $b_n = 1$ for all n, one can verify by induction on n that

$$A_n = \begin{bmatrix} 2^{-1} & & & & \\ -2^{-1} & 2^{-2} & & & \\ & -2^{-2} & & & \\ & & \ddots & & \\ & & & 2^{-n} & \\ & & & -2^{-n} & 1 \end{bmatrix}. \tag{3.3}$$

With $X_{\min}$ given (uniquely) as the set of all infinite paths in the graph



(3.4)

we have, for any $(\alpha_1,\dots,\alpha_{n+1})^T \in \mathbb{Z}^{n+1}$, that

$$\Phi_n(\alpha_1,\dots,\alpha_{n+1})^T = \left(\sum_{l=1}^{n} \alpha_l 2^{-l}\chi_{B(l,n)}\right) + \alpha_{n+1}\chi_{B(n+1,n)}. \tag{3.5}$$

By Corollary 3.2, every order-preserving automorphism ψ of $K_0(\mathfrak{A})$ must be of the form $\psi(f) = f \circ \theta^{-1}$ for some homeomorphism θ of $X_{\min}$, where $f \in K_0(\mathfrak{A}) \subset C(X_{\min}, \mathbb{Q})$. The set $X_{\min}$ is easily seen to be homeomorphic to $\overline{\{1/n : n \in \mathbb{Z}^+\}}$, and thus, any homeomorphism θ of $X_{\min}$ must fix the point corresponding to 0. It follows that if θ is a nonidentity homeomorphism of $X_{\min}$, then there exist $n \geq 1$ and $1 \leq l_1 < l_2 \leq n$ such that $\theta(B(l_1,n)) = B(l_2,n)$, where here, $B(l_1,n)$ and $B(l_2,n)$ are singletons. But then, for the function $f = 2^{-l_1}\chi_{B(l_1,n)} + 2^{-l_2}\chi_{B(l_2,n)} \in K_0(\mathfrak{A})$, we see that

$$\psi(f) = f \circ \theta^{-1} = 2^{-l_1}\chi_{B(l_2,n)} + 2^{-l_2}\chi_{B(l_1,n)} \notin K_0(\mathfrak{A}). \tag{3.6}$$

Hence, we conclude that the only order-preserving automorphism of $K_0(\mathfrak{A})$ is the identity map.

Consequently, the converse of Corollary 3.5 is not true. After all, we have just shown that the only order-preserving automorphism of $K_0(\mathfrak{A})$ is the identity map. So again, by [5, Theorem IV.5.7], $\mathrm{Aut}(\mathfrak{A}) = \overline{\mathrm{Inn}(\mathfrak{A})}$. However, $X_{\min}$ clearly has many nontrivial homeomorphisms.

As some of the results here generalize certain well-known results from the theory of AF algebras, a natural next question is how much the results in the present paper can be generalized further. It is possible to define a generalization of the set $X_{\min}$, such as in [8] for ordered Bratteli diagrams, for an arbitrary AF algebra. The nonuniqueness of $X_{\min}$ in this case is problematic, but it may still be possible to generalize some of the results here to fit into this context.

Furthermore, it would be interesting to know if being able to embed an AF algebra's dimension group into a dimension group of the form $(C(X,\mathbb{Q}),C(X,\mathbb{Q}^+),\chi_X)$, where X is a 0-dimensional compact Hausdorff space, is sufficient for its having a Bratteli diagram satisfying Property 1.1.

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