CHARACTERIZATIONS OF FIXED POINTS OF NONEXPANSIVE MAPPINGS

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Using the notion of Banach limits, we discuss the characterization of fixed points of nonexpansive mappings in Banach spaces. Indeed, we prove that the two sets of fixed points of a nonexpansive mapping and some mapping generated by a Banach limit coincide. In our discussion, we may not assume the strict convexity of the Banach space.

1. Introduction

Let C be a closed convex subset of a Banach space E. A mapping T on C is called a *nonexpansive mapping* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T. Kirk [17] proved that $F(T)$ is nonempty in the case that C is weakly compact and has normal structure. See also [2, 3, 5, 6, 11] and others.

Convergence theorems to fixed points are also proved by many authors; see [1, 7, 8, 9, 10, 13, 15, 18, 23, 30] and others. Very recently, the author proved the convergence theorems for two nonexpansive mappings without the assumption of the strict convexity of the Banach space. To prove this, the author proved the following theorem, which plays an extremely important role in [26].

Theorem 1.1 (see [26]). *Let C be a compact convex subset of a Banach space E and let T be a nonexpansive mapping on C. Then $z \in C$ is a fixed point of T if and only if*

$$\liminf_{n\to\infty} \left\| \frac{1}{n}\sum_{i=1}^{n} T^i z - z \right\| = 0 \tag{1.1}$$

holds.

The author also proved the following theorem. Using it, we give one nonexpansive retraction onto the set of all fixed points.

Theorem 1.2 (see [27]). *Let E be a Banach space with the Opial property and let C be a weakly compact convex subset of E. Let T be a nonexpansive mapping on C. Put*

$$M(n,x) = \frac{1}{n}\sum_{i=1}^{n} T^i x \tag{1.2}$$

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for $n \in \mathbb{N}$ and $x \in C$. Then for $z \in C$, the following are equivalent:

(i) *z is a fixed point of T;*

(ii) *$\{M(n,z)\}$ converges weakly to z;*

(iii) *there exists a subnet $\{M(\nu_\beta,z) : \beta \in D\}$ of a sequence $\{M(n,z)\}$ in C converging weakly to z.*

In this paper, using the notion of Banach limits, we generalize Theorems 1.1 and 1.2. We remark that the proofs of our results are simpler than the proofs of Theorems 1.1 and 1.2. In our discussion, we may not assume the strict convexity of the Banach space.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers. For a subset A of $\mathbb{N}$, we define a function I_A from $\mathbb{N}$ into $\mathbb{R}$ by

$$I_A(n) = \begin{cases} 1 & \text{if } n \in A, \\ 0 & \text{if } n \notin A. \end{cases} \tag{2.1}$$

Let E be a Banach space. We denote by E^* the dual of E. We recall that E is said to have the *Opial property* [21] if for each weakly convergent sequence $\{x_n\}$ in E with weak limit x_0, $\liminf_n \|x_n - x_0\| < \liminf_n \|x_n - x\|$ for all $x \in E$ with $x \neq x_0$. All Hilbert spaces, all finite-dimensional Banach spaces, and $\ell^p (1 \leq p < \infty)$ have the Opial property. A Banach space with a duality mapping which is weakly sequentially continuous also has the Opial property; see [12]. We know that every separable Banach space can be equivalently renormed so that it has the Opial property; see [31]. Gossez and Lami Dozo [12] prove that every weakly compact convex subset of a Banach space with the Opial property has normal structure. See also [19, 20, 22, 25] and others.

We denote by ℓ^∞ the Banach space consisting of all bounded functions from $\mathbb{N}$ into $\mathbb{R}$ (i.e., all bounded real sequences) with supremum norm. We recall that $\mu \in (\ell^\infty)^*$ is called a *mean* if $\|\mu\| = \mu(I_{\mathbb{N}}) = 1$. It is equivalent to

$$\inf_{n \in \mathbb{N}} a(n) \leq \mu(a) \leq \sup_{n \in \mathbb{N}} a(n) \tag{2.2}$$

for all $a \in \ell^\infty$. We also know that if $a(n) \leq b(n)$ for all $n \in \mathbb{N}$, then $\mu(a) \leq \mu(b)$ holds. Sometimes, we denote by $\mu_n(a(n))$ the value $\mu(a)$. $\mu \in (\ell^\infty)^*$ is called a *Banach limit* if the following hold:

(i) μ is a mean;

(ii) $\mu(a) = \mu_n(a(n+1))$ for all $a \in \ell^\infty$. That is, putting $b(n) = a(n+1)$ for $n \in \mathbb{N}$, we have $\mu(a) = \mu(b)$.

It is obvious that

$$\mu(a) = \mu_n(a(n+k)) \tag{2.3}$$

for a Banach limit μ, $a \in \ell^\infty$, and $k \in \mathbb{N}$. We know that Banach limits exist; see [4]. We also know that

$$\liminf_{n\to\infty} a(n) \le \mu(a) \le \limsup_{n\to\infty} a(n) \tag{2.4}$$

for all $a \in \ell^\infty$.

Let T be a nonexpansive mapping on a weakly compact convex subset C of a Banach space E. Let μ be a Banach limit. Then we know that for $x \in C$, there exists a unique element x_0 of C satisfying

$$\mu_n(f(T^n x)) = f(x_0) \tag{2.5}$$

for all $f \in E^*$; see [14, 19]. Following Rodé [24], we denote such x_0 by $T_\mu x$. We also know that T_μ is a nonexpansive mapping on C.

We now prove the following lemmas, which are used in Section 3.

Lemma 2.1. *Let $a, b \in \ell^\infty$ and let μ be a Banach limit. Then the following hold.*

(i) *If there exists $n_0 \in \mathbb{N}$ such that $a(n) \le b(n)$ for all $n \ge n_0$, then $\mu(a) \le \mu(b)$ holds.*

(ii) *If there exists $n_0 \in \mathbb{N}$ such that $a(n) = b(n)$ for all $n \ge n_0$, then $\mu(a) = \mu(b)$ holds.*

Proof. We first show (i). We note that $a(n_0+n) \le b(n_0+n)$ for all $n \in \mathbb{N}$. Since μ is a Banach limit, we have

$$\mu_n(a(n)) = \mu_n(a(n_0+n)) \le \mu_n(b(n_0+n)) = \mu_n(b(n)). \tag{2.6}$$

It is obvious that (ii) follows from (i). This completes the proof. □

Lemma 2.2. *Let $A_1, A_2, A_3, \dots, A_k$ be subsets of $\mathbb{N}$ and let μ be a Banach limit. Put*

$$A = \bigcap_{j=1}^{k} A_j, \qquad \alpha = \sum_{j=1}^{k} \mu(I_{A_j}) - k + 1. \tag{2.7}$$

Suppose that $\alpha > 0$. Then,

$$\mu(I_A) \ge \alpha \tag{2.8}$$

holds and

$$\{n \in \mathbb{N} : n \ge n_0\} \cap A \ne \varnothing \tag{2.9}$$

holds for all $n_0 \in \mathbb{N}$.

Proof. It is obvious that $n \in A$ if and only if

$$\sum_{j=1}^{k} I_{A_j}(n) = k, \tag{2.10}$$

and $n \in \mathbb{N} \setminus A$ if and only if

$$\sum_{j=1}^{k} I_{A_j}(n) \le k-1. \tag{2.11}$$

Therefore we obtain

$$I_A(n) \ge \sum_{j=1}^{k} I_{A_j}(n) - k + 1 \tag{2.12}$$

for all $n \in \mathbb{N}$. Hence,

$$\begin{aligned} \mu(I_A) &\ge \mu_n\left(\sum_{j=1}^{k} I_{A_j}(n) - k + 1\right) \\ &= \sum_{j=1}^{k} \mu(I_{A_j}) - k + 1 \\ &= \alpha > 0. \end{aligned} \tag{2.13}$$

We suppose that $\{n \in \mathbb{N} : n \ge n_0\} \cap A = \varnothing$ for some $n_0 \in \mathbb{N}$. Then $I_A(n) = 0$ for $n \ge n_0$. So, from Lemma 2.1, we obtain

$$0 < \alpha \le \mu(I_A) = \mu(0) = 0. \tag{2.14}$$

This is a contradiction. This completes the proof. □

3. Main results

In this section, we prove our main results.

We first prove the following theorem, which plays an important role in this paper.

Theorem 3.1. *Let E be a Banach space and let C be a weakly compact convex subset of E. Let T be a nonexpansive mapping on C. Let μ be a Banach limit. Suppose that $T_\mu z = z$ for some $z \in C$. Then there exist sequences $\{p_n\}$ in $\mathbb{N}$ and $\{f_n\}$ in E^* such that*

$$\begin{gathered} p_{n+1} > p_n, \\ \|T^{p_n}z - z\| \ge \lambda - \frac{1}{3^{n+1}}, \\ f_\ell(T^{p_n}z - z) \le \frac{2^{\ell+1}}{3^{\ell+1}} \quad \text{for } \ell = 1,2,\ldots,n-1, \\ \|f_n\| = 1, \qquad f_n(T^{p_n}z - z) = \|T^{p_n}z - z\| \end{gathered} \tag{3.1}$$

for all $n \in \mathbb{N}$, where

$$\lambda = \limsup_{n\to\infty} \|T^n z - z\|. \tag{3.2}$$

Before proving this theorem, we need some preliminaries. In the following lemmas and the proof of Theorem 3.1, we put

$$A(f,\varepsilon) = \{n \in \mathbb{N} : f(T^n z - z) \le \varepsilon\} \tag{3.3}$$

for $f \in E^*$ and $\varepsilon > 0$, and

$$B(\varepsilon) = \{n \in \mathbb{N} : \|T^n z - z\| \ge \lambda - \varepsilon\} \tag{3.4}$$

for $\varepsilon > 0$.

Lemma 3.2. *For every $n \in \mathbb{N}$,*

$$\|T^n z - z\| \le \lambda \tag{3.5}$$

holds.

Proof. Since μ is a Banach limit, we have $\mu_n(\|T^n z - z\|) \le \lambda$. Fix $m \in \mathbb{N}$. By the Hahn-Banach theorem, there exists $f \in E^*$ such that

$$\|f\| = 1, \qquad f(T^m z - z) = \|T^m z - z\|. \tag{3.6}$$

For $n \in \mathbb{N}$, we have

$$\begin{aligned}
\|T^m z - z\| &= f(T^m z - z) \\
&= f(T^m z - T^{m+n} z) + f(T^{m+n} z - z) \\
&\le \|f\| \|T^m z - T^{m+n} z\| + f(T^{m+n} z - z) \\
&= \|T^m z - T^{m+n} z\| + f(T^{m+n} z - z) \\
&\le \|T^n z - z\| + f(T^{m+n} z - z).
\end{aligned} \tag{3.7}$$

Hence

$$\begin{aligned}
\|T^m z - z\| &= \mu_n(\|T^m z - z\|) \\
&\le \mu_n(\|T^n z - z\| + f(T^{m+n} z - z)) \\
&= \mu_n(\|T^n z - z\|) + \mu_n(f(T^{m+n} z)) - f(z) \\
&= \mu_n(\|T^n z - z\|) + \mu_n(f(T^n z)) - f(z) \\
&= \mu_n(\|T^n z - z\|) + f(T_\mu z) - f(z) \\
&= \mu_n(\|T^n z - z\|) \\
&\le \lambda.
\end{aligned} \tag{3.8}$$

This completes the proof. □

Lemma 3.3. *Suppose that $m \in \mathbb{N}$, $f \in E^*$, and $\delta > 0$ satisfy that*

$$\|f\| = 1, \qquad f(T^m z - z) = \|T^m z - z\| \geq \lambda - \delta. \tag{3.9}$$

Then

$$\mu(I_{A(f,\varepsilon)}) \geq \frac{\varepsilon}{\varepsilon + \delta} \tag{3.10}$$

holds for all $\varepsilon > 0$.

Proof. For $n > m$, by Lemma 3.2, we have

$$\begin{aligned} f(T^n z - z) &= f(T^n z - T^m z) + f(T^m z - z) \\ &\geq -\|f\|\|T^n z - T^m z\| + f(T^m z - z) \\ &= -\|T^n z - T^m z\| + \|T^m z - z\| \\ &\geq -\|T^{n-m} z - z\| + \|T^m z - z\| \\ &\geq -\lambda + \lambda - \delta = -\delta. \end{aligned} \tag{3.11}$$

On the other hand, by the definition of $A(f,\varepsilon)$,

$$f(T^n z - z) > \varepsilon \tag{3.12}$$

for all $n \in \mathbb{N} \setminus A(f,\varepsilon)$. Therefore, for $n \in \mathbb{N}$ with $n > m$, we have

$$\begin{aligned} f(T^n z - z) &\geq -\delta I_{A(f,\varepsilon)}(n) + \varepsilon I_{\mathbb{N}\setminus A(f,\varepsilon)}(n) \\ &= -\delta I_{A(f,\varepsilon)}(n) + \varepsilon I_{\mathbb{N}\setminus A(f,\varepsilon)}(n) + (\varepsilon - \varepsilon) I_{A(f,\varepsilon)}(n) \\ &= -(\delta + \varepsilon) I_{A(f,\varepsilon)}(n) + \varepsilon I_{\mathbb{N}}(n) \\ &= -(\delta + \varepsilon) I_{A(f,\varepsilon)}(n) + \varepsilon. \end{aligned} \tag{3.13}$$

By Lemma 2.1, we have

$$\begin{aligned} 0 &= f(T_\mu z) - f(z) \\ &= \mu_n(f(T^n z)) - f(z) \\ &= \mu_n(f(T^n z - z)) \\ &\geq \mu_n(-(\delta + \varepsilon)\, I_{A(f,\varepsilon)}(n) + \varepsilon) \\ &= -(\delta + \varepsilon)\, \mu(I_{A(f,\varepsilon)}) + \varepsilon. \end{aligned} \tag{3.14}$$

Hence, we obtain

$$\mu(I_{A(f,\varepsilon)}) \geq \frac{\varepsilon}{\varepsilon + \delta}. \tag{3.15}$$

This completes the proof. □

LEMMA 3.4. *$\mu(I_{B(\varepsilon)}) = 1$ holds for all $\varepsilon > 0$.*

Proof. We fix $\varepsilon > 0$ and $\eta \in \mathbb{R}$ with $1/2 < \eta < 1$ and put

$$\delta = \frac{\varepsilon(1-\eta)}{2\eta}. \tag{3.16}$$

We note that $0 < \delta < \varepsilon/2$. By the definition of λ, there exists $m \in \mathbb{N}$ such that

$$\|T^m z - z\| \geq \lambda - \delta. \tag{3.17}$$

Fix $f \in E^*$ with

$$\|f\| = 1, \qquad f(T^m z - z) = \|T^m z - z\|. \tag{3.18}$$

So, using Lemma 3.3, we have

$$\mu(I_{A(f,\varepsilon/2)}) \geq \frac{\varepsilon/2}{\varepsilon/2+\delta} = \eta. \tag{3.19}$$

For $n \in \mathbb{N}$ with $m+n \in A(f,\varepsilon/2)$, we have

$$\begin{aligned}\|T^n z - z\| &\geq \|T^m z - T^{m+n} z\| \\ &\geq f(T^m z - T^{m+n} z) \\ &= f(T^m z - z) + f(z - T^{m+n} z) \\ &= \|T^m z - z\| + f(z - T^{m+n} z) \\ &\geq \lambda - \delta - \frac{\varepsilon}{2} \\ &\geq \lambda - \varepsilon,\end{aligned} \tag{3.20}$$

and hence $n \in B(\varepsilon)$. Therefore

$$I_{B(\varepsilon)}(n) \geq I_{A(f,\varepsilon/2)}(m+n) \tag{3.21}$$

for all $n \in \mathbb{N}$. So we obtain

$$\begin{aligned}\mu(I_{B(\varepsilon)}) &\geq \mu_n(I_{A(f,\varepsilon/2)}(m+n)) \\ &= \mu_n(I_{A(f,\varepsilon/2)}(n)) \\ &\geq \eta.\end{aligned} \tag{3.22}$$

Since η is arbitrary, we obtain the desired result. □

Proof of Theorem 3.1. By the definition of λ, there exists $p_1 \in \mathbb{N}$ such that

$$\|T^{p_1} z - z\| \geq \lambda - \frac{1}{3^2}. \tag{3.23}$$

Fix $f_1 \in E^*$ with

$$\|f_1\| = 1, \qquad f_1(T^{p_1} z - z) = \|T^{p_1} z - z\|. \tag{3.24}$$

By Lemma 3.3, we have

$$\mu(I_{A(f_1,(2/3)^2)}) \geq \frac{2^2}{2^2+1}. \tag{3.25}$$

We now define inductively sequences $\{p_n\}$ in $\mathbb{N}$ and $\{f_n\}$ in E^*. Suppose that $p_k \in \mathbb{N}$ and $f_k \in E^*$ are known. Since

$$\begin{aligned}
&\mu(I_{B(1/3^{k+2})}) + \sum_{\ell=1}^{k} \mu(I_{A(f_\ell,(2/3)^{\ell+1})}) - k \\
&\quad \geq 1 + \sum_{\ell=1}^{k} \frac{2^{\ell+1}}{2^{\ell+1}+1} - k \\
&\quad \geq 1 + \sum_{\ell=1}^{k} \frac{2^{\ell+1}-1}{2^{\ell+1}} - k = 1 + \sum_{\ell=1}^{k} \frac{-1}{2^{\ell+1}} \\
&\quad > \frac{1}{2} > 0,
\end{aligned} \tag{3.26}$$

we have

$$\{m \in \mathbb{N} : m \geq p_k + 1\} \cap B\left(\frac{1}{3^{k+2}}\right) \cap \bigcap_{\ell=1}^{k} A\left(f_\ell, \left(\frac{2}{3}\right)^{\ell+1}\right) \neq \varnothing \tag{3.27}$$

by Lemma 2.2. So we can choose $p_{k+1} \in \mathbb{N}$ such that $p_{k+1} > p_k$,

$$||T^{p_{k+1}}z - z|| \geq \lambda - \frac{1}{3^{k+2}}, \qquad f_\ell(T^{p_{k+1}}z - z) \leq \frac{2^{\ell+1}}{3^{\ell+1}} \tag{3.28}$$

for $\ell = 1, 2, \ldots, k$. Fix $f_{k+1} \in E^*$ with

$$||f_{k+1}|| = 1, \qquad f_{k+1}(T^{p_{k+1}}z - z) = ||T^{p_{k+1}}z - z||. \tag{3.29}$$

Note that

$$\mu(I_{A(f_{k+1},(2/3)^{k+2})}) \geq \frac{2^{k+2}}{2^{k+2}+1} \tag{3.30}$$

by Lemma 3.3. Hence we have defined $\{p_n\}$ and $\{f_n\}$. □

Now, we prove our main results.

Theorem 3.5. *Let C be a weakly compact convex subset of a Banach space E with the Opial property. Let T be a nonexpansive mapping on C. Let μ be a Banach limit. Then $z \in C$ is a fixed point of T if and only if $T_\mu z = z$.*

Proof. We first assume that $z \in C$ is a fixed point of T. Then, we have

$$f(T_\mu z) = \mu_n(f(T^n z)) = \mu_n(f(z)) = f(z) \tag{3.31}$$

for all $f \in E^*$, and hence $T_\mu z = z$. Conversely, we assume that $T_\mu z = z$. By Theorem 3.1, there exist sequences $\{p_n\}$ in $\mathbb{N}$ and $\{f_n\}$ in E^* satisfying the conclusion of Theorem 3.1. We put $\lambda = \limsup_n \|T^n z - z\|$. Since C is weakly compact, there exists a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ such that $\{T^{p_{n_k}} z\}$ converges weakly to some point $u \in C$. If $n_k > \ell$, then

$$f_\ell(T^{p_{n_k}} z - z) \leq \frac{2^{\ell+1}}{3^{\ell+1}}. \tag{3.32}$$

So we obtain

$$f_\ell(u - z) \leq \frac{2^{\ell+1}}{3^{\ell+1}} \tag{3.33}$$

for all $\ell \in \mathbb{N}$. Since

$$\begin{aligned} ||T^{p_\ell} z - u|| &= ||f_\ell|| ||T^{p_\ell} z - u|| \\ &\geq f_\ell(T^{p_\ell} z - u) \\ &= f_\ell(T^{p_\ell} z - z) + f_\ell(z - u) \\ &= ||T^{p_\ell} z - z|| + f_\ell(z - u) \\ &\geq \lambda - \frac{1}{3^{\ell+1}} - \frac{2^{\ell+1}}{3^{\ell+1}} \end{aligned} \tag{3.34}$$

for $\ell \in \mathbb{N}$, we have

$$\liminf_{\ell \to \infty} ||T^{p_\ell} z - u|| \geq \lambda, \tag{3.35}$$

and hence

$$\begin{aligned} \liminf_{k \to \infty} ||T^{p_{n_k}} z - z|| \leq \lambda \leq \liminf_{\ell \to \infty} ||T^{p_\ell} z - u|| \\ \leq \liminf_{k \to \infty} ||T^{p_{n_k}} z - u||. \end{aligned} \tag{3.36}$$

By the Opial property of E, we obtain $z = u$. We also have

$$\liminf_{k \to \infty} ||T^{p_{n_k}} z - Tz|| \leq \liminf_{k \to \infty} ||T^{p_{n_k} - 1} z - z|| \leq \lambda, \tag{3.37}$$

and hence $Tz = u$. Therefore $Tz = z$. This completes the proof. □

Theorem 3.6. *Let C be a compact convex subset of a Banach space E. Let T be a nonexpansive mapping on C. Let μ be a Banach limit. Then $z \in C$ is a fixed point of T if and only if $T_\mu z = z$.*

Proof. From the proof of Theorem 3.5, we know that $Tz = z$ implies that $T_\mu z = z$. Conversely, we assume that $T_\mu z = z$. By Theorem 3.1, there exist sequences $\{p_n\}$ in $\mathbb{N}$ and $\{f_n\}$ in E^* satisfying the conclusion of Theorem 3.1. We put $\lambda = \limsup_n \|T^n z - z\|$. Since C is compact, there exists a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ such that $\{T^{p_{n_k}} z\}$ converges strongly to some point $u \in C$. As in the proof of Theorem 3.5, we obtain $\liminf_\ell \|T^{p_\ell} z - u\| \geq \lambda$. This implies that $\lambda = 0$, and hence $\{T^n z\}$ converges to z. So we have

$$Tz = T\Big(\lim_{n\to\infty} T^n z\Big) = \lim_{n\to\infty} T^{n+1} z = z. \tag{3.38}$$

This completes the proof. □

Appendix

In some sense, Theorems 3.5 and 3.6 are generalizations of Theorems 1.2 and 1.1, respectively. To show this, we use the notion of universal nets. We recall that a net $\{y_\beta : \beta \in D\}$ in a topological space Y is universal if for each subset A of Y, there exists $\beta_0 \in D$ satisfying either of the following:

(i) $y_\beta \in A$ for all $\beta \in D$ with $\beta \geq \beta_0$; or

(ii) $y_\beta \in Y \setminus A$ for all $\beta \in D$ with $\beta \geq \beta_0$.

For every net $\{y_\beta : \beta \in D\}$, by the axiom of choice, there exists a universal subnet $\{y_{\beta_\gamma} : \gamma \in D'\}$ of $\{y_\beta : \beta \in D\}$. If f is a mapping from Y into a topological space Z and $\{y_\beta : \beta \in D\}$ is a universal net in Y, then $\{f(y_\beta) : \beta \in D\}$ is also a universal net in Z. If Y is compact, then a universal net $\{y_\beta : \beta \in D\}$ in Y always converges. See [16] and others for details.

Proposition A.1. *Let $\{\nu_\beta : \beta \in D\}$ be a universal subnet of a sequence $\{n : n \in \mathbb{N}\}$ in $\mathbb{N}$. Define a function μ from ℓ^∞ into $\mathbb{R}$ by*

$$\mu(a) = \lim_{\beta\in D} \frac{1}{\nu_\beta} \sum_{i=1}^{\nu_\beta} a(i) \tag{A.1}$$

for all $a \in \ell^\infty$. Then μ is a Banach limit.

Proof. We note that μ is well defined because $\{\nu_\beta : \beta \in D\}$ is universal. It is obvious that μ is linear. For $a \in \ell^\infty$, we have

$$\mu(a) = \lim_{\beta\in D} \frac{1}{\nu_\beta} \sum_{i=1}^{\nu_\beta} a(i) \leq \lim_{\beta\in D} \frac{1}{\nu_\beta} \sum_{i=1}^{\nu_\beta} \|a\| = \lim_{\beta\in D} \|a\| = \|a\|. \tag{A.2}$$

Similarly, we obtain $\mu(a) \geq -\|a\|$. Hence $\|\mu\| \leq 1$. Since $\mu(I_{\mathbb{N}}) = 1$, we have $\|\mu\| = \mu(I_{\mathbb{N}}) = 1$, that is, μ is a mean on ℓ^∞. We also have

$$\begin{aligned} \mu_n(a(n+1)) &= \lim_{\beta\in D} \frac{1}{\nu_\beta} \sum_{i=1}^{\nu_\beta} a(i+1) \\ &= \lim_{\beta\in D} \left(\frac{1}{\nu_\beta} \sum_{i=1}^{\nu_\beta} a(i) - \frac{a(1)}{\nu_\beta} + \frac{a(\nu_\beta+1)}{\nu_\beta} \right) \\ &= \mu(a) \end{aligned} \tag{A.3}$$

for all $a \in \ell^\infty$. This completes the proof. □

Proposition A.2. *Let C be a weakly compact convex subset of a Banach space E and let T be a nonexpansive mapping on C. Define $M(\cdot,\cdot)$ as in Theorem 1.2. Take a universal subnet $\{\nu_\beta : \beta \in D\}$ of a sequence $\{n : n \in \mathbb{N}\}$ in $\mathbb{N}$ and define a mapping U on C by*

$$Ux = \underset{\beta\in D}{\text{weak-lim}}\, M(\nu_\beta, x) \tag{A.4}$$

for all $x \in C$. Then there exists a Banach limit μ satisfying $T_\mu x = Ux$ for all $x \in C$.

Proof. Define a Banach limit μ as in Proposition A.1. Then for $x \in C$ and $f \in E^*$, we have

$$\begin{aligned} f(Ux) &= \lim_{\beta\in D} f\left(\frac{1}{\nu_\beta} \sum_{i=1}^{\nu_\beta} T^i x \right) \\ &= \lim_{\beta\in D} \frac{1}{\nu_\beta} \sum_{i=1}^{\nu_\beta} f(T^i x) \\ &= \mu_n(f(T^n x)) \\ &= f(T_\mu x). \end{aligned} \tag{A.5}$$

Since f is arbitrary, we have $Ux = T_\mu x$ for all $x \in C$. This completes the proof. □

Using Proposition A.2, we obtain the following proposition.

Proposition A.3. *Let C be a weakly compact convex subset of a Banach space E and let T be a nonexpansive mapping on C. Then if $z \in C$ satisfies Theorem 1.2(iii), then there exists a Banach limit μ satisfying $T_\mu z = z$.*

Proof. Define $M(\cdot,\cdot)$ as in Theorem 1.2. Take a universal subnet $\{\nu_{\beta_\gamma} : \gamma \in D'\}$ of $\{\nu_\beta : \beta \in D\}$. We note that $\{\nu_{\beta_\gamma} : \gamma \in D'\}$ is also a universal subnet of a sequence $\{n : n \in \mathbb{N}\}$ in $\mathbb{N}$. We also note that $\{M(\nu_{\beta_\gamma}, z) : \gamma \in D'\}$ converges weakly to z because $\{M(\nu_\beta, z) : \beta \in D\}$ does. Define a mapping U on C by

$$Ux = \underset{\gamma\in D'}{\text{weak-lim}}\, M(\nu_{\beta_\gamma}, x) \tag{A.6}$$

for all $x \in C$. It is obvious that $Uz = z$. By Proposition A.2, there exists a Banach limit μ satisfying $U = T_\mu$. Then we have $T_\mu z = z$. This completes the proof. □

In the case that E is a Hilbert space, or E is a uniformly convex Banach space with a Fréchet differentiable norm, T_μ itself is a nonexpansive retraction from C onto $F(T)$; see Baillon [1] and Bruck [8]. In general, this does not hold. We finally give an example.

Example A.4 (see [27, 28]). Define a compact convex subset C of $(\mathbb{R}^2, \|\cdot\|_\infty)$ by

$$C = \{(x_1,x_2) : 0 \le x_2 \le 1, -x_2 \le x_1 \le x_2\}. \tag{A.7}$$

Define a nonexpansive mapping T on C by

$$T(x_1,x_2) = (-x_1, |x_1|) \tag{A.8}$$

for all $(x_1,x_2) \in C$. Then, $F(T) = \{(0,0)\}$ and

$$T_\mu(x_1,x_2) = (0, |x_1|) \tag{A.9}$$

for $(x_1,x_2) \in C$ and a Banach limit μ. That is, T_μ is not a nonexpansive retraction from C onto $F(T)$.

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